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ARBITRAGE PRICING THEORY 50 YEARS AFTER BLACK MERTON SCHOLES

Robert A. Jarrow

In the past 50 years, the Black Merton Scholes option pricing methodology has advanced in three directions: the mathematical foundations, modifying its assumptions, and applications to new derivatives. This lecture reviews the advances with respect to the mathematical foundations and the modifications of its assumptions. The key insight of this review is that the BMS methodology is very robust to modifying all of its assumptions, except for a relaxation of competitive markets.

1 Introduction

The purpose of this paper is to review arbitrage pricing theory 50 years after the publication of the Black Merton Scholes (BMS) (Black and Scholes, 1973; Merton, 1973) pricing methodology. The BMS methodology planted a seed that grew into a massive tree with strong roots. The roots are the formal mathematical foundations of the BMS model—the fundamental theorems of asset pricing that characterize the notion of no-arbitrage and complete markets. The trunk and branches correspond to the extensions (the relaxation of the assumptions) and the applications (the pricing of various derivatives) of the BMS model. This review will discuss the roots and extensions, not the applications. The key insight of this review is that the BMS methodology is very robust to modifying all of its assumptions, except one. It is not robust with respect to the relaxation of the competitive markets assumption.

An outline for this paper is as follows. First, we review the BMS model. Second the fundamental theorems, and third the extensions. The extensions correspond to the relaxations of the BMS assumptions. The extensions covered include alternative risky asset price evolutions, the Heath *et al.* (1992) model, the reduced form credit risk model (see Jarrow and Turnbull, 1992, 1995), market frictions, incomplete markets, and non-competitive markets.

Samuel Curtis Johnson Graduate School of Management, Cornell University, Ithaca, NY 14853, USA.
E-mail: raj15@cornell.edu

2 The BMS Model (The Seed)

This section reviews the BMS model so that we can understand the extensions introduced and the motivation for these extensions.

2.1 The assumptions

The assumptions in the Black and Scholes (1973) paper are the following:

Assumption 1. Competitive markets. This means that all traders act as price takers, believing that their trades have no quantity impact on the price process.

Assumption 2. Frictionless markets. This means that there are no transaction costs (e.g. bid/ask spreads, fees for trading) and no trading constraints (e.g. short sales or margin requirements).

Assumption 3. Continuous time and trading over a finite horizon with time denoted $t \in [0, T]$.

Assumption 4. No arbitrage.[1]

Assumption 5. Constant default-free interest rates, i.e. the value of a money market account at time $t \in [0, T]$ is represented as

$$B_t = e^{-rt}$$

where the default-free spot rate is $r > 0$, a constant.

Assumption 6. The stock price process is geometric Brownian motion, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with S_0, $\mu, \sigma > 0$ constants and W_t a standard Brownian motion initialized at 0. In addition, we assume that there are no cash flows from the stock (this is easily relaxed so that it will not be included in the subsequent extensions studied).

In Merton's (1973) paper, he modified the previous two assumptions to include:

Assumption 5. A stochastic default-free zero-coupon bond maturing on the option's maturity date.

Assumption 6. A stochastic drift μ_t and a deterministic volatility $\sigma_t > 0$.

2.2 The implications

Given the above assumptions, we can value and hedge derivatives. The original papers focused on pricing European call options, whose lead we follow next. Let C_t denote the time t value of a European call with maturity T and strike K on the stock S_t. Under the above structure, it can be shown that risk-neutral valuation holds, i.e.

$$\frac{C_t}{B_t} = E_t^{\mathbb{Q}}\left[\frac{\max(S_T - K, 0)}{B_T}\right]$$

where $\mathbb{Q}$ is the risk neutral probability measure and $E_t^{\mathbb{Q}}[\cdot]$ is conditional expectation under $\mathbb{Q}$. A calculation gives the famous BMS formula (with constant r, σ).

$$C_t = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\log(S_t/K) + \left(r + \frac{1}{2}\sigma^2(T-t)\right)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$N(\cdot)$ is the standard cumulative normal distribution function.

Most importantly, the market is complete (the definition will be provided below) and there exists a dynamic trading strategy that replicates the option's payoffs at time T. This is known as synthetic replication, and it is perhaps the key insight underlying the various extensions of the BMS methodology. The position in the stock at time t that replicates the call option's payoff at maturity is called the hedge ratio, and it is the partial derivative of the option value with respect to the time t stock price.

2.3 The limitations of BMS

A model works well when the assumptions are good approximations of the market's structure. As a consequence, the BMS model applies well to large volume, actively traded, and liquid markets for the underlying (approximating frictionless and competitive), and where there are only a finite number of assets trading whose price processes' volatilities are constant or deterministic (approximating geometric Brownian motion). Unfortunately, most markets do not satisfy these conditions.

In markets for most underlyings (equities, foreign currencies, commodities, interest rates), volatilities are typically stochastic and prices may exhibit jumps. In addition, for some asset classes there is a term structure of prices (interest rates, forwards, futures, calls and puts with different strikes and maturities), implying an infinite number of correlated assets trade. Furthermore, markets are not often frictionless, given that transaction costs and trading constraints exist. And finally, non-competitive trading situations can arise where large trade sizes affect the price paid or received per share, and the impact lasts for a non-trivial time interval.

To value derivatives in these various and different market settings, we need to modify the BMS assumptions. In this regard, I will discuss modifying each of these assumptions separately and independently. Of course, one can combine many of the extensions together, which is often done in practice. Before that, however, we need to discuss the mathematical foundations of arbitrage pricing theory.

3 The Mathematical Foundations (The Roots)

The mathematical foundations of the BMS methodology has been advanced over the past 50 years. The basic model structure and its implications are discussed next. This is a brief overview of the relevant issues, for the formal definitions and development of the following concepts see Jarrow (2022).

3.1 The set-up

The randomness in the market is characterized by a filtered probability space with $\mathbb{P}$ the statistical probability measure. Traded continuously in time and over a finite horizon with $t \in [0, T]$ are a finite number of risky assets and a money market account (mma). The mma is used as a numeraire. Trading strategies using these securities are stochastic processes that are predictable, self-financing, and admissible (i.e. there exists a uniform lower bound on the value of any trading strategy). The assumptions, analogous to those underlying the BMS model, are the following.

Assumption 1. Competitive markets.

Assumption 2. Frictionless markets (no transaction costs, no trading constraints).

Assumption 3. A finite number of asset price processes that are semimartingales (allows discontinuous sample path processes).

Let B_t denote the value of the money market account at time $t \in [0, T]$ that is assumed to be of finite variation and strictly positive. For simplicity of the notation, we consider only one risky asset with time $t \in [0, T]$ price S_t.

Before introducing the remaining assumptions, we need some definitions.

No Simple Arbitrage Opportunities (*NA*). The market contains no admissible self-financing trading strategies with zero initial investment, that never lose value, and with strictly positive probability have a strictly positive value at some future date.

No Free Lunch with Vanishing Risk (*NFLVR*). The market contains no admissible self-financing trading strategies that are approximate simple arbitrage opportunities.

No Dominance (*ND*). The market contains no assets whose payoffs can be obtained at a lower initial cost by an admissible self-financing trading strategy.

Complete Markets with respect to a probability measure $\mathbb{Q} \sim \mathbb{P}$ *that makes* S/B *a* $\mathbb{Q}$ *local martingale*. Any derivative's payoffs that are appropriately integrable with respect to such a $\mathbb{Q}$ can be replicated with an admissible self-financing trading strategy whose value process is a $\mathbb{Q}$ martingale.

3.2 The fundamental theorems

There are three fundamental theorems. For proofs and additional discussion see Jarrow (2022).

Theorem 1. *NFLVR if and only if there exists an equivalent local martingale measure* $\mathbb{Q}$ (*i.e.*, S/B *is a* $\mathbb{Q}$ *local martingale*).

Theorem 2. *NFLVR and ND if and only if there exists an equivalent martingale measure* (*i.e.*, S/B *is a* $\mathbb{Q}$ *martingale*).

Theorem 3. *Assume NFLVR and ND. Choose an equivalent martingale measure* $\mathbb{Q}$. *The martingale measure is complete with respect to* $\mathbb{Q}$ *if and only if the equivalent martingale measure* $\mathbb{Q}$ *is unique.*

Although confusing, Theorem 3 is known as the second fundamental theorem, and Theorem 2 is known as the third fundamental theorem. The reason for this order reversal is that historically, Theorem 2 was the last of the three theorems proven in the literature.

An important corollary of these three theorems is the following:

Corollary (Risk-Neutral Valuation) *Assume NFLVR*, *ND*, *and a complete market with respect to the equivalent martingale measure* $\mathbb{Q}$. *Then, the time t price of any derivative with an appropriately measurable and integrable payoff* $\frac{X_{\mathcal{T}}}{B_{\mathcal{T}}}$ *at a time* $\mathcal{T} \in [0, T]$ *with* $\mathcal{T} \geq t \geq 0$ *satisfies*

$$X_t = E_t^Q \left[\frac{X_{\mathcal{T}}}{B_{\mathcal{T}}} \right] B_t$$

and synthetic construction holds, *i.e. there exists an admissible self-financing trading strategy such that its value process* $V_t = X_t$ *for all t a.e.* $\mathbb{P}$.

Consequently, when using this formal methodology to invoke risk-neutral valuation, one needs to add the following assumptions on the market so that risk-neutral valuation can be invoked.

Assumption 4. NFLVR and ND.

Assumption 5. Complete markets.

3.3 Some new economic insights – Asset price bubbles

The formal mathematical foundations led to some new insights. In particular, given just NFLVR, the difference between the normalized risky asset price process being a strict local martingale versus a martingale is equivalent to the existence of an asset price bubble, defined by

$$\frac{\beta_t}{B_t} := \frac{S_t}{B_t} - E_t^{\mathbb{Q}} \left(\frac{S_T}{B_T} \right) > 0$$

where $E_t^{\mathbb{Q}}\left(\frac{S_T}{B_T}\right)$ is the asset's fundamental value, which is the risk-adjusted discounted cash flow from holding the asset and not reselling it before the model's horizon time T. The fundamental value as given is the standard definition from the classical economics literature.

Given the existence of asset price bubbles, the following facts result.

Fact 1. NFLVR and complete markets are consistent with asset price bubbles (when ND is violated).

Fact 2. NFLVR, ND, and complete markets imply no asset price bubbles.

Fact 3. NFLVR, ND, and incomplete markets are consistent with asset price bubbles.

Given these facts, under certain market structures, we can still price various derivatives (e.g. calls and puts) given price bubbles exist. We will explore these insights shortly.

Remark. The Original BMS Model implies no Price Bubbles

It can be shown that the BMS asset price evolution is consistent with NFLVR and ND, which implies the existence of a martingale measure. The evolution also implies that the market is complete. Hence, by fact 2, the BMS stock price process implies, by construction, a market with no stock price bubbles. If one believes price bubbles exist, one cannot use the original BMS model. This completes the remark.

4 The Extensions (The Trunk and Branches)

This section relaxes the assumptions underlying the BMS model sequentially, starting with different asset price evolutions.

4.1 Alternative risky asset price evolutions

This extension was already explored in the mathematical foundations of arbitrage pricing theory in the preceding section, so the discussion will be brief. First the assumptions.

Assumption 1. A finite number of semimartingale price processes.

These price processes are very general, for example they can have discontinuous sample paths and stochastic volatilities. The "finite" is essential here. Infinite dimensional asset price processes are discussed below (the HJM model).

Assumption 2. NFLVR and ND.

Given this assumption, by the third fundamental theorem (Theorem 2 above), we get the existence of an equivalent martingale measure.

Assumption 3. Complete Markets.

By the corollary, this implies that risk-neutral valuation and synthetic construction applies. Completeness in a market depends on the number of assets trading and the number of martingale shocks underlying the asset price processes. Essentially, for a market to be complete, martingale representation needs to hold for the randomness in the market, and a subset of the assets must have a one-to-one and onto transformation from the asset price processes to the martingale processes. Examples of markets satisfying this condition include stochastic volatilities (if additional securities trade) and jump processes with a finite number of jump magnitudes (e.g. a single jump amplitude which is a market crash).

New Insights: Asset Price Bubbles

The following two market structures are consistent with asset price bubbles:

Market 1. NFLVR, complete (if ND violated).

Market 2. NFLVR, ND, and an incomplete market.

Modified option pricing results for these two market structures have been proven. They are:

Market 1. (a) A buy-and-hold trading strategy in an asset is dominated by a dynamic

self-financing admissible trading strategy using the asset and a money market account.
(b) Risk-neutral valuation typically fails for calls (calls can have bubbles), but is valid for puts (puts have no bubbles).

Market 2. (a) Synthetic construction and risk-neutral valuation fails for both calls and puts (due to the incomplete market structure).
(b) After the selection of a unique local martingale measure for pricing in an incomplete market, calls have bubbles (risk-neutral valuation fails for calls) and puts do not have bubbles (risk-neutral valuation is valid for puts). Here, the call option's bubble equals the bubble in the underlying asset.

Much more is known about markets with price bubbles and how to empirically test for their existence.

4.2 Heath Jarrow Morton (HJM)

This extension corresponds to an infinite number of asset price processes. The canonical example is the term structure of default-free interest rates, which is the HJM (Heath *et al.*, 1992) model. The following assumptions apply.

Assumption 1. Competitive markets.

Assumption 2. Frictionless markets (no transaction costs, no trading constraints).

Assumption 3. A stochastic term structure of default-free interest rates. Trading are a continuum of default-free zero-coupon bonds $p(t, \mathcal{T})$ paying a sure \$1 at time $\mathcal{T}$ with $0 \le t \le \mathcal{T} \le T$ (an infinite dimensional asset price process) and a money market account $B_t = e^{\int_0^t r_s ds}$ where r_t is the default-free spot rate of interest. Given is an initial forward rate curve

$$f(0, \mathcal{T}) \quad \text{for } 0 \le \mathcal{T} \le T$$

where $r_t := f(t, t)$ and the forward rate's evolution is

$$f(t, \mathcal{T}) = f(0, \mathcal{T}) + \int_0^t \mu(s, \mathcal{T}) ds + \sum_{i=1}^{D} \int_0^t \sigma_i(s, \mathcal{T}) dW_i(s) \quad \text{for } 0 \le t \le \mathcal{T} \le T$$

where $W_i(t)$ for $i = 1, \ldots, D$ are standard independent Brownian motions and $\mu(t, \mathcal{T})$ and $\sigma_i(t, \mathcal{T})$ for $i = 1, \ldots, D$ appropriately measurable and integrable.

It is easy to generalize this evolution to include jumps in interest rates.

Assumption 4. NFLVR and ND.

Assumption 5. Complete markets.

Assumption 4 implies that the HJM drift condition on $\mu(t, \mathcal{T})$ must be satisfied. Assumption 5 is satisfied by assuming a non-singular volatility matrix $[\sigma_i(t, \mathcal{T}) : i \times \mathcal{T}]$, appropriately defined for a finite subset (D elements) of the traded zero-coupon bonds.

Given Assumptions 1–4, it can be shown that there exists a unique equivalent martingale measure $\mathbb{Q}$ such that

$$p(t, \mathcal{T}) = E_t^{\mathbb{Q}}\left[e^{-\int_t^{\mathcal{T}} r_s ds}\right].$$

Adding Assumption 5, risk-neutral valuation applies, i.e. given any interest rate derivative's payoff $\frac{X_{\mathcal{T}}}{B_{\mathcal{T}}}$ at a time $\mathcal{T} \in [0, T]$ with $\mathcal{T} \ge t \ge 0$ that is appropriately measurable and integrable satisfies

$$X_t = E_t^{\mathbb{Q}}\left[X_{\mathcal{T}} e^{-\int_t^{\mathcal{T}} r_s ds}\right].$$

Furthermore, one can synthetically construct the derivative's payoff in numerous ways (using different finite subsets of the zero-coupon bonds). These implications imply that interest rate derivatives can be priced and hedged.

Remark. Extensions of HJM to Other Asset Classes.

The same logic underlying the HJM model can be applied to alternative term structures of asset prices. Examples include: (i) foreign currency term structures, (ii) nominal and real term structures, (iii) the term structure of commodity futures prices, (iv) the strike and maturity structures of European call and put options, and (v) the term structure of credit risky bonds. This completes the remark.

4.3 The reduced-form credit risk model

This is an extension of the HJM model to allow for default risk. Although not previously stated, in all of the previous models, there is an implicit assumption that there is no counterparty risk when trading derivatives. Counterparty risk is the risk that one side of the transaction will fail to execute on the derivative's contracted payments. Recall that the trading of derivatives is a zero supply market, and the previous valuation methodologies only apply if both sides of the market execute the terms of the derivative's contract payoffs with probability one. The reduced-form credit risk model, originally created by Jarrow and Turnbull (1992, 1995), relaxes this assumption.

Assumption 1. The HJM model assumptions hold.

Assumption 2. Trading is a risky zero-coupon bond $D(t, \mathcal{T})$ paying a promised \$1 at time $\mathcal{T}$ with $0 \leq t \leq \mathcal{T} \leq T$ issued by a credit entity. Let τ be the random default time of the credit entity (a stopping time). If the credit entity defaults before $\mathcal{T}$, the zero coupon bond pays a random recovery rate of $\delta \leq 1$ dollars.

Assumption 3. NFLVR and ND.

Assumption 4. Complete markets.

Given Assumptions 1–3, there exists a unique equivalent martingale measure $\mathbb{Q}$ such that

$$p(t, \mathcal{T}) = E_t^{\mathbb{Q}}\left[e^{-\int_t^{\mathcal{T}} r_s ds}\right]$$

$$D(t, \mathcal{T}) = E_t^{\mathbb{Q}}\Big[1_{\tau > \mathcal{T}} e^{-\int_t^{\mathcal{T}} r_s ds} + \delta 1_{\tau \leq \mathcal{T}} e^{-\int_t^{\tau} r_s ds}\Big].$$

Adding Assumption 4, risk-neutral valuation applies, i.e. given any credit derivative's payoff $\frac{X_{\mathcal{T}}}{B_{\mathcal{T}}}$ at a time $\mathcal{T} \in [0, T]$ with $\mathcal{T} \geq t \geq 0$ that is appropriately measurable and integrable satisfies

$$X_t = E_t^{\mathbb{Q}}\left[X_{\mathcal{T}} e^{-\int_t^{\mathcal{T}} r_s ds}\right].$$

And, one can synthetically construct the credit derivative's payoffs. Hence, the reduced-form credit risk model enables one to price and hedge credit derivatives, examples include risky coupon bonds, risky coupon bonds with embedded options, and credit default swaps.

Remark. Complications.

Given the default time is typically an inaccessible stopping time, and the recovery rate is a diffuse random variable with a continuum of possible magnitudes, the market is typically incomplete. In this case, to obtain unique pricing of credit derivatives, one can assume that an expanded market including the trading of credit derivatives is complete, which then determines a unique martingale measure $\mathbb{Q}$ that can be used for pricing. However, in this expanded market, synthetic construction only works when the credit derivatives are included in the replicating admissible self-financing trading strategy.

If this alternative assumption does not apply, and the expanded market is still incomplete, then one can price using the methods for pricing in an incomplete market (discussed below). This completes the remark.

4.4 Market frictions

The relaxation of the frictionless market assumption implies introducing either transaction costs or trading constraints into the methodology.

4.4.1 Transaction costs

There are two types of transaction costs: (i) a fixed fee for each trade and (ii) a variable cost per each unit traded. Given fixed costs per each trade, continuous trading is infeasible because the trading costs are infinite when continuously trading over any finite interval. Hence, the previous methodology does not apply. Under this market structure, trading strategies are necessarily limited to buying/selling at only a finite number of stopping times over $[0, T]$. This implies that the market is incomplete. We discuss valuation in incomplete markets later.

Given variable costs per each unit traded, this is equivalent to a temporary quantity impact on the price when trading. More formally, if a trader buys a larger quantity, the average price paid per share increases relative to the no-trade price. If a trader sells a larger quantity, the average price paid per share decreases relative to the no-trade price. Interestingly, one can show that with continuous trading, e.g. high-frequency traders, one can (in the limit) avoid these variable costs and the standard pricing methodology still approximately applies. This is a special case of liquidity risk, which is discussed later.

4.4.2 Trading constraints

Trading constraints, e.g. short sale restrictions, typically make the market incomplete. If this is the case, then the standard pricing and hedging methodology fails. One can value derivatives using an incomplete market method, which is discussed next.

4.5 Incomplete markets

This section reviews what can be done when a market is incomplete and synthetic construction of a derivative's non-linear payoffs fail. From the second fundamental theorem (Theorem 3), there exists an infinite number of equivalent martingale measures. Any one of which gives an arbitrage-free price for a derivative. It can be shown that the range of these arbitrage-free prices is determined by the super- and sub-replication cost of a derivative's payoffs. Unfortunately, this range of potential prices is typically too large to be of any use in practice.

Consequently, to value derivatives in an incomplete market, we need a rule for choosing a unique element from the set of equivalent martingale measures. One approach is to specify an objective function with respect to the hedging error and choose the cost of the trading strategy that optimizes the objective function. The objective function could be a trader's utility function or a risk measure.

A second approach can be used if the hedging error is idiosyncratic, e.g. jump risk with a jump-diffusion asset evolution is diversifiable. Here, the unique martingale measure is determined only by the risks that can be hedged, because the non-hedged idiosyncratic risks have no risk premium.

The problem with both of the existing approaches in practice is that specifying an objective function is difficult, and the second approach often does not apply. A new approach to solving this problem, still assuming no trading constraints, uses ideas from filtration reduction and the fundamental theorems of asset pricing (see Grigorian and Jarrow, 2024a, 2024b). The intuition behind this valuation method is the following.

- The trader examines the possible risks remaining from a partial synthetic replication, and decides which of those risks to hedge exactly, and which to hedge only on average.
- The exactly hedged risks (a subset) determine a filtration reduction and a hypothetical complete market.
  - In the hypothetical complete market the traded prices correspond to the conditional expectation under the statistical probability measure of the original price processes, given the reduced filtration.
  - The reduced filtration represents the risks generated by these traded asset prices that are hedged.
  - This hypothetical market is complete, hence there is a unique martingale measure for pricing derivatives.
- Given a price consistency condition, this unique martingale measure can be uniquely uplifted to the original incomplete market.
  - Interestingly, the price determined by this uplifted martingale measure in the original incomplete market is the cost of constructing the partial hedge in the original market generated by using the exact hedge from the hypothetical complete market.
  - It can be shown that the hedging error of this partial hedge has zero value under this uplifted martingale measure.

This approach appears promising because it is easily applied in practice without explicitly specifying an objective function.

4.6 Non-competitive markets

A non-competitive market is one where traders have a quantity impact on the price process and they do not act as price takers. This is the notion of "liquidity risk," which is characterized by a supply curve for buying and selling. More formally, if a trader buys a larger quantity, the average price paid per share increases. If a trader sells a larger quantity, the average price paid per share decreases. There are two cases to consider in a market with liquidity risk: a temporary or permanent quantity impact on the price.

A temporary quantity impact is defined to be the situation where the quantity impact disappears after the trade is executed. With continuous trading, e.g. high-frequency traders, one can avoid these costs and the standard pricing methodology still approximately applies (see Cetin *et al.*, 2004). The variable transaction cost per unit trade that was previously discussed is a special case of this set-up.

A permanent quantity impact is defined to be the situation where the quantity impact lasts for a finite time period after the initial trade. If there is a permanent quantity impact, then trader's can trade strategically, to obtain "manipulative" profits. In such a market setting, the BMS methodology fails. A simple example shows why. Suppose we are at an instant just before a call option matures and the stock price is just out-of-the money. Suppose all of the remaining BMS model assumptions hold. Assuming competitive markets, this implies that the call option is nearly worthless and the hedge ratio is near zero. Next, suppose a manipulator can buy the underlying stock, pushing it in-the-money. In this case, the call option is worth the in-the-money value and the hedge ratio is one. The non-competitive market's valuation and hedging is seriously in error. The BMS methodology dramatically fails.

In theory, one can still value options in such a market with a permanent quantity impact on the price, but one needs to know the manipulator's trading strategy and their quantity impact on the price. In practice, such information is almost never available.

5 Conclusion

This paper reviews asset pricing theory's advances in the past 50 years. The BMS model was a seed from which a massive tree grew. As evidenced above, the BMS methodology is robust to modifying all of its assumptions to obtain arbitrage-free values and (partial) hedging of the relevant risks, except for one. The BMS methodology fails when the competitive market assumption is violated and there is a permanent quantity impact on the price from trading, so that strategic trading exists. Fortunately, as a first approximation, most markets are not subject to such a violation of the competitive market assumption, which explains why the BMS methodology is one of the most successful applications of economic theory to practice.

Endnote

[1] Complete markets is implied by the stock price process.

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