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EQUIVALENT EXPECTATION MEASURES FOR RISK AND RETURN ANALYSIS OF CONTINGENT CLAIMS

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Nearly half-a-century after the advent of equivalent martingale measures (EMMs), Nawalkha and Zhuo (2022, 2023) generalize these measures to obtain equivalent expectation measures (EEMs) for analyzing risk and return of portfolios of contingent claims over a finite horizon date. The new measures allow the derivation of analytical solutions of the physical moments and co-moments of contingent claim returns until before the horizon date, and serve as pricing measures on or after that date. This novel approach allows Markowitz's (1952) mean–variance optimization to be applied to equity portfolios embedded with options as well as fixed-income portfolios with or without options. This is useful in the investment management of equity funds, bond funds, and hedge funds, for managing risk–return trade-offs more effectively over finite planning horizons.

Modern portfolio theory invented by the late Nobel Laureate Harry Markowitz (1952) has been applied almost exclusively to stocks over the past 70 years. The extension of modern portfolio theory by Sharpe (1964) to an equilibrium in which all investors hold the market portfolio requires additional assumptions besides mean–variance utility, such as homogeneous beliefs and the existence of a "complete" market in which all risks can be traded and hedged. If the market is not complete and/or investors hold divergent beliefs about the direction of the market or volatility risk, then the assumption of mean–variance utility for all investors does not guarantee the efficiency of the market portfolio. Under such conditions, the composition of the risky part of the optimal portfolios for different investors could vary significantly even if all investors have mean–variance utility functions. Optimistic investors may buy call options and write put options, while pessimistic investors may buy put options and write call options, even though the market portfolio will have zero allocation in options. Moreover, since volatility is not traded, more risk-averse investors will demand a higher premium for volatility risk,

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and be long in straddles and strangles perceiving options to be underpriced. The opposite would be true for less risk-averse investors who would demand a lower premium for volatility risk, and be short in straddles and strangles perceiving options to be overpriced. Now, consider how different investors might perform mean–variance optimization over a specified finite horizon under these conditions where the market portfolio is not optimal and the conventional Capital Asset Pricing Model (CAPM) does not hold.

Consider another example of a portfolio manager who holds corporate bonds of a certain risk class, who in addition to measuring the performance of the fund against a comparable bond index with similar maturity and risk profile, is interested in computing the ex-ante Sharpe ratio of the fund over a finite horizon. Using individual bond characteristics and historical data on market prices, the manager is able to fit a structural corporate bond pricing model to value the bonds in the portfolio, but cannot find a simple and straightforward way to estimate the expected returns, variances, and covariances of bond returns over a finite horizon. How would the corporate bond portfolio manager estimate the ex-ante Sharpe ratio over a finite horizon? As another example, consider a portfolio manager who holds a portfolio of Treasuries with bonds of varying maturities. How would the Treasury portfolio manager estimate the ex-ante Sharpe ratio over a finite horizon? Finally, as a more complicated example, consider another hedge fund manager who holds a variety of interest rate derivatives, such as caps and swaptions. How would the hedge-fund manager estimate the ex-ante Sharpe ratio over a finite horizon?

In each of the above examples, whether it's an equity portfolio with embedded options, a corporate bond portfolio, a Treasury bond portfolio, or a portfolio of interest rate derivatives, a large variety of state-of-the-art models exist that can be used for the valuation of the underlying securities. It is also possible to estimate parameters under both the physical measure and the risk-neutral measure under most of these models using advanced econometric techniques, such as the generalized method of moments or Markov Chain Monte Carlo methods.

However, analytical solutions for expected returns and risk measures, such as variance and covariance of returns, did not exist until Nawalkha and Zhuo (2022) introduced the *equivalent expectation measures (EEM)* theory. This theory serves as a generalization of martingale pricing theory, allowing derivation of the analytical solutions of expected returns for all contingent claims that admit analytical solutions to their current prices using equivalent martingale measures. Subsequently, Nawalkha and Zhuo (2023) extended the EEM theory to compute variance, covariance, and other risk measures for these claims over a finite horizon. The EEM theory of Nawalkha and Zhuo (2022, 2023) has opened up the possibility of extending Markowitz's (1952) mean–variance optimization to portfolios that also include contingent claims, since analytical solutions of expected returns, and variances and covariances of returns, over a finite horizon, have become available for the first time for virtually every contingent claim valuation model in finance.

This paper offers a condensed overview of the equivalent expectation measures introduced by Nawalkha and Zhuo (2022), aiming to make these concepts more accessible to a broader audience, including portfolio managers, risk managers, and other industry practitioners. The paper is structured as follows: Section 1 presents the fundamental intuition behind equivalent expectation measures by utilizing a binomial tree in discrete-time. This method is employed to compute the means, variances, and covariance of returns associated with two European options on different stocks, using an equivalent expectation measure denoted as $\mathbb{R}$. Sections 2 and 3 highlight significant discoveries from Nawalkha and Zhuo's (2022) research in continuous-time, using $\mathbb{R}$ and another equivalent expectation measure $\mathbb{R}_1^T$. These two measures are illustrated using the models of Black and Scholes (1973) and Merton (1973b), respectively.

1 The $\mathbb{R}$ Measure—A Discrete-Time Example

The intuition underlying the EEM theory of Nawalkha and Zhuo (2022, 2023) can be exemplified with the construction of a simple discrete-time example to compute the expected returns and variance and covariance of returns of European call options. We begin by constructing an equivalent expectation measure as follows. Define a new probability measure $\mathbb{R}$, which remains the physical measure $\mathbb{P}$ until before time H, and becomes the risk-neutral measure $\mathbb{Q}$ on or after time H, for a specific future horizon date H, that lies between the current time t and a future maturity date T. By construction, the $\mathbb{R}$ measure provides the physical expectation of the claim's time-H future price until before time H, and serves as the pricing (or the equivalent martingale) measure on or after time H.

Using this new measure, we show how to construct a binomial tree to obtain the expected future price of the European call option C, which matures at time T with strike price K, written on an underlying asset price process S. Given $C_T = \max(S_T - K, 0) = (S_T - K)^+$ as the terminal payoff from the call option and using a constant interest rate r, the future price of the call option at time H can be computed under the $\mathbb{Q}$ measure, as follows:

$$C_H = \mathbb{E}_H^{\mathbb{Q}}[\mathrm{e}^{-r(T-H)}(S_T - K)^+].$$

Taking the physical expectation of the future call price at the current time $t \leq H$, gives[1]

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{P}}[\mathbb{E}_H^{\mathbb{Q}}[\mathrm{e}^{-r(T-H)}(S_T - K)^+]]. \tag{1}$$

Suppose we can force the construction of the new equivalent probability measure $\mathbb{R}$ by satisfying the following two conditions:

(i) the $\mathbb{P}$ transition probabilities at time t of all events until time H are equal to the corresponding $\mathbb{R}$ transition probabilities of those events, and
(ii) the $\mathbb{Q}$ transition probabilities at time H of all events until time T are equal to the corresponding $\mathbb{R}$ transition probabilities of those events.

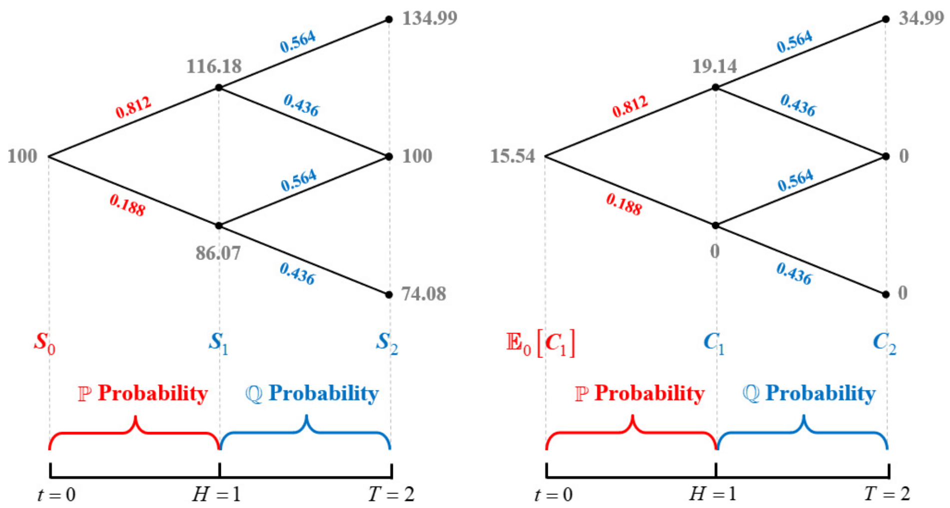
Using this construction, the law of iterated expectations immediately gives:

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}}[\mathrm{e}^{-r(T-H)}(S_T - K)^+]. \tag{2}$$

Note that for each value of the horizon date H—from the current time t until the claim's maturity date T—Equation (2) uses a different $\mathbb{R}$ measure specific to that horizon. The stochastic process under the $\mathbb{R}$ measure evolves under the physical measure $\mathbb{P}$ until before time H, and under the risk-neutral measure $\mathbb{Q}$ on or after time H.

Now, consider a discrete 2-year binomial tree illustrated in Figure 1, in which the current time $t = 0$, $H = 1$ years, $T = 2$ years, and the 1-year short rate is a constant.

To construct the discrete 2-year binomial tree for the stock price process under the $\mathbb{R}$ measure, assume that the current price S_0 is \$100, the stock's annualized expected return μ is 0.1, the annualized risk-free rate r is 0.03, the annualized stock return volatility σ is 0.15, and the discrete interval $\Delta t = 1$ year. The stock prices



Panel A Stock price Panel B Expected call price

Figure 1 The binomial tree of stock S and option C under the $\mathbb{R}$ measure.

This figure shows a 2-year binomial tree structure for the stock price S and its expected call price under the $\mathbb{R}$ measure. The $\mathbb{R}$ measure follows physical $\mathbb{P}$ distribution until before $H = 1$ year, and risk-neutral $\mathbb{Q}$ distribution beginning at time $H = 1$ year. The following parameters are used: $S_t = 100$, $K = 100$, $\mu = 0.1$, $r = 0.03$, $\sigma = 0.15$, $\Delta t = 1$. The up and down node values of the future stock price evolution are computed using the binomial-tree method given by Cox *et al.* (1979). The up and down $\mathbb{R}$ probabilities are the same as the corresponding up and down $\mathbb{P}$ probabilities at time 0, equal to 0.812 and 0.188, respectively; and the up and down $\mathbb{R}$ probabilities are the same as the corresponding up and down $\mathbb{Q}$ probabilities at time 1, equal to 0.564 and 0.436, respectively.

at different nodes are identical under all three measures, $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$, and are given by the binomial model of Cox *et al.* (1979): the stock price either moves up by the multiplicative factor $u = e^{\sigma\sqrt{\Delta t}} = 1.1618$, or moves down by the multiplicative factor $d = e^{-\sigma\sqrt{\Delta t}} = 0.8607$ over each time step, as shown in Figure 1A.

The $\mathbb{R}$ measure is constructed by satisfying conditions (i) and (ii) given above. To satisfy condition (i), the next-period up and down $\mathbb{R}$ probabilities are equal to the corresponding up and down $\mathbb{P}$ probabilities, respectively, at time $t = 0$, and are calculated as:

$$p^u = \frac{e^{\mu\Delta t} - d}{u - d} = \frac{e^{0.1\times 1} - 0.8607}{1.1618 - 0.8607} = 0.812,$$

$$p^d = 1 - p^u = 0.188.$$

To satisfy condition (ii), the next-period up and down $\mathbb{R}$ probabilities are equal to the corresponding up and down $\mathbb{Q}$ probabilities, respectively, at time $H = 1$, and are calculated as:

$$q^u = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.03\times 1} - 0.8607}{1.1618 - 0.8607} = 0.564,$$

$$q^d = 1 - q^u = 0.436.$$

Now, consider the computation of the expected future price $\mathbb{E}_0[C_1]$ of a 2-year European call option written on this stock with a strike price $K = \$100$. The expected future call price can be computed using the $\mathbb{R}$ measure as shown in Figure 1B.

The option prices C_2 at the terminal nodes of the tree are calculated by the payoff function $\max(S_2 - K, 0)$. The option prices C_1 at time

$H = 1$ year are computed using risk-neutral discounting (see Cox *et al.* (1979)). Specifically, the option value C_1 at the up node is

$$\begin{aligned} C_1^u &= e^{-r\Delta t}(q^u \times \$34.99 + q^d \times \$0) \\ &= e^{-0.03\times 1}(0.564 \times \$34.99 + 0.436 \times \$0) \\ &= \$19.14. \end{aligned}$$

Similar calculations give $C_1^d = \$0$ at the down node. Then at the current time $t = 0$, the expected future call price $\mathbb{E}_0[C_1]$ is calculated by taking the expectation of call prices C_1^u and C_1^d, using the up and down $\mathbb{P}$ probabilities, as follows:

$$\begin{aligned} \mathbb{E}_0[C_1] &= p^u \times C_1^u + p^d \times C_1^d \\ &= 0.812 \times \$19.14 + 0.188 \times \$0 \\ &= \$15.54. \end{aligned}$$

Therefore, the expected return of the option over horizon date $H = 1$ can be given as[2]

$$\begin{aligned} \mathbb{E}_0[R_C] &= \frac{\mathbb{E}_0[C_1]}{C_0} - 1 \\ &= \frac{\$15.54}{\$10.47} - 1 = 48.40\%. \end{aligned}$$

Now consider the computation of the variance and covariance of option returns over the 1-year horizon. Consider another stock S', with the current price equal to \$100, annualized expected return μ' equal to 0.15, and annualized stock return volatility σ' equal to 0.2, and the correlation ρ between the two stocks S and S' equal to 0.5. Consider the expected future price of a 2-year call option with a strike price of \$100, written on this stock S', over the 1-year horizon date. The expected future price can be computed using the call prices at the end of 1 year, which, in turn, are computed in a

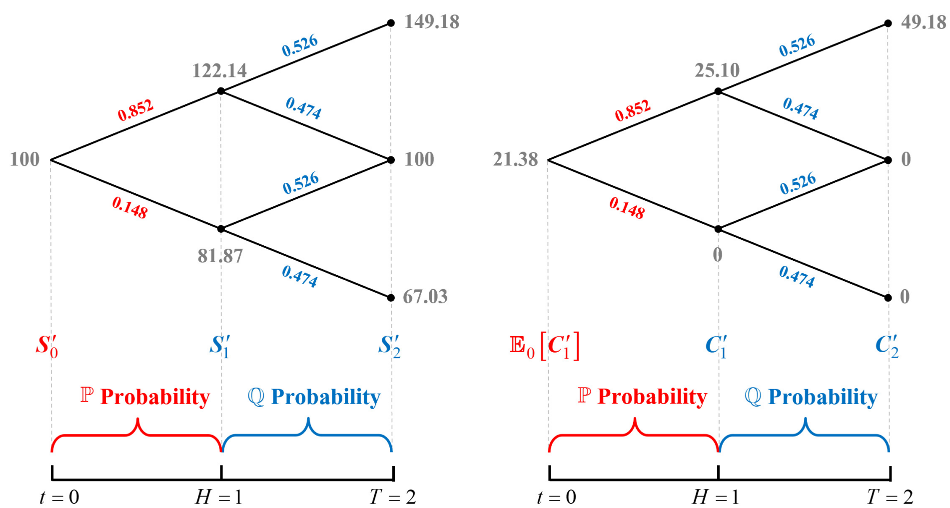


Figure 2 The binomial tree of stock S' and option C' under the $\mathbb{R}$ measure.

This figure shows a 2-year binomial tree structure for the stock price S' and its expected call price under the $\mathbb{R}$ measure. The $\mathbb{R}$ measure follows physical $\mathbb{P}$ distribution until before $H = 1$ year, and risk-neutral $\mathbb{Q}$ distribution beginning at time $H = 1$ year. The following parameters are used: $S'_t = 100$, $K' = 100$, $\mu' = 0.15$, $r = 0.03$, $\sigma' = 0.2$, $\Delta t = 1$. The up and down node values of the future stock price evolution are computed using the binomial-tree method given by Cox *et al.* (1979). The up and down $\mathbb{R}$ probabilities are the same as the corresponding up and down $\mathbb{P}$ probabilities at time 0, equal to 0.852 and 0.148, respectively; and the up and down $\mathbb{R}$ probabilities are the same as the corresponding up and down $\mathbb{Q}$ probabilities at time 1, equal to 0.526 and 0.474, respectively.

similar way in Figure 2 as they are for the first stock in Figure 1.

As in Figure 2B, the 1-year call prices in the up and down states are \$25.10 and 0, respectively. Using these prices, the expected future price $\mathbb{E}_0[C_1']$ is given as follows

$$\begin{aligned}\mathbb{E}_0[C_1'] &= p^{u'} \times C_1^{u'} + p^{d'} \times C_1^{d'}\\ &= 0.852 \times \$25.10 + 0.148 \times \$0\\ &= \$21.38.\end{aligned}$$

Then the expected return on this option over time $H = 1$ is given as

$$\begin{aligned}\mathbb{E}_0[R_{C'}] &= \frac{\mathbb{E}_0[C_1']}{C_0'} - 1\\ &= \frac{\$21.38}{\$12.81} - 1 = 66.99\%.\end{aligned}$$

To obtain the variances of the future prices of the two options over $H = 1$ year, it suffices for us to compute the second moments of the future prices, as follows

$$\begin{aligned}\mathbb{E}_0[C_1^2] &= p^u \times (C_1^u)^2 + p^d \times (C_1^d)^2\\ &= 0.812 \times (\$19.14)^2 + 0.188 \times (\$0)^2\\ &= 297.37,\\ \mathbb{E}_0[C_1'^2] &= p^{u'} \times (C_1^{u'})^2 + p^{d'} \times (C_1^{d'})^2\\ &= 0.852 \times (\$25.10)^2 + 0.148 \times (\$0)^2\\ &= 536.63.\end{aligned}$$

Hence the variance of the future option prices can be given as follows

$$\begin{aligned}\mathrm{Var}_0[C_1] &= \mathbb{E}_0[C_1^2] - (\mathbb{E}_0[C_1])^2\\ &= 297.37 - 15.54^2 = 55.96,\\ \mathrm{Var}_0[C_1'] &= \mathbb{E}_0[C_1'^2] - (\mathbb{E}_0[C_1'])^2\\ &= 536.63 - 21.38^2 = 79.39.\end{aligned}$$

Then we can obtain the variance of holding each of these two call options' returns, which are given by

$$\begin{aligned}\mathrm{Var}_0[R_C] &= \mathrm{Var}_0\left[\frac{C_1}{C_0} - 1\right]\\ &= \frac{\mathrm{Var}_0[C_1]}{C_0^2} = \frac{55.96}{10.47^2} = 0.511,\\ \mathrm{Var}_0[R_{C'}] &= \mathrm{Var}_0\left[\frac{C_1'}{C_0'} - 1\right]\\ &= \frac{\mathrm{Var}_0[C_1']}{C_0'^2} = \frac{79.39}{12.81^2} = 0.484.\end{aligned}$$

To compute the covariance of the future prices of these two options, we need the joint up–up, up–down, down–up, down–down $\mathbb{R}$ probabilities for the two stocks, which can be calculated as:

$$\begin{aligned}p^{uu} &= \frac{1}{2}(p^u + p^{u'}) + \frac{1}{4}\rho - \frac{1}{4} = 0.707,\\ p^{ud} &= \frac{1}{2}(p^u + p^{d'}) - \frac{1}{4}\rho - \frac{1}{4} = 0.105,\\ p^{du} &= \frac{1}{2}(p^d + p^{u'}) - \frac{1}{4}\rho - \frac{1}{4} = 0.145,\\ p^{dd} &= \frac{1}{2}(p^d + p^{d'}) + \frac{1}{4}\rho - \frac{1}{4} = 0.043.\end{aligned}$$

Hence, the second co-moment of the future prices of these two options at $H = 1$ year, is given as

$$\begin{aligned}\mathbb{E}_0[C_1C_1'] &= p^{uu} \times C_1^uC_1^{u'} + p^{ud} \times C_1^uC_1^{d'}\\ &\quad + p^{du} \times C_1^dC_1^{u'} + p^{dd} \times C_1^dC_1^{d'}\\ &= 0.707 \times \$19.14 \cdot \$25.10 + 0.105\\ &\quad \times \$19.14 \cdot \$0 + 0.145\\ &\quad \times \$0 \cdot \$25.10 + 0.043 \times \$0 \cdot \$0\\ &= 339.55.\end{aligned}$$

Using the second co-moment, the covariance can be given as

$$\begin{aligned}\mathrm{Cov}_0[C_1, C_1'] &= \mathbb{E}_0[C_1 C_1'] - \mathbb{E}_0[C_1]\cdot\mathbb{E}_0[C_1'] \\ &= 339.55 - 15.54 \times 21.38 = 7.31.\end{aligned}$$

Therefore, the covariance of option returns over the 1-year horizon is given as

$$\begin{aligned}\mathrm{Cov}_0[R_C, R_{C'}] &= \mathrm{Cov}_0\left[\frac{C_1}{C_0}, \frac{C_1'}{C_0'}\right] \\ &= \frac{\mathrm{Cov}_0[C_1, C_1']}{C_0 \cdot C_0'} \\ &= \frac{7.31}{10.47 \cdot 12.81} = 0.055.\end{aligned}$$

The above 2-year binomial tree can be generalized to multiple periods (of any length) with any specific horizon H between the current time and the option expiration date, by using the one-period up and down $\mathbb{R}$ transition probabilities as the corresponding $\mathbb{Q}$ transition probabilities for all periods between time H and time T, and as the corresponding $\mathbb{P}$ transition probabilities for all periods between time t and time H.

In addition, the above discrete-time example used the money market account as the numeraire and the risk-neutral measure $\mathbb{Q}$ as the corresponding equivalent martingale measure for valuation on or after the horizon date H. The theory underpinning the above example can be generalized to continuous-time models that use alternative numeraires for the valuation of contingent claims. The mathematical intuition underlying this new theory is explained in Section 2 for the $\mathbb{R}$ measure, which uses the money market account as the numeraire, and in Section 3 for the $\mathbb{R}_1^T$ measure, which uses the T-maturity pure discount bonds as a numeraire.

2 The $\mathbb{R}$ Measure in Continuous-Time

Assuming a similar form for the physical and risk-neutral processes, the $\mathbb{R}$ measure can be used for the derivation of the analytical solutions of expected future prices of contingent claims under all models that admit an analytical solution to the claim's price using the risk-neutral measure $\mathbb{Q}$. Nawalkha and Zhuo (2022) present $\mathbb{R}$ as a dynamic change of measure which generalizes Equation (2) to value any contingent claim F, with a payoff F_T at time T as follows:

$$\mathbb{E}_t[F_H] = \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u} F_T\right]. \tag{3}$$

In the following we demonstrate the use of $\mathbb{R}$ measure for continuous-time models using the example of the Black and Scholes (1973) option pricing model. The geometric Brownian motion for the asset price process under the Black and Scholes (1973) model is given as follows:

$$\frac{\mathrm{d}S_s}{S_s} = \mu \mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{P}}, \tag{4}$$

where μ is the drift, σ is the volatility, and $W^{\mathbb{P}}$ is the Brownian motion under the physical measure $\mathbb{P}$. Denote L_s as the Radon–Nikodým derivative process of the risk-neutral measure $\mathbb{Q}$ with respect to the physical measure $\mathbb{P}$, when the numeraire is the money market account. For this case, L_s under the Black and Scholes (1973) model is given as

$$\begin{aligned}L_s &\triangleq \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_s} \\ &= \exp\left(-\int_0^s \gamma \mathrm{d}W_u^{\mathbb{P}} - \frac{1}{2}\int_0^s \gamma^2 \mathrm{d}u\right),\end{aligned} \tag{5}$$

where $\gamma = (\mu - r)/\sigma$ is the market price of risk (MPR) and r is the constant risk-free rate. Then the Girsanov theorem allows the transformation of Brownian motions under two equivalent probability measures. Defining $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma \mathrm{d}u$, the asset price process under the risk-neutral measure $\mathbb{Q}$ is given by

$$\frac{\mathrm{d}S_s}{S_s} = r\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{Q}}. \tag{6}$$

Then denote $\mathcal{L}_s(H)$ as the Radon–Nikodým derivative process of the equivalent expectation measure $\mathbb{R}$ with respect to the physical measure $\mathbb{P}$. As shown by Nawalkha and Zhuo (2022), $\mathcal{L}_s(H)$ can be given as follows:

$$\begin{aligned}\mathcal{L}_s(H) &\triangleq \frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_s} \\ &= \begin{cases} \dfrac{L_s}{L_H} = \exp\left(-\displaystyle\int_H^s \gamma \mathrm{d}W_u^{\mathbb{P}} - \frac{1}{2}\int_H^s \gamma^2 \mathrm{d}u\right), \\ \qquad\qquad \text{if } H \le s \le T, \\ 1, \quad \text{if } 0 \le s < H, \end{cases} \\ &= \exp\left(-\int_0^s \gamma_u(H)\mathrm{d}W_u^{\mathbb{P}} - \frac{1}{2}\int_0^s (\gamma_u(H))^2 \mathrm{d}u\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_s(H) &\triangleq 1_{\{s \ge H\}}\gamma \\ &= \begin{cases} \gamma, & \text{if } H \le s \le T, \\ 0, & \text{if } 0 \le s < H. \end{cases}\end{aligned}$$

Therefore the Brownian motion under the $\mathbb{R}$ measure is derived as follows:

$$W_s^{\mathbb{R}} = W_s^{\mathbb{P}} + \int_0^s 1_{\{u \ge H\}}\gamma \mathrm{d}u.$$

Substituting $W_s^{\mathbb{P}}$ from the above equation into Equation (4) gives the asset price process under the $\mathbb{R}$ measure as follows:

$$\frac{\mathrm{d}S_s}{S_s} = (r + \sigma\gamma 1_{\{s<H\}})\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{R}} \tag{7}$$

$$= (\mu 1_{\{s<H\}} + r 1_{\{s \ge H\}})\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{R}}, \tag{7a}$$

where $1_{\{\cdot\}}$ is an indicator function which equals 1 if the condition is satisfied (and 0 otherwise).

A highly intuitive property of the $\mathbb{R}$ measure is that any stochastic process under this measure can be obtained by a simple inspection of that process under the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure. By construction, the stochastic process under the $\mathbb{R}$ measure is the same as the physical stochastic process *before the horizon* H, and it switches to the risk-neutral process *at or after the horizon* H. Therefore, the asset price process in Equation (7a) (which is equivalent to Equation (7) since $\mu = r + \gamma\sigma$) can be derived by simply comparing the asset price process under the physical measure and the risk-neutral measure, given by Equations (4) and (6), respectively. This insight is not limited to the Black–Scholes model and applies more generally to any contingent claim model that uses the $\mathbb{Q}$ measure for valuation.

To demonstrate the usefulness of the $\mathbb{R}$ measure, consider a European call option C on the asset, maturing at time T with strike price K. Substituting $F_T = C_T = \max(S_T - K, 0) = (S_T - K)^+$ as the terminal payoff from the European call option, Equation (3) gives

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}}[\mathrm{e}^{-r(T-H)}(S_T - K)^+], \tag{8}$$

where C_H is the price of the call option at time H. Since Equation (7a) implies that S_T is lognormally distributed under the $\mathbb{R}$ measure, the expected call price can be solved easily and is given as

$$\begin{aligned}\mathbb{E}_t[C_H] &= S_t \mathrm{e}^{\mu(H-t)}\mathcal{N}(\hat{d}_1) \\ &\quad - K\mathrm{e}^{-r(T-H)}\mathcal{N}(\hat{d}_2),\end{aligned} \tag{9}$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function, and

$$\hat{d}_1 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\hat{d}_2 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

While Rubinstein (1984) first derived the expected price formula under the Black–Scholes model by using the property of double integrals of normally distributed variables, our derivation is much simpler. Unlike the Rubinstein (1984) method, which applies only to the Black–Scholes model, the $\mathbb{R}$ measure can be used to derive the expected future price under all contingent claim models that admit an analytical solution to the current price using the $\mathbb{Q}$ measure. Generally, under these models, either one or the other condition given below holds[3]:

(i) the payoff F is stochastic and the short rate r is constant, or
(ii) the payoff F is constant and the short rate r is stochastic.

Some of these models include the equity option pricing models of Black and Scholes (1973), Cox and Ross (1976), Merton (1976), Hull and White (1987), and Rubinstein (1991); the corporate debt pricing models of Merton (1974), Black and Cox (1976), and Leland and Toft (1996); the term structure models of Dai and Singleton (2000, 2002), Ahn *et al.* (2002), Leippold and Wu (2003), and Collin-Dufresne *et al.* (2008) for pricing default-free bonds; the credit default swap pricing model of Longstaff *et al.* (2005); the VIX futures and the variance swaps models of Dew-Becker *et al.* (2017), Eraker and Wu (2017), Johnson (2017), and Cheng (2019); the currency option model of Garman and Kohlhagen (1983); and the various Fourier transform-based contingent claim models of Heston (1993), Duffie *et al.* (2000), Carr *et al.* (2002), and others. Nawalkha and Zhuo (2023) extend the EEM theory of Nawalkha and Zhuo (2022) to allow for the derivation of analytical solutions of variance, covariance, skewness, and other higher-order moments and co-moments of contingent claim returns under the above models.

3 The $\mathbb{R}_1^T$ Measure

A variety of contingent claim models exist in finance under which *both* the payoff F and the short rate r are stochastic. Generally, under such models, the risk-neutral measure $\mathbb{Q}$ cannot be used to derive the analytical solution of the current price of claim, and the equivalent expectation measure $\mathbb{R}$ cannot be used to derive the analytical solution of the expected future price of claim.[4] The most famous of all such models is obviously the stochastic interest rate-based option pricing model of Merton (1973b), whose PDE for the option price is consistent with an expectation under the forward measure $\mathbb{Q}^T$, that uses the T-maturity pure discount bonds as the numeraire asset.[5]

Nawalkha and Zhuo (2022) show how to obtain the equivalent expectation measure $\mathbb{R}_1^T$ corresponding to the forward measure $\mathbb{Q}^T$. Assuming similar forms for the stochastic processes under the physical measure $\mathbb{P}$ and the forward measure $\mathbb{Q}^T$, the $\mathbb{R}_1^T$ measure allows the derivation of analytical solutions of the expected future prices of all claims that admit an analytical solution to the claim's current price under the forward measure $\mathbb{Q}^T$. Also, $\mathbb{R}_1^T$ converges to $\mathbb{Q}^T$ when $t = H$, so the forward measure $\mathbb{Q}^T$ is nested in $\mathbb{R}_1^T$, similar to how the risk-neutral measure $\mathbb{Q}$ is nested in $\mathbb{R}$. The $\mathbb{R}_1^T$ measure can be derived using Equation (3), as follows:

$$\begin{aligned}\mathbb{E}_t[F_H] &= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u} F_T\right] \\ &= \mathbb{E}_t^{\mathbb{R}}[P(H,T)] \\ &\quad \cdot \mathbb{E}_t^{\mathbb{R}}\left[\frac{\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}}{\mathbb{E}_t^{\mathbb{R}}[P(H,T)]} \cdot F_T\right] \\ &= \mathbb{E}_t^{\mathbb{P}}[P(H,T)] \cdot \mathbb{E}_t^{\mathbb{R}_1^T}[F_T],\end{aligned} \tag{10}$$

where $\mathbb{E}_t^{\mathbb{P}}[P(H,T)] = \mathbb{E}_t^{\mathbb{R}}[P(H,T)] = \mathbb{E}_t^{\mathbb{R}}\big[\mathbb{E}_H^{\mathbb{R}}\big[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}\big]\big] = \mathbb{E}_t^{\mathbb{R}}\big[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}\big]$, and for any random variable Z_T,

$$\mathbb{E}_t^{\mathbb{R}_1^T}[Z_T] = \mathbb{E}_t^{\mathbb{R}}\left[\frac{\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}}{\mathbb{E}_t^{\mathbb{R}}[P(H,T)]} \cdot Z_T\right].$$

This section considers applications of the $\mathbb{R}_1^T$ measure, which is perhaps the next most important EEM after the $\mathbb{R}$ measure, and is derived as a generalization of the forward measure $\mathbb{Q}^T$. We present the $\mathbb{R}_1^T$ measure using the example of the Merton (1973b) option pricing model. As in Merton (1973b), we assume the asset price S is described by

$$\frac{\mathrm{d}S_s}{S_s} = \mu(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_{1s}^{\mathbb{P}}, \tag{11}$$

where $W_{1s}^{\mathbb{P}}$ is the Brownian motion under $\mathbb{P}$, $\mu(s)$ is the instantaneous expected return which may be stochastic, and $\sigma(s)$ is the instantaneous volatility which is assumed to be deterministic.

For a T-maturity pure discount bond $P(\cdot, T)$, we assume its price follows

$$\frac{\mathrm{d}P(s,T)}{P(s,T)} = \mu_P(s,T)\mathrm{d}s + \sigma_P(s,T)\mathrm{d}W_{2s}^{\mathbb{P}}, \tag{12}$$

where $W_{2s}^{\mathbb{P}}$ is the Brownian motion associated with the T-maturity pure discount bond, which has a deterministic correlation with $W_{1s}^{\mathbb{P}}$, given by the coefficient $\rho(s)$; $\mu_P(s,T)$ is the instantaneous expected return, which may be stochastic; and $\sigma_P(s,T)$ is the instantaneous volatility which is assumed to be deterministic.

Consider a European call option C written on the asset S with a strike price of K, and an option expiration date equal to T. Using the T-maturity pure discount bond as the numeraire, the expected future price of this option is given by Equation (10), with $F_T = C_T = (S_T - K)^+$ as the terminal payoff, given as follows:

$$\begin{aligned}\mathbb{E}_t[C_H] &= \mathbb{E}_t^{\mathbb{P}}[P(H,T)]\mathbb{E}_t^{\mathbb{R}_1^T}[(S_T-K)^+] \\ &= \mathbb{E}_t^{\mathbb{P}}[P(H,T)]\mathbb{E}_t^{\mathbb{R}_1^T}\left[\frac{(S_T-K)^+}{P(T,T)}\right] \\ &= \mathbb{E}_t^{\mathbb{P}}[P(H,T)]\mathbb{E}_t^{\mathbb{R}_1^T}[(V_T-K)^+],\end{aligned} \tag{13}$$

where $V = S/P(\cdot, T)$ is the asset price normalized by the numeraire.

The derivation of an analytical solution of the expected future call price (of a similar form as in Merton, 1973b) using Equation (13) requires that the asset price normalized by the bond price be distributed lognormally under the $\mathbb{R}_1^T$ measure. Nawalkha and Zhuo (2022) show that *sufficient* assumptions for this to occur are:

(i) the physical drift of the asset price process is of the form $\mu(\mathrm{s}) = r_s + \gamma(\mathrm{s})$, where the risk premium $\gamma(\mathrm{s})$ is deterministic, and
(ii) the short rate process r_s, and the bond price process in Equation (12) are consistent with the various multifactor Gaussian term structure models, such that the physical drift of the bond price process is of the form $\mu_P(s,T) = r_s + \gamma_P(s,T)$, where the risk premium $\gamma_P(s,T)$ is deterministic.

Under these two assumptions, Nawalkha and Zhuo (2022) derive the solution of the expected future price under the Merton's (1973b) model as:

$$\begin{aligned}\mathbb{E}_t[C_H] &= \mathbb{E}_t[S_H]\mathcal{N}(\hat{d}_1) \\ &\quad - K\mathbb{E}_t[P(H,T)]\mathcal{N}\left(\hat{d}_2\right),\end{aligned} \tag{14}$$

where

$$\hat{d}_1 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t[S_H]}{\mathbb{E}_t[P(H,T)]K} + \frac{v_p}{2},$$

$$\hat{d}_2 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t[S_H]}{\mathbb{E}_t[P(H,T)]K} - \frac{v_p}{2},$$

$$v_p = \sqrt{\begin{array}{l}\int_t^T [\sigma(u)^2 - 2\rho(u)\sigma(u)\sigma_P(u,T) \\ \quad + \sigma_P(u,T)^2]\mathrm{d}u.\end{array}}$$

The analytical solution nests the Merton's (1973b) solution for the current call price for the special case of $H = t$, under which the $\mathbb{R}_1^T$ measure becomes the $\mathbb{Q}^T$ forward measure (see Geman, 1989). Equation (14) also nests the solution of the expected future price of a call option written on a pure discount bond under all multifactor Gaussian term structure models.

The definition of v_p remains the same as under the Merton's (1973b) model, and the specific solutions of $\mathbb{E}_t[S_H]$ and $\mathbb{E}_t[P(H,T)]$ can be derived with the appropriate specifications of the asset price process and the bond price process in Equations (11) and (12), respectively. These solutions can be shown to be consistent with various multifactor asset pricing models, such as the Merton's (1973a) ICAPM or the APT (see Ross, 1976; Connor and Korajczyk, 1989), which allow stochastic interest rates but with the additional restrictions of non-stochastic volatilities and non-stochastic risk premiums.

The $\mathbb{R}_1^T$ measure can be used to derive the expected future prices of contingent claims under the following models: the stochastic interest rate-based equity option pricing model of Merton (1973b) and its extensions; the stochastic interest rate-based corporate debt pricing models of Longstaff and Schwartz (1995), Jarrow *et al.* (1997), and Collin-Dufresne and Goldstein (2001); various term structure models for pricing bond options and caps, such as Dai and Singleton (2000, 2002), Collin-Dufresne *et al.* (2008), Ahn *et al.* (2002) and Leippold and Wu (2003); Heath *et al.* (1992), Miltersen *et al.* (1997), Brace *et al.* (1997), and Jamshidian (1997); and the currency option pricing models of Grabbe (1983) Amin and Jarrow (1991), and Hilliard *et al.* (1991), among others.

Nawalkha and Zhuo (2023) generalize the $\mathbb{R}_1^T$ measure further for the derivation of the analytical solutions of variance and covariance measures of contingent claim returns under the above models. They propose a second-order EEM $\mathbb{R}_2^T$, for the derivation of analytical solutions of the second-order moments and co-moments of contingent claim returns under all models that admit an analytical solution to the current price of the claim using the forward measure $\mathbb{Q}^T$. Using the second-order moments and co-moments, the analytical solutions of variance and covariance measures of contingent claim returns can be obtained under the above models.

4 Conclusion

While the revolution in martingale pricing theory (see Cox and Ross, 1976; Harrison and Kreps, 1979) has provided a plethora of valuation models for contingent claims, the *equivalent expectation measures* theory of Nawalkha and Zhuo (2022, 2023) obtains expected returns, variances, covariances, and other higher-order moments and co-moments of contingent claim returns over a finite horizon. The equivalent expectation measures also nest the corresponding equivalent martingale measures, when the horizon date H equals the current time t. This paper reviews two equivalent expectation measures $\mathbb{R}$ and $\mathbb{R}_1^T$, given as generalizations of two widely used equivalent martingale measures: the risk-neutral measure $\mathbb{Q}$ and the forward measure $\mathbb{Q}^T$, respectively. The $\mathbb{R}$ and $\mathbb{R}_1^T$ measures allow the derivation of analytical solutions of expected returns and variances and covariances of returns, under virtually all contingent claim models in finance. The availability of analytical solutions of expected returns and variances and covariances of returns of contingent claims allows these claims to be included in Markowitz's (1952) portfolio theory for mean–variance portfolio optimization.

These analytical solutions can help the managers of mutual funds and hedge funds, as well as risk managers of large financial institutions in doing risk and returns analysis of portfolios embedded with financial derivatives, such as options, futures, and swaps. They can also allow estimation of ex-ante Sharpe ratios for portfolio consisting of U.S. Treasuries or corporate bonds, or equity strategies such as covered call, protective put, straddle, butterfly spread, and others. This is useful when markets are incomplete and/or investors have heterogeneous beliefs, conditions under which the market portfolio is not mean–variance efficient, and is not held as the optimal portfolio by all investors.

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Endnotes

1 We use $\mathbb{E}[\cdot]$ and $\mathbb{E}^{\mathbb{P}}[\cdot]$ interchangeably to denote expectation under the physical measure $\mathbb{P}$. In addition, we denote by $\mathbb{E}_t$ the conditional expectation with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, that is, $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$.

2 The current price can be obtained using the traditional binomial tree pricing method, and given as

$$\begin{aligned} C_0 &= \mathrm{e}^{-r\Delta t}(q^u \times C_1^u + q^d \times C_1^d) \\ &= \mathrm{e}^{-0.03\times 1}(0.564 \times \$19.14 + 0.436 \times \$0) \\ &= \$10.47. \end{aligned}$$

3 This is not true for Fourier inversion-based models, such as Heston (1993), Pan (2002), Duffie *et al.* (2000), and others, in which $\mathbb{Q}$ and $\mathbb{R}$ measures can be used when both the payoff F and the short rate r are stochastic.

4 See Footnote 3.

5 It is well known that the asset price normalized by the numeraire asset (T-maturity pure discount bonds), $V = S/P(\cdot, T)$, is a martingale under the $\mathbb{Q}^T$ measure. This implicitly follows from Merton's (1973b) PDE using Feynman–Kac theorem, and explicitly by applying the $\mathbb{Q}^T$ measure as in Geman *et al.* (1995).

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