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LIMITING INVESTMENT OPPORTUNITY SETS, ASSET PRICING, AND THE ROLL CRITIQUE

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We consider the impact of low volatility assets on the investment opportunity set (IOS) and resultant asset pricing. The limiting IOS and its finite investable proxy imply an asset pricing model that differs from standard asset pricing models. The Sharpe (1964)–Lintner (1965) CAPM with a unique market portfolio is not descriptive of asset pricing and the zero-beta rate of the Black model, converges to the exogenous riskless rate. Spanning tests show that the limiting IOS, with estimated slope and upper bound Sharpe ratio of 0.18, is given by the linear limiting IOS asymptotes, implying multiple efficient portfolios of risky assets. We find no evidence of any efficient portfolio with only positive weights, implying that the market portfolio is not mean–variance efficient.

This paper is motivated by the literature on bounded Sharpe ratios, the Roll (1977) critique of asset pricing tests, and the efficiency of market portfolios. The analysis is applied to the theoretical construct of a limiting Investment Opportunity Set (*IOS*) that is well-specified by a *proxy IOS*, without a riskless asset, and consisting of market traded risky assets, with a non-singular covariance matrix. The simplified mathematics and estimators of the limiting *IOS* differ from the traditional non-limiting efficient set, with substantial economic and managerial implications.[1]

Our analysis results in an estimated market risk price given by an upper Sharpe ratio bound of 0.18 and a maximum likelihood (ML) estimate of 0.25 that apply to all risky asset portfolios lying on the linear, limiting, and efficient *IOS* proxy. The limiting *IOS* has the largest Sharpe ratio compared to all *IOS* derived from proper subsets. Our Sharpe ratio bound provides supportive evidence for the MacKinlay (1995) opinion of the perfect market upper Sharpe bound of 0.18.[2] However, our convergence bound requires substantially fewer assets in the bounding sequence.

MacKinlay considers market imperfections including biases in empirical methodology, the existence of market frictions, or the presence of irrational investors. In contrast, we determine a reliable estimate of the maximum Sharpe ratio for market representative asset sets, given the market's implicit structure and lack of perfection. In our model, expanded or reduced asset

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sets are important because they change the empirically observed Sharpe ratio. We demonstrate a strong, finite convergence to the maximum Sharpe ratio with a limiting proxy risky asset set (without the inclusion of a riskless asset or very large asset sets). Our limiting asset set includes diverse equity assets, and long- and short-term fixed income assets, resulting in a more inclusive *IOS* than the MacKinlay/Fama French (1993) sets.

We differ from MacKinlay's (1995) analysis by not requiring sufficient multifactor portfolios and market imperfections. In his analyses, the asset pricing model is considered well-specified when the tangency portfolio is formable from a linear combination of factor portfolios, irrespective of whether the allocations result in nonpositive allocations. Our limit does not result in a unique tangency portfolio, because efficient portfolios have equal Sharpe ratios. We require that an efficient market portfolio must have only positive weights. We verify that every limiting efficient proxy portfolio contains some negative asset weights and conclude that positive weight market portfolio proxies are not efficient and the traditional CAPM cannot be descriptive of asset prices. However, the Merton (1973) ICAPM remains a potential candidate for the underlying asset pricing model.

We develop the asset pricing implications of a limiting *IOS*, in which the *IOS* converges to its limiting asymptotes, including a renewed analysis of Roll's (1977) critique.[3] With a limiting *IOS*, the Black (1972) asset pricing model converges to the Sharpe–Lintner (S-L) asset pricing model. However, unlike the S-L CAPM, all efficient portfolios on the limiting *IOS* generate asset pricing models that have equivalent expected return for any selected asset. Thus, the market portfolio is not unique in determining asset prices, and therefore, the strict S-L CAPM does not apply. In our limiting *IOS*, betas will be inversely dependent on the standard deviation of a proxy limiting portfolio. Like Roll, we do not find evidence of a limiting, efficient portfolio with only positive weights and conclude that a proxy market portfolio, constructed solely from the underlying assets, is inefficient and therefore does not determine asset prices.

When weights on expanded asset sets are unknown or are estimated with error, Roll's requirement of a positive weight efficient portfolio may be difficult to determine.[4] However, our necessary condition for limiting efficiency is that a proposed market proxy has a Sharpe ratio greater than or equal to what is believed to be the market's limiting Sharpe ratio of, for example, 0.18. That condition is testable without knowledge of the weights on efficient limiting portfolios, although market proxies will have positive weights.

Our research extends and complements Korkie and Turtle (2021) who present a shrinkage estimator for the limiting *IOS* slope, equivalently the market price of risk. They consider similar asset sets and present a battery of robustness tests for a wide variety of alternative equity asset sets including 5, 10, 38, or 48 industry portfolios.[5] An objective of our research is to evaluate the asset pricing implications of enhanced asset sets that expand from the oft-studied Fama–French risky assets to include other equities and fixed incomes. We consider only assets with lengthy times series to mitigate estimation errors.

We discuss the implications of our limiting *IOS* in relation to Merton's Intertemporal CAPM. The market portfolio in the Merton model must have all positive weights and is the result of combining an efficient limiting proxy portfolio with a portfolio that hedges (factor) shifts in the *IOS*. We find that a positive weight Merton market portfolio can be consistent with hedging an efficient portfolio even if the efficient portfolio does not have strictly positive weights. However, that entirely depends on the weights in the hedge portfolio and the amount of the hedge.[6]

1 Limiting Investment Opportunity Sets

In principle, the market Investment Opportunity Set (*IOS*) should consist of all non-redundant assets or a good proxy thereof. Appendix 1 summarizes selected efficient set mathematics of Lintner (1965), Merton (1972), Roll (1977), and Section 1.2 extends them to the upper bounded, limiting efficient sets. The following Section 1.1 parameterizes representative equity only *IOS* with commonly used empirical data and Section 1.2 demonstrates the convergence of the equity *IOS* to a limiting *IOS*, as assets are added in the sequence.[7]

1.1 All-equity parameterization

To compare *IOS*s for different types and numbers of assets, N, we begin with a monthly proxy *IOS* parameterized by 10 value-weight decile portfolios and 10 value-weight industry portfolios obtained from Kenneth French's monthly data library, over the period 01/1950 to 12/2019. The parameterized *IOS* is shown in Table 1, panel A, using ML estimates of the table's entries. Table 2, panel A, uses the same data but computes the estimates with unbiased plug-in estimates of the *IOS* components (as developed in Appendix 2). For brevity, only Table 1 is discussed in this section.

From panel A of Table 1, the determinant of the information matrix is $\hat{P} = 24.8$, indicative of a full rank matrix. We report summary statistics for the exogenous efficient portfolio standard deviation, σ_p ranging from the global minimum risk, vertex portfolio's standard deviation 3.10% (see initial bold entry in panel A) to an arbitrary 5%/month, with resultant efficient portfolio means, $\mu_p = \hat{\mu}_p$, ranging from 0.88% to 1.48%/month and asymptote mean returns ranging, $\mu_A = \hat{\mu}_A$, from 1.35% to 1.65%/month. The mean return difference between the asymptote and hyperbola, $\hat{\mu}_A - \hat{\mu}_p$, is a maximum 0.48%, at the vertex portfolio, and monotonically declines to 0.17%/month at $\sigma_p = 5\%$, indicative of this all-equity proxy *IOS* that is not close to its upper limit, defined by the asymptote.

An *IOS* proxy Sharpe ratio, $\frac{\hat{\mu}_p - r_f}{\sigma_p}$, is defined as an efficient proxy portfolio's mean return in excess of an exogenous riskless 1-month Tbill return, r_f, all divided by the portfolio's standard deviation, σ_p. The 1-month Tbill return is used solely to measure the Sharpe ratio and is riskless for a 1-month horizon; none of the *IOS*s contain a 1-month Tbill. In contrast, a 2-month Tbill or any longer maturity Treasury bill or bond is risky over a 1-month horizon because its price at the end of the investment horizon is unknown. Similarly, a Tbill with 15 days left to maturity is risky because the reinvestment return for the remainder of the month is unknown. The closer the maturity is to 1-month, the smaller its volatility risk.

Inputs to the reported Sharpe ratio in Table 1 use the portfolio sample mean, $\hat{\mu}_p$, the exogenous riskless rate, $r_f = 0.338\%$, and the conditional portfolio standard deviation, σ_p. An asymptote portfolio's Sharpe ratio, $\frac{\hat{\mu}_A - r_f}{\sigma_p}$, is analogously reported with $\hat{\mu}_A$, r_f, and σ_p. Because the specified 1-month Tbill return is less than the vertex mean, the tangent portfolio's Sharpe ratio is on the upper portion of the *IOS*. In panel A, the *IOS* proxy ML Sharpe ratio is 0.173 at the vertex, rises to 0.232 and then declines to 0.228. The tangency Sharpe ratio is also 0.232, with a volatility of 4.14% and mean 0.13%. The *IOS* proxy Sharpe ratios differ from the asymptote's Sharpe ratios, over all reported volatilities, again indicative of a proxy *IOS* that is not close to its upper limit, except at very large volatilities (not shown).

Table 1 A comparison of the limiting *IOS*, a proxy all-equity *IOS*, and an expanded proxy *IOS* for a 1-month investment horizon, using ML estimates.

Panel A: Proxy equity only *IOS* from 10 value-weight decile and 10 value-weight industry portfolios: $\hat{\mu}_g = 0.88\%$, $\hat{\sigma}_g = 3.096\%$, $\hat{\psi}^2 = 0.0238$, $\hat{P} = 24.81$

	Standard deviation (% values), σ_p					
	3.096%	3.500%	4.000%	**4.359**%	4.500%	5.000%
IOS proxy efficient mean	0.875%	1.127%	1.266%	1.348%	1.379%	1.481%
Asymptote mean	1.352%	1.415%	1.492%	1.547%	1.569%	1.646%
Difference in means	0.477%	0.288%	0.226%	0.199%	0.190%	0.166%
IOS proxy Sharpe ratio	0.173	0.225	0.232	0.232	0.231	0.228
Asymptote Sharpe ratio	0.328	0.308	0.288	0.277	0.274	0.262

Panel B: Proxy limiting *IOS* from 10 value-weight decile; 10 value-weight industry portfolios; 2-month Treasury bills, 1-, 2-, 5-, and 10-year Treasury bonds; and long-term corporate bonds: $\hat{\mu}_g = 0.36\%$, $\hat{\sigma}_g = 0.264\%$, $\hat{\psi}^2 = 0.0628$, $\hat{P} = 9016.9$

	Standard deviation (% values), σ_p										
	0.264%	1.000%	1.500%	2.000%	2.500%	3.000%	**3.392**%	3.500%	4.000%	4.500%	5.000%
IOS proxy efficient mean	0.362%	0.604%	0.732%	0.859%	0.985%	1.111%	1.210%	1.237%	1.362%	1.488%	1.613%
Asymptote mean	0.428%	0.613%	0.738%	0.863%	0.989%	1.114%	1.212%	1.239%	1.364%	1.490%	1.615%
Difference in means	0.066%	0.009%	0.006%	0.004%	0.003%	0.003%	0.003%	0.002%	0.002%	0.002%	0.002%
IOS proxy Sharpe ratio	0.092	0.266	0.263	0.260	0.259	0.258	0.257	0.257	0.256	0.255	0.255
Asymptote Sharpe ratio	0.342	0.275	0.267	0.263	0.260	0.259	0.258	0.257	0.257	0.256	0.255

The efficient set constants are parameterized from monthly asset data, over the period 1/31/1950 to 12/31/2019. Specifically, the 10 value-weight decile portfolios and 10 value-weight industry portfolios are from Kenneth French's monthly data library. The 2-month Tbill returns, and 1-, 2-, 5-, and 10-year constant maturity Tbonds are from CRSP. Long-term corporate bond returns are from the SBBI (Stocks, Bonds, Bills, and Inflation) yearbook. ML estimates of the parameters are used in constructing the Table. Each panel reports the *IOS* properties, efficient proxy portfolio mean and Sharpe ratio, asymptote mean and Sharpe ratio, conditional on each standard deviation in the second row. The 1-month Tbill return is 0.338%. Panel A reports results for the proxy *IOS* created solely from equity portfolios. Panel B reports results for the proxy limiting *IOS* developed from 26 equity and fixed income assets. Bold entries identify the standard deviations for the vertex portfolio or the equal-weight portfolio of a panel's assets.

Table 2 A comparison of the limiting *IOS*, a proxy all-equity *IOS*, and an expanded proxy *IOS* for a 1-month investment horizon, using unbiased plug-in estimates.

Panel A: Proxy equity only *IOS* from 10 value-weight decile and 10 value-weight industry portfolios: $\breve{\mu}_g = 0.88\%$, $\breve{\sigma}_g = 3.133\%$, $\breve{\psi}^2 = 0.000558$, $\breve{P} = 0.5685$

	Standard deviation (% values), σ_p					
Standard deviation	**3.133**%	3.500%	4.000%	**4.362**%	4.500%	5.000%
IOS proxy efficient mean	0.875%	0.912%	0.934%	0.947%	0.952%	0.967%
Asymptote mean	0.949%	0.958%	0.970%	0.978%	0.982%	0.993%
Difference in means	0.074%	0.046%	0.036%	0.031%	0.030%	0.026%
IOS proxy Sharpe ratio	0.171	0.164	0.149	0.140	0.136	0.126
Asymptote Sharpe ratio	0.195	0.177	0.158	0.147	0.143	0.131

Panel B: Proxy limiting *IOS* from 10 value-weight decile; 10 value-weight industry portfolios; 1-, 2-, 5-, and 10-year Treasury bonds; long-term corporate bonds; and 2-month Treasury bills: $\breve{\mu}_g = 0.36\%$, $\breve{\sigma}_g = 0.268\%$, $\breve{\psi}^2 = 0.0310$, $\breve{P} = 4313$

	Standard deviation (% values), σ_p										
Standard deviation	**0.268**%	1.000%	1.500%	2.000%	2.500%	3.000%	**3.394**%	3.500%	4.000%	4.500%	5.000%
IOS proxy efficient mean	0.362%	0.532%	0.622%	0.711%	0.800%	0.888%	0.958%	0.977%	1.065%	1.153%	1.241%
Asymptote mean	0.410%	0.538%	0.626%	0.714%	0.802%	0.890%	0.960%	0.978%	1.066%	1.154%	1.242%
Difference in means	0.047%	0.006%	0.004%	0.003%	0.003%	0.002%	0.002%	0.002%	0.002%	0.001%	0.001%
IOS proxy Sharpe ratio	0.090	0.194	0.189	0.187	0.185	0.183	0.183	0.182	0.182	0.181	0.181
Asymptote Sharpe ratio	0.266	0.200	0.192	0.188	0.186	0.184	0.183	0.183	0.182	0.181	0.181

The efficient set constants are parameterized from monthly asset data, over the period 1/31/1950 to 12/31/2019. Specifically, the 10 value-weight decile portfolios and 10 value-weight industry portfolios are from Kenneth French's monthly data library. The 1-, 2-, 5-, and 10-year constant maturity Tbond and 2-month Tbill returns are from CRSP. Long-term corporate bond returns are from the SBBI (Stocks, Bonds, Bills, and Inflation) yearbook. Unbiased estimates of the parameters are used in constructing the table. Each panel reports the *IOS* properties, efficient proxy portfolio mean and Sharpe ratio, asymptote mean and Sharpe ratio, conditional on each standard deviation in the second row. The 1-month Tbill return is 0.338%. Panel A reports results for the proxy *IOS* created solely from equity portfolios. Panel B reports results for the proxy limiting *IOS* developed from 26 equity and fixed income assets. Bold entries identify the standard deviations for the vertex portfolio or the equal-weight portfolio of a panel's assets.

Conditional on the Table 1 panel A parameters, none of the efficient portfolios have strictly non-negative weights calculated from Appendix 1, Equation (A2). (See Section 1.2.2 and Endnote 16 for the method of proof.) Contrarily, the positive and equal weight portfolio, comprised of $1/N$ weights on each of the 20 size and industry portfolios, has a volatility of 4.36% (see bold entry in the initial row, fourth column of panel A) and a mean of 1.07% (not reported in the table) compared to the ML efficient portfolio mean of 1.35%, indicative of an inefficient equal weight portfolio.

The following section expands the asset set from 20 to 26 assets, including fixed income assets, and shows the results in Table 1 panel B for ML estimates and Table 2 panel B for unbiased plug-in estimates of the *IOS* components. From this panel and the following discussion, we see that the panel A, equity-only proxy *IOS* is a poor representation of the unobserved, inclusive market *IOS*.

1.2 Limiting IOS proxies

1.2.1 Discussion and formulas

The conclusions from this section's analysis are based on the assumption (and our demonstration) of convergence of the point sequence of elements from the fundamental information matrix,[8] $\{\mu_g, \sigma_g^2, \psi^2\}_{N=1}^{\infty}$, with limit $(r_f, 0, \psi_u^2)$ that is closely approximated using a finite proxy for ψ_u^2, the upper bounded price of variance risk for efficient portfolios. In this limit, with known riskless rate, ψ_u^2 is the only unknown defining the limiting efficient set of only risky assets.

The Section 1.1 parameterization consists solely of equities and does not achieve the definition of a good proxy *IOS*. An expanded asset set is more representative of the *IOS* facing investors than an *IOS* constructed solely from equities. It is critical to recognize that an expanded asset set is a superset of the all-equity subset and therefore offers more mean–variance opportunities than the subset. This implies that a subset's *IOS* must lie inside the superset's *IOS* for all finite opportunities. In turn, this implies that the Sharpe ratios of superset efficient portfolios must dominate the Sharpe ratios of the subset efficient portfolios. Therefore, there is a computable financial advantage knowing the dominant risky asset set that excludes the riskless asset. This set has a maximum Sharpe ratio that exceeds the maximum Sharpe ratios of its proper subsets, as we demonstrate in this section.

The addition of low-risk assets, such as fixed income Federal Government securities, with remaining maturities *approaching the investment horizon from above or below*, causes the *IOS* vertex portfolio to converge to $(r_f, 0)$, where r_f is the horizon's riskless return. That is, $\sigma_p^2 = \sigma_g^2$ converges to zero and by the law of one price, $\mu_p = \mu_g$ converges to the riskless return, thereby preventing riskless arbitrage. Because the hyperbola's asymptotes emanate from the point $(\mu_g, \sigma_p = 0)$ on the mean axis, the efficient portion of the hyperbola converges to its upper asymptote, with slope equal to the upper-bounded Sharpe ratio, ψ_u. We define this asset set's *IOS* as the "limiting *IOS*."

Because the efficient portion of the hyperbola converges to its upper asymptote, the limiting expected return from efficient portfolio, p, is simplified relative to Equation (A1) of Appendix 1 and is given by,

$$\mu_{pu} = r_f + \psi_u \sigma_p, \tag{1}$$

where p is any portfolio on the limiting and linear efficient set. The limiting asset pricing model is given by,

$$\begin{aligned}\mu_j &= r_f + \psi_u \frac{\sigma_{j,p}}{\sigma_p} \\ &= r_f + \beta_{j,p}(\mu_{pu} - r_f) \end{aligned} \tag{1a}$$

for any asset $j's$ mean return, μ_j; covariance, $\sigma_{j,p}$, with any efficient portfolio, p; and beta, $\beta_{j,p}$, relative to the efficient portfolio. Because the mean return on an asset j is fixed, it must have the same value for every efficient portfolio, p. Therefore, the beta must adjust such that,

$$\beta_{j,p} = \frac{\sigma_{j,p}}{\sigma_p^2} = \frac{(\mu_j - r_f)}{(\mu_{pu} - r_f)} = \frac{(\mu_j - r_f)}{\psi_u \sigma_p}. \tag{1b}$$

The convergence of σ_g^2 to zero also causes the zero-beta rate, $\mu_{z,p} = \frac{\psi^2\sigma_g^2}{(\mu_g-\mu_p)} + \mu_g$, defined in Equation (A5), to converge to the limiting mean return on the global minimum portfolio, $\mu_g = r_f$, for every limiting efficient portfolio, p. The zero-beta rate convergence to the risk-free rate implies from Equation (A6) that the Black (1972) asset pricing model converges to the asset pricing model given by Equation (1a). This theoretical limiting asset pricing model is like the CAPM model because it contains the riskless rate. However, it differs because the asset pricing model exists only at the limit and holds for *every risky efficient portfolio*, p, with its corresponding upper bounded mean return, μ_{pu}, risk price, ψ_u, and resultant betas, $\beta_{j,p}$. One of these portfolios might be the traditional market portfolio, m, but it is not unique in describing the mean returns of inefficient, risky assets.[9]

In these limiting *IOS* expressions, the sole parameter is the upper bound on the total risk price, ψ_u, that is also the upper bound Sharpe ratio, Sh_u, for any efficient portfolio, p.[10] The covariance between a limiting efficient portfolio's return and any asset, j, is

$$\sigma_{j,p} = \frac{(\mu_j - r_f)\sigma_p}{\psi_u}, \tag{2}$$

and the correlation is the ratio of the Sharpe ratios,

$$\rho_{j,p} = \frac{Sh_j}{Sh_u}. \tag{3}$$

If $j = s$ is efficient, the covariance between any two limiting efficient portfolios is the product of their volatilities,

$$\sigma_{s,p} = \sigma_s \sigma_p, \tag{4}$$

and they are therefore perfectly positively correlated.

There is extensive literature that limits Sharpe ratios.[11] An interesting question is how close we can get to the limiting *IOS* with selections of the *IOS* assets that preserve the convenient standard formulas in Appendix 1 and Section 1.2 and provide confidence in related empirical work.[12] Essentially, we want an investable proxy *IOS* and its market risk price, ψ, that are sufficiently close to the limiting values such that reliable empirical inferences of asset pricing and portfolio efficiency are possible.

1.2.2 Data and tests with augmented asset sets

Table 1 panel B computes the ML values parameterized from 10 value-weight decile portfolios and 10 value-weight industry portfolios, augmented by 2-month Tbills and 1-, 2-, 5-, and 10-year Treasury bonds plus a long-term corporate bond portfolio, for the 1/31/1950 to 12/31/2019 data period.[13]

The efficient portfolio standard deviations, σ_p, range from the vertex portfolio's volatility, 0.26% (see bold entry in the initial row of Panel B), to 5%/month with resultant efficient *IOS* portfolio means, $\hat{\mu}_p$, increasing from 0.36% to 1.61%/month. This *IOS* proxy mean return difference from the asymptote mean, $\hat{\mu}_A - \hat{\mu}_p$, is a maximum 0.066%, at the vertex portfolio, and decreases quickly to 0.009%/month at $\sigma_p = 1\%$. The *IOS* proxy Sharpe ratio, $\frac{\hat{\mu}_p - r_f}{\sigma_p}$, is approximately 0.26 and differs from the asymptote Sharpe ratio, $\frac{\hat{\mu}_A - r_f}{\sigma_p}$, by at most 0.01, for volatilities of 1% or greater. The tangency Sharpe ratio, 0.27, occurs at a very low volatility 0.77% and is close to the proxy Sharpe ratio, at the same volatility. The portfolio's low volatility and mean, 0.54%, is unlikely to be a targeted portfolio. Because 1-month Tbill returns are *not used* in constructing the proxy or limiting *IOS*, the proxy vertex portfolio mean return, 0.362%, only approaches the historical mean 1-month Tbill return of 0.338%/month, and the vertex portfolio's return volatility, 0.264%, is not 0%.

Conditional on the Table 1 panel B parameters, none of the efficient portfolios have strictly nonnegative weights calculated from Appendix 1, Equation (A2). This was verified by solving Equation (A3b) for the vertex portfolio, g, and Equation (A3) for the $\sigma_p = 5\%$ efficient portfolios' weights, conditional on the standard deviation (weights are not reported for brevity).[14] Three equities and the 10-year Tbond have negative weights in both portfolios. For example, the vertex portfolio weights on the fourth decile equity and 10-year Tbond are -0.01 and -0.03, respectively, whereas the corresponding $\sigma_p = 5\%$ portfolio weights are -0.88 and -0.83, respectively. Because the weights change monotonically between the two portfolios (Roll Corollary 7) and one or more assets has negative weights in both the vertex and the $\sigma_p = 5\%$ portfolio, that is sufficient to conclude that an efficient nonnegative weight portfolio does not exist in the *IOS*. For example, the tangent maximum Sharpe ratio portfolio lies between the vertex and the $\sigma_p = 5\%$ portfolios, with mean 0.54%, and with negative weights -0.13 and -0.14, respectively on the fourth equity and 10-year Tbond.

The positive weight, but inefficient, equal weight portfolio comprised of weights $1/N$ on equities and fixed income assets has a mean return of 0.927% (not tabled) and volatility of 3.392%/month. At this volatility, the limiting and proxy portfolio means, $\hat{\mu}_A$ and $\hat{\mu}_p$, respectively, are 1.212%/month and 1.210%/month, and exceed the portfolio's actual mean return, 0.927%. The equal weight Sharpe ratio is 0.173, well below the proxy Sharpe ratio ML estimate of 0.257. Therefore, the equal weight portfolio remains inefficient in this expanded ML *IOS*, as was the case for the all-equity *IOS*.

At this point, our principal interest is to show the convergence of the proxy *IOS* to a limiting *IOS*, as the number of risky assets increases and with different estimators of the risk price. Whereas Table 1 is based on ML estimators, Table 2 shows the convergence properties of the *IOS* based on unbiased plug-ins of its components' values. Overall, the results are very similar to the ML results, from panel A to panel B in terms of the proxy *IOS* means converging to the asymptote means and the equality of the asymptote and *IOS* proxy Sharpe ratios. In Table 2 panel B, the *IOS* proxy Sharpe ratio differs from the asymptote Sharpe ratio by at most 0.006, for volatilities of 1% or greater. However, the *unbiased* plug-in estimates of the Sharpe ratios are substantially smaller at approximately 0.18, compared to the Table 1 panel B ML estimates of approximately 0.26, which are well known to be positively biased.

A natural question is why an investor would prefer the *IOS* proxy over a more efficient portfolio on the limiting *IOS*, which consists of the riskless asset and the largest Sharpe ratio portfolio in the *IOS*.[15] In response, portfolios at the limit are not available to an investor; however, proxy *IOS* portfolios are investable, and those portfolios do not require the 1-month Tbill as an invested asset. A comparison of the all-equity *IOS* and the expanded *IOS* in Table 1 panel A shows the maximum proxy Sharpe ratio of the all-equity *IOS* as 0.23 and panel B shows the expanded asset set's maximum Sharpe ratio as 0.27, a percentage increase of approximately 15%. Similarly, panels

A and B of Table 2 show the maximum Sharpe ratio percentage increase of 13%.

Therefore, there is considerable financial benefit from knowing the contents of a good *IOS* proxy that yields the maximum Sharpe ratio. In addition, there is no need to identify a maximum Sharpe ratio tangent portfolio because the *IOS* proxy has approximately a constant Sharpe ratio over the entire risk range and does not require leverage at the riskless rate. For example, the proxy Sharpe ratio from Table 2, panel B using unbiased inputs, is approximately 0.18, for *all efficient portfolios* with volatilities exceeding 1%.[16]

In comparison, MacKinlay (1995, Table 1) estimates the maximum Sharpe ratio, for four equity portfolios and a long maturity bond portfolio, as 0.12 ($=\sqrt{0.0145}$) with ML inputs, and 0.05 ($=\sqrt{0.0021}$) with unbiased plug-ins, over July 1963 to December 1991, compared to our analogous equity and fixed income portfolio maximum Sharpe ratios of 0.26 and 0.18, respectively from panel B of Tables 1 and 2. The differences are likely due to sample periods, sample size and as we stress, asset sets. Our larger asset set, including short-term fixed income assets, increases the all-equity ML Sharpe ratio in Table 1, from approximately 0.23 to 0.26.

Figure 1 shows the Table 1 limiting asymptote and its *IOS* proxy, in comparison with the equity only *IOS*, based on the ML estimators. The locations of the equal weight portfolio of the equity only 20 assets set, and the more inclusive 26-assets set are identified with an asterisk and circle, respectively. Both equal weight portfolios are inefficient in their respective asset sets. The larger asset set is very close to the limiting asymptote, especially for risks greater than 1%/month. Therefore, the ray emanating from the riskless rate to *any* efficient portfolio, with volatility greater than 1%, is virtually equivalent to the proxy limiting *IOS* and its asymptote, which have a nearly constant and

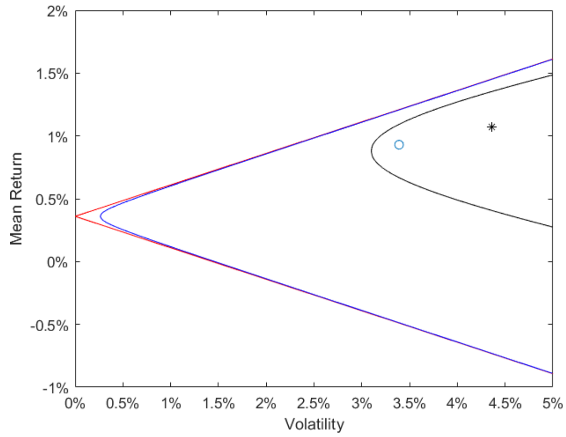

Figure 1 Two proxy *IOS* sets with 1-month horizons and related equal-weighted portfolios.

The proxy *IOS*, plotted in blue, is constructed from 10 value-weight decile portfolios and 10 value-weight industry portfolios; 2-month Tbills; 1-, 2-, 5-, and 10-year constant maturity Tbonds; and a long-term corporate bond portfolio. The limiting *IOS*, defined by the asymptotes, is plotted in red. The all-equity *IOS* constructed from the 10 value-weight decile portfolios and 10 value-weight industry portfolios is plotted in black. The equal weight portfolio of all assets of the proxy *IOS* is represented by the blue circle, and the equal weight portfolio of the equity only assets is represented by the black asterisk. Inputs for the figure are based on the ML estimators $\hat{\mu}_g$, $\hat{\sigma}_g^2$, and $\hat{\psi}^2$, reported in the descriptions of Table 1 Panels A and B.

maximum Sharpe ratio. The risky asset *IOS*, theoretically a hyperbola, converges virtually to a “<” shape.

1.3 Limiting Sharpe ratio and risk price estimates

We are interested in the proxy *IOS* with an upper limit of the risk price, denoted, ψ_u, that is also the slope of the asymptote to the proxy *IOS*. Equivalently, that risk price is the Sharpe ratio of the proxy limiting *IOS* and its theoretical limiting Sharpe ratio. Tables 1 and 2 use estimated risk prices to demonstrate the closeness of a proxy *IOS* to its linear asymptote. We find that a proxy *IOS* containing equities and fixed income assets of various maturities is very close to the upper bound given by its asymptote. This result was

Table 3 A comparison of estimates of the limiting risk price, $\psi = \sqrt{a - \frac{b^2}{c}}$.

Estimator	$N = 20$	$N = 26$
Maximum likelihood, $\hat{\psi}$	0.154	0.251
Half-moment, ψ_{HM}	0.034	0.179
Plug-in estimate of $\breve{\psi}$	0.024	0.176

Uses monthly asset data, over the period 1/31/1950 to 12/31/2019, $T = 840$, for 10 value-weight decile portfolios, 10 value-weight industry portfolios in the $N = 20$ asset set, plus the fixed income securities 2-month Treasury bills and 1-, 2-, 5-, and 10-year Treasury bonds, and long-term corporate bonds in the $N = 26$ asset set. $a, b,$and c are the efficient set constants, constructed from an N-asset set.

obtained using the ML estimator, $\hat{\psi}$, in Table 1, and by the Table 2 plug-in estimator, $\breve{\psi}$, defined in Appendix 2. The latter is equivalent to the square root of the unbiased estimator of the squared Sharpe ratio. In Table 3, we also investigate one additional risk price estimator, based on the general moment formula from Johnson *et al.* (1995), defined as our half-moment estimator denoted by ψ_{HM}.[17]

In Table 3 the (biased) ML estimate of the risk price, $\hat{\psi}$, changed from 0.154 to 0.251 with the addition of the fixed income assets, as the asset set increased from $N = 20$ to $N = 26$. The plug-ins estimate of the risk price increased from 0.024 to 0.176 with the addition of the fixed income assets. The half-moment estimate increased from 0.034 to 0.179 with the addition of the fixed income assets and its value is slightly larger than the plug-in estimate, 0.176, but considerably less than the biased ML estimate of 0.251.

The Table 3 risk price estimates for $N = 26$ are approximately equal to the related Sharpe ratios for the *IOS* proxy and asymptote as reported in Tables 1 and 2, Panel B. The small differences likely arise in part because the risk price estimates are calculated relative to the estimated global minimum variance portfolio; not relative to a riskless asset in the usual Sharpe ratio estimator.[18]

In conclusion, because the *IOS* proxy is very close to the limiting and linear *IOS*, for the $N = 26$ asset set, our estimate of the limiting Sharpe ratio is 0.18. This estimate is based on values of the unbiased plug-ins, 0.176 and the half-moment estimator, 0.179. A remaining question is whether the limiting Sharpe ratio from *IOS*s that include other traded or non-traded assets, such as foreign assets, commodity ETFs, or human capital, are materially larger in parameter space. Based on the preceding analyses, this seems unlikely, but possible. The following section implements some statistical spanning tests, but the introduction of other traded and non-traded assets is left to future research.

1.4 Spanning significance tests

In the limit, portfolios on the limiting *IOS* frontier are infeasible because there are no portfolio

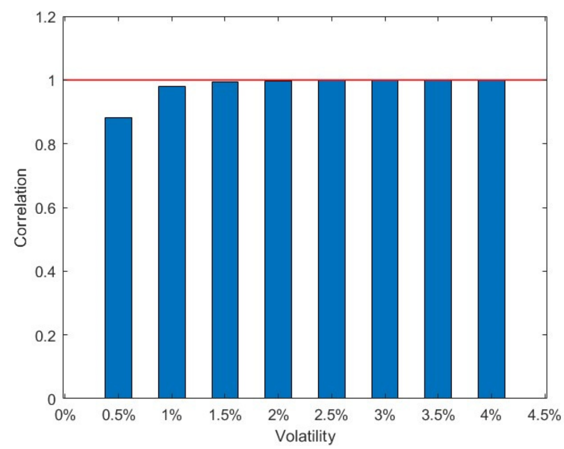

Figure 2 Correlations between representative efficient proxy portfolios.

Vertical bars are correlations between an efficient proxy *IOS* portfolio, s, with volatility 4% per month and efficient proxy *IOS* portfolios, p, with volatility ranging from 0.5% to 4%/month. Correlations are calculated using the covariance Equation (A6). The upper bound correlation, 1, is the correlation between any efficient limiting *IOS* portfolios, from Equation (4). A good proxy *IOS* has correlations close to 1.

weight vectors that result in the limiting means. Nonetheless, because the larger asset set shown in panel B of Tables 1 and 2 appears to converge to within a very small neighborhood of the estimated limiting *IOS*, our posterior belief is that the 26-asset *IOS* is a good, feasible proxy to the true limiting *IOS*, with a Sharpe ratio upper bound that is very near the limiting theoretical Sharpe ratio upper bound.

As an indication of the convergence of the proxy *IOS* to the limiting *IOS*, we plot the behavior of correlations between two efficient portfolios from our limiting proxy *IOS*. In Figure 2, we calculate covariances between an efficient proxy portfolio with volatility of 4%/month, and another efficient proxy portfolio with volatilities ranging from 0.5% to 4%/month from Equation (A6), and then we compute the related correlations. In the

Table 4 Some spanning test results.

Asset Universe, N	*Asset subset, N_1*	*F*	*p-Value*
$N = 26$	$N_1 = 20$		
10 decile; 10 industry; 2-mo TBills; 1-, 2-, 5-, and 10-year TBonds; and LT corp bonds	10 decile, 10 industry	1,486.2	<0.0001
$N = 26$	$N_1 = 25$		
10 decile; 10 industry; 2-mo TBills; 1-, 2-, 5-, and 10-year TBonds; and LT corp bonds	10 decile; 10 industry; 2-mo TBills; 1-, 2-, and 5-year TBonds; and LT corp bonds	2.8423	0.059
$N = 27$	$N_1 = 26$		
10 decile; 10 industry; 2-mo TBills; 1-, 2-, 5-, and 10-year TBonds; LT corp bonds; and the SBBI high yield corp bond	10 decile; 10 industry; 2-mo TBills; 1-, 2-, 5-, and 10-year TBonds; and LT corp bonds	1.0894	0.337

We report spanning test results for various asset sets and required ML inputs. For the initial row, the spanning test statistic is F-distributed with $2(N - N_1)$ and $2(T - N)$ degrees of freedom under the null hypothesis, and is given by,

$$\frac{(T-N)}{(N-N_1)} \frac{\left(\sqrt{\hat{P} + \frac{1}{\hat{\sigma}_g^2}} - \sqrt{\hat{P}_1 + \frac{1}{\hat{\sigma}_{g1}^2}}\right)}{\sqrt{\hat{P}_1 + \frac{1}{\hat{\sigma}_{g1}^2}}},$$

where $\hat{P} = \frac{\hat{\psi}^2}{\hat{\sigma}_g^2}$, $\hat{P}_1 = \frac{\hat{\psi}_1^2}{\hat{\sigma}_{g1}^2}$, for $(N - N_1) \geq 2$. For rows 2 and 3, the spanning test statistic is F-distributed with 2 and $T - N$ degrees of freedom under the null hypothesis, and is given by,

$$\frac{(T-N)}{(N-N_1)} \frac{\left(\left(\hat{P} + \frac{1}{\hat{\sigma}_g^2}\right) - \left(\hat{P}_1 + \frac{1}{\hat{\sigma}_{g1}^2}\right)\right)}{\left(\hat{P}_1 + \frac{1}{\hat{\sigma}_{g1}^2}\right)},$$

for $(N - N_1) = 1$. Tests are constructed for various subsets of the following assets: 10 value-weight decile portfolios; 10 value-weight industry portfolios; 2-month Treasury bills; 1-, 2-, 5-, and 10-year Treasury bonds; long-term corporate bonds; and SBBI high-yield corporate bonds. All tests are based on the $T = 840$ monthly observations over the period 1/31/1950 to 12/31/2019.

limiting case, as shown in Equation (4), these correlations should be 1. As shown in the figure, for volatilities of 1%/month or more, we observe proxy correlations that are very close to 1.

To test our prior that the *IOS* converges to the true limiting *IOS*, we present three spanning tests.[19] The initial two tests examine if subsets of the $N = 26$ asset *IOS* span the proxy limiting *IOS*. The first row of Table 4 presents results of whether the $N_1 = 20$ equity portfolios, considered in panels A of Tables 1 and 2, span the broader 26-asset set in panels B that include the additional fixed income securities. The reported F-statistic of 1,486.2 strongly rejects the hypothesis that equities alone are sufficient to span our broader limiting *IOS*. We next examine the impact of eliminating a single fixed income security from the 26-asset set—the 10-year constant maturity bond. The reported F-statistic of 2.842 is marginally significant with a p-value of 0.059, marginally indicative of a lack of spanning of the 26-asset set by the 25-asset set.

Our final spanning test is whether the addition of a non-redundant asset results in an *IOS* that is significantly and economically different from (not spanned by) a proper subset. If it is significantly different, the increase in the limiting Sharpe ratio should be economically small. The SBBI high-yield bond index is added to the 26-asset set, resulting in 27 assets. As shown in the final row of Table 4, the resulting F-statistic of 1.0894 is insignificant with a p-value of 0.337.

In sum, the spanning tests suggest that the risk price of the 26-asset set is in the close neighborhood of the true unconditional risk price.[20]

1.5 *Implications of many efficient portfolios for inefficient asset betas*

The existence of more than one limiting efficient portfolio implies that risky asset betas are sensitive to the chosen reference efficient portfolio. The resultant betas, for an inefficient asset, will have substantially different values dependent on the reference efficient portfolio. This is due to the large range in efficient portfolio volatilities in our proxy limiting *IOS*.

Figure 3 plots the resultant betas for a risky asset, j, with expected return, $\mu_j = 0.8\%$/month, relative to risky efficient portfolios, p, with volatilities ranging from 1%/month to 5%/month. From Equation (1b), $\beta_{j,p} = \frac{(\mu_j - r_f)}{\psi_u \sigma_p}$, we observe that larger reference portfolio volatility, σ_p, implies smaller resultant betas. We evaluate the betas using the Table 3, $N = 26$ assets plug-in estimate of $\psi_u = 0.176$ and with r_f given by the one-month TBill rate, 0.338%/month, for the red curve. The blue curve uses the same values except the riskless rate, r_f, is given by the limiting vertex mean, $\hat{\mu}_g = \breve{\mu}_g = 0.36\%$/month.

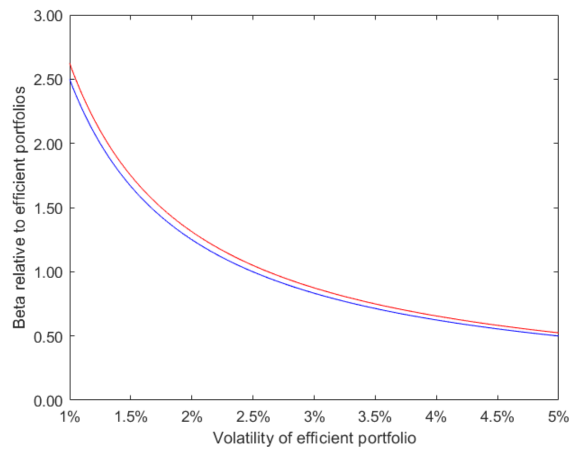

Figure 3 Changes in an asset's beta relative to different limiting efficient portfolios.

We plot the beta for a single inefficient asset, with a mean of $\mu_j = 0.8\%$/month, versus proxy limiting efficient portfolios with volatilities, σ_p, ranging from 1% to 5%, in the $N = 26$ asset case. Betas are given by Equation (1b), $\beta_{j,p} = \frac{(\mu_j - r_f)}{\psi_u \sigma_p}$, where we replace ψ_u with the plug-in estimate from Table 3 of 0.176. The riskless rate, r_f, in the $\beta_{j,p}$ formula, is given by the one-month TBill mean of 0.338%/month for the red curve and by the limiting vertex mean of $\hat{\mu}_g = \breve{\mu}_g = 0.36\%$/month for the blue curve. The small vertical separation indicates very similar beta values for the two values of the riskless rate.

For presentation purposes, the resultant betas are not plotted for efficient portfolios, p, with very small volatilities, because those betas are very large.

We conclude that the betas of an inefficient asset, j, will be larger, the smaller volatility of an efficient proxy portfolio, p.

2 The Roll Critique and Limiting IOS Conclusions

Our proxy *IOS*, from 26 traded assets, seems sufficiently close to the true, unconditional, limiting *IOS* that correct conclusions may result from large sample statistical tests and estimators. The hypotheses that can be reliably tested are subject to the Roll (1977) critique, which in summary says "…any valid test presupposes complete knowledge of the true market portfolio's composition. This implies, *inter alia*, that every individual asset must be included in a correct test."[21] However, there are characteristics of the limiting *IOS* and its proxy *IOS*, which may extend or modify some of the Roll critique, as discussed in this section.

The following conclusions and managerial implications result from our analyses:

(1) The limiting portfolio efficient set is linear with one unknown, ψ_u, the limiting Sharpe ratio of all efficient portfolios that is closely approximated by a proxy *IOS* of traded equities and fixed income assets. Our nearly unbiased, limiting, unconditional Sharpe ratio is 0.18, substantially larger than the all-equity value and less than the upwardly biased limiting maximum likelihood (ML) estimate of 0.25.

(2) The limiting Sharpe ratio is likely time varying, as indicated by its changes in equivalent assets data sets from different time periods.[22]

(3) Due to the *IOS* convergence to its limit and the zero-beta rate to the riskless rate, the Black (1972) asset pricing model converges to the Sharpe–Lintner (S-L) asset pricing model.

(4) An efficient market portfolio could be any portfolio on the efficient portion of the proxy *IOS*, implying that the market portfolio does not play a unique role in determining asset prices, and therefore the betas, computed with respect to any efficient portfolio, are not unique.

(5) The efficacy of borrowing and lending at the horizon's risk-free rate is not in question, because all efficient risk-return combinations on the proxy *IOS* are obtainable with investments in only risky assets. However, all resultant proxy portfolio allocations involve costly short positions, implying that their expected returns should reflect short position costs. This has serious implications for the inclusion of both long and short versions of the same asset in a covariance matrix and for *IOS* spanning tests.[23] This is an analogous problem to costless borrowing at the riskless rate in the S-L CAPM.

(6) In the event, that other assets are found that result in a different limiting *IOS* and its proxy and that have one or more strictly positive weight portfolio(s), those portfolios do not play a unique role in determining asset prices.

(7) Evidence confirming the existence of an efficient proxy *IOS* portfolio, with strictly positive weights, does not exist. Consistent with Roll (pp. 154, 173–174), the lack of an only positive weight efficient portfolio implies that a hypothesized proxy market portfolio is inefficient, and the strict S-L asset pricing model does not apply. In addition, the S-L and Black models do not apply because there is more than one limiting efficient portfolio.

(8) When weights on expanded asset sets are unknown or at least are estimated with error, Roll's requirement of a positive weight efficient portfolio may be difficult to determine. However, a necessary condition for limiting efficiency is that a proposed market proxy has a Sharpe ratio greater than or equal to what is believed to be the market's limiting Sharpe ratio of, for example 0.18. That condition is testable, typically using a limiting ML value like 0.25, without knowledge of the true weights on efficient limiting portfolios or asset betas.

(9) Despite the lack of a limiting positive weight proxy for the unknown market portfolio, it is possible that the unknown market portfolio is efficient with a Sharpe ratio not significantly different from 0.18. That is, our *IOS* proxy spans the true limiting *IOS* and its proxy *IOS* and that statement seems testable, using expanded asset sets.

(10) A test of the parametric efficiency of any managed portfolio is simply obtained by comparing the portfolio Sharpe ratio with the Sharpe ratio derived from an acceptable proxy *IOS*.[24] No other parameters require estimation. This test may be preferable to the alternative "intersection" tests of portfolio efficiency because the proposed test may be against a constant and does not require proper subsets in the test.

(11) A high-level conclusion is that the test of a hypothesized market portfolio's limiting efficiency and resultant, but not unique asset pricing, presupposes knowledge *only* of the market's limiting Sharpe ratio. The test does not require positive sample weights, proxy portfolios, or the inclusion of every individual asset that is contained in the true market portfolio.

(12) In Roll's (pp. 171–172) discussion of a zero minimum variance portfolio, one asset is eliminated from each pair that is perfectly correlated. The resulting *IOS* is then combined with investment in the single riskless asset, implying that the efficient portfolios are perfectly, positively correlated. The difference, in our case, is that the limiting efficient set of assets is linear, perfectly correlated, and its proxy comprised of *only* risky assets.

(13) Limiting efficient proxy portfolios are almost, but not perfectly, positively correlated because the proxy efficient set is almost but not perfectly linear, as shown in Figure 2. This is true regardless of whether the portfolios are market proxies or not.

(14) Inefficiency of the Merton (1973) market portfolio might be consistent with a positive weight market portfolio, even if an efficient portfolio to be hedged does not have strictly positive weights. However, the resulting hedged portfolio requires strictly positive weights to be a market portfolio. As a result of Roll (corollaries 5 and 7), strictly positive weights require that the hedging portfolio is located on the negatively sloped side of its two-portfolio hyperbola and offsets the efficient portfolio's negative weights.

(15) The hedging requirements of an optimal portfolio are significantly reduced because there are only two variables to hedge, shifts in the horizon's riskless return and shifts in the limiting Sharpe ratio. However, these variables may be functions of commonly used economic factor variables.

(16) A limiting *IOS* proxy, containing equities and long- and short-term debt securities, dominates smaller subsets containing only equities, for example. The dominant limiting asset set has a maximum and nearly constant Sharpe ratio for all efficient portfolios of sufficient volatility. This eliminates the need for an investor to search for the maximum Sharpe ratio portfolio in a dominant asset set. In addition, the small sample

problem, of tangents from the riskless rate lying on the lower segment of the *IOS*, is avoided.

(17) That portfolio choice is based solely on the desired volatility risk, for a fixed investment horizon. As in classic portfolio selection, the risk choice is subject to uncertainty due to the uncertainty of its portfolio weights. However, for limiting proxy portfolios, the risk is (hypothetically) smaller because conditionally smaller risk portfolios may be selected that offer the same Sharpe ratio.

Appendix 1 Investment Opportunity Sets and Estimators

This appendix summarizes the mathematics of Markowitz (1952) efficient sets, as developed by Lintner (1965), Merton (1972), Roll (1977), and more recently, others.

Consider the population parameters of an *IOS* containing the return means and variances of all possible portfolios, formed from N assets, and for an investment horizon of H. The equation describing the mean–standard deviation efficient *IOS* portfolios is well-known to be the upper portion of the hyperbola given by[25]

$$\mu_p = \mu_g + \psi\sqrt{\sigma_p^2 - \sigma_g^2}, \qquad \text{(A1)}$$

for all efficient portfolios, p, where μ_p and $\sigma_p^2 \geq \sigma_g^2$, are the expected return and variance of portfolio, p. Respectively, $\mu_g = b/c$ and $\sigma_g^2 = 1/c$ are the mean return and variance of the global minimum variance (vertex) portfolio, g. The price of an efficient portfolio's square root of variance risk, σ_p^2, in excess of σ_g^2, is $\psi = \sqrt{a - \frac{b^2}{c}}$, which is also the slope of the asymptote to the *IOS*, and all are specific to an *IOS* with its N and H. The efficient set constants are also well-known to be $a = \mu'\Sigma^{-1}\mu$, $b = \iota'\Sigma^{-1}\mu$, and $c = \iota'\Sigma^{-1}\iota$, where μ is the $(N \times 1)$ vector of asset expected returns, ι is the $(N \times 1)$ vector of ones, and Σ^{-1} is the $(N \times N)$ matrix of inverse elements of the non-singular covariance matrix, Σ.

The asset allocations for an efficient portfolio, p, are

$$X_p = \Sigma^{-1}(\mu\iota)A^{-1}(\mu_p 1)', \qquad \text{(A2)}$$

where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is the fundamental information matrix from Roll (1977, Equation (A8)) and therefore

$$A^{-1} = \frac{\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}}{ac - b^2} = \frac{\begin{bmatrix} 1 & -\mu_g \\ -\mu_g & \psi^2\sigma_g^2 + \mu_g^2 \end{bmatrix}}{\psi^2}.$$

Substituting for μ_p and A^{-1} and evaluating Equation (A2) gives

$$X_p = \Sigma^{-1}(\mu\iota)\frac{\begin{bmatrix} 1 & -\mu_g \\ -\mu_g & \psi^2\sigma_g^2 + \mu_g^2 \end{bmatrix}}{\psi^2} \times \left(\mu_p\mu_g + \psi\sqrt{\sigma_p^2 - \sigma_g^2}\right)'$$

$$= \Sigma^{-1}\mu\left[\frac{\sqrt{\sigma_p^2 - \sigma_g^2}}{\psi}\right] + \Sigma^{-1}\iota\left[\sigma_g^2 - \frac{\mu_g\sqrt{\sigma_p^2 - \sigma_g^2}}{\psi}\right],$$

for all efficient portfolios, p. (A3a) and if p is the vertex portfolio, its weights simplify to

$$X_g = \sigma_g^2 \Sigma^{-1} \iota. \tag{A3b}$$

The determinant of the information matrix is

$$P = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 = \frac{\psi^2}{\sigma_g^2}, \tag{A4}$$

which is a spanning measure of the N asset set. A positive difference,

$$P - P_1 = \left(\frac{\psi^2}{\sigma_g^2} - \frac{\psi_1^2}{\sigma_{g1}^2} \right) > 0,$$

is indicative of the lack of spanning of an N asset set by a proper subset of $N_1 < N$ assets, where the 1 subscript indicates respective values for the subset of N_1 assets.[26]

The CAPM pricing model is given by $\mu_j = r_f + \beta_{j,m}(\mu_m - r_f)$, where m is the unique market portfolio and $\beta_{j,m}$ is calculated with respect to m. The Black (1972) asset pricing model implied by Equation (A1) is given by $\mu_j = \mu_{z,p} + \beta_{j,p}$ $(\mu_p - \mu_{z,p})$, where $\beta_{j,p}$ is the beta of asset j with respect to efficient portfolio p and $\mu_{z,p} = \frac{(a - b\mu_p)}{(b - c\mu_p)}$ (Roll, 1977; Equations (A15) and (A19)) is the zero-beta rate corresponding to efficient portfolio p. Substituting for the efficient set constants, in terms of μ_g, σ_g^2, and ψ results in the zero beta rate,

$$\mu_{z,p} = \frac{\psi^2 \sigma_g^2}{(\mu_g - \mu_p)} + \mu_g. \tag{A5}$$

Substituting, the Black asset pricing model may be written as

$$\begin{aligned} \mu_j = {} & \frac{\psi^2 \sigma_g^2}{(\mu_p - \mu_g)} (\beta_{j,p} - 1) + \mu_g \\ & + \beta_{j,p}(\mu_p - \mu_g), \quad j = 1, 2, \ldots N. \end{aligned} \tag{A6}$$

Equation (A6) implies that the covariance, $\sigma_{j,p}$, between an efficient portfolio's return and any asset, j, is a linear function of μ_j, conditional on μ_p and σ_p^2, and the *IOS* parameters $\mu_g, \sigma_g,$ and ψ^2 (Roll Corollary 6). If $j = s$ is an efficient portfolio, the covariance between any two efficient portfolios is positive (Roll Corollary 4).[27] If one of the portfolios is the vertex portfolio, the covariance between the vertex and an efficient portfolio, p, is the vertex portfolio's variance (Roll, 1977, p. 161),

$$\sigma_{g,p} = \sigma_g^2. \tag{A7}$$

The asymptote to the *IOS* has expected return,

$$\mu_A = \mu_g + \psi \sigma_p, \tag{A8}$$

and the expected return difference between the asymptote and the efficient portfolio with variance, σ_p^2, is

$$\mu_A - \mu_p = \psi \left(\sqrt{\sigma}_p^2 - \sqrt{\sigma_p^2 - \sigma_g^2} \right). \tag{A9}$$

This distance must be positive to ensure a full-rank covariance matrix, Σ, underlying the *IOS*.

Assuming normally distributed returns, unbiased estimators of the *IOS* parameters, μ_g, σ_g^2 and ψ^2, or equivalently, a, b, and c, are well known and could be used in estimating the corresponding *IOS* parameters. An unbiased estimator of the risk price parameter, ψ, has, to our knowledge, not been published elsewhere.[28] Appendix 2 contains our development of estimators for ψ, P, and μ_p, including the ML estimators and estimators based on unbiased plug-ins of their components.

Appendix 2 ML Estimators, Plug-In Estimators, and Unbiased Estimators

The ML estimators of μ_g, σ_g^2, ψ^2, and P are denoted $\hat{\mu}_g = \frac{\hat{b}}{\hat{c}}$, $\hat{\sigma}_g^2 = \frac{1}{\hat{c}}$, $\hat{\psi}^2 = \hat{a} - \frac{\hat{b}^2}{\hat{c}}$, and $\hat{P} = \frac{\hat{\psi}^2}{\hat{\sigma}_g^2}$, where $\hat{a} = \hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu}$, $\hat{b} = \iota'\hat{\Sigma}^{-1}\hat{\mu}$, and $\hat{c} = \iota'\hat{\Sigma}^{-1}\iota$, and where $\hat{\mu}$ and $\hat{\Sigma}$ are the

ML estimators of the mean vector and covariance matrix, respectively. We are particularly interested in estimators of the risk price, ψ, and its upper bound.

Information matrix determinant

The information matrix determinant is $P = ac - b^2 = \frac{\psi^2}{\sigma_g^2}$, with ML estimator, $\hat{P} = \frac{\hat{\psi}^2}{\hat{\sigma}_g^2}$, and its expected value, $E(\hat{P}) = E(\hat{\psi}^2)E\left(\frac{1}{\hat{\sigma}_g^2}\right)$, where[29]

$$E(\hat{\psi}^2) = \frac{N - 1 + T\psi^2}{T - N - 1},$$

$$E\left(\frac{1}{\hat{\sigma}_g^2}\right) = \frac{T}{(T - N - 2)\sigma_g^2},$$

$$E(\hat{\mu}_g) = \mu_g \quad \text{and} \quad \mathrm{Cov}(\hat{\psi}^2, \hat{\sigma}_g^2) = 0.$$

Solve for ψ^2 and $\frac{1}{\sigma_g^2}$, resulting in $\psi^2 = \frac{E(\hat{\psi}^2)(T-N-1)-(N-1)}{T}$, and $\frac{1}{\sigma_g^2} = \frac{T-N-2}{T}E\left(\frac{1}{\hat{\sigma}_g^2}\right)$. Substitute in P and the unbiased estimator of P is therefore,

$$P_{unbiased} = \frac{\hat{\psi}^2(T - N - 1)(T - N - 2) - (N - 1)(T - N - 2)}{T^2\hat{\sigma}_g^2}.$$

The plug-in estimator is given by,

$$\breve{P} = \frac{\breve{\psi}^2}{\breve{\sigma}_g^2} = \frac{[\hat{\psi}^2(T - N - 1) - (N - 1)](T - N)}{T^2\hat{\sigma}_g^2}$$

where $E(\hat{\sigma}_g^2) = \frac{T-N}{T}\hat{\sigma}_g^2$ and so, $\breve{\sigma}_g^2 = \frac{T}{T-N}\hat{\sigma}_g^2$

Asymptote slope and risk price

The unbiased estimator of the squared risk price is

$$\psi^2_{unbiased} = \frac{\hat{\psi}^2(T - N - 1) - (N - 1)}{T}$$

and the plug-in estimator of the risk price is obtained by simply taking its square root,

$$\breve{\psi} = \sqrt{\psi^2_{unbiased}} = \sqrt{\frac{\hat{\psi}^2(T - N - 1) - (N - 1)}{T}}.$$

This plug-in estimator is biased, and, to our knowledge, an unbiased estimator has not been published.

Our half-moment estimator, ψ_{HM}, of the risk price, ψ, is obtained from Johnson *et al*. (1995) and given by

$$E(\hat{\psi}) = \frac{\Gamma\left(\frac{T-N}{2}\right)\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{T-N+1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)} \times {}_1F_1\left(-\frac{1}{2}, \frac{N-1}{2}, -\frac{T\psi^2}{2}\right),$$

where $\left(\frac{T-N+1}{N-1}\right)\hat{\psi}^2$ follows a non-central F-distribution with $N - 1$ and $T - N + 1$ degrees of freedom, non-centrality parameter, $T\psi^2$, $\Gamma(.)$ denotes the gamma function, and ${}_1F_1\left(-\frac{1}{2}, \frac{N-1}{2}, -\frac{T\psi^2}{2}\right)$ denotes the confluent hypergeometric function.

An estimate is obtained by replacing $E(\hat{\psi})$ with $\hat{\psi}$ on the left-hand side, and then solving numerically for $\psi_{HM} = \psi \geq 0$. Because the confluent hypergeometric function equals 1 when $\psi = 0$, positive ψ_{HM} risk prices occur only if, $\hat{\psi} \geq \frac{\Gamma\left(\frac{T-N}{2}\right)\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{T-N+1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}$. For our limiting $N = 26$ proxy asset sets, the critical value for the ML estimate is $\psi_c \geq 0.1736$. In cases when $\hat{\psi}$ is less than this critical value, there is no half-moment estimate that results in a positive risk price.

The unconditional expectation of $\hat{\psi}$ is a weighted average of conditional expectations above and below the critical value, $\psi_c = \frac{\Gamma\left(\frac{T-N}{2}\right)\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{T-N+1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}$, and is given by

$$E(\hat{\psi}) = E(\hat{\psi} \mid \hat{\psi} \geq \psi_c)\text{Prob}(\hat{\psi} \geq \psi_c) + E(\hat{\psi} \mid \hat{\psi} < \psi_c)\text{Prob}(\hat{\psi} < \psi_c),$$

which results in an upward biased estimate of $\psi_{HM} > 0$, if $\text{Prob}(\hat{\psi} < \psi_c) > 0$. For our $N = 26$ asset case, we estimate the $\text{Prob}(\hat{\psi} < \psi_c)$ to be 0.006. We therefore conclude that the half-moment, ψ_{HM}, is unlikely to be meaningfully impacted by observations below the critical value in our limiting proxy *IOS*. Korkie and Turtle (2021) provide further discussion of a related issue in which the familiar unbiased estimator for the squared risk price is often impossibly negative (especially in smaller time series samples of equity portfolios).

Efficient portfolio mean return

The efficient portfolio mean return is

$$\mu_p = \mu_g + \psi\sqrt{\sigma_p^2 - \sigma_g^2},$$

with ML estimator

$$\hat{\mu}_g + \sqrt{\hat{\psi}^2}\sqrt{\sigma_p^2 - \hat{\sigma}_g^2}$$

and plug-in estimator

$$\breve{\mu}_g + \sqrt{\breve{\psi}^2}\sqrt{\sigma_p^2 - \breve{\sigma}_g^2},$$

where

$$\breve{\mu}_g = \hat{\mu}_g, \quad \breve{\sigma}_g^2 = \frac{T\hat{\sigma}_g^2}{(T-N)}, \quad \text{and}$$

$$\sqrt{\breve{\psi}^2} = \sqrt{\frac{\hat{\psi}^2(T-N-1) - (N-1)}{T}}.$$

If desired, the unbiased expectation of $\sqrt{\sigma_p^2 - \hat{\sigma}_g^2}$ might be developed using the law of the unconscious statistician (and may be implemented using numerical integration in MATLAB, for example). Let $\hat{x} = \frac{T\hat{\sigma}_g^2}{\sigma_g^2} \sim \chi^2_{T-N}$, with density $f(\hat{x})$. Also, let $y = g(\hat{x}) = \sqrt{\sigma_p^2 - \sigma_g^2\frac{\hat{x}}{T}}$, providing expectation,

$$E[g(\hat{x})] = \int_0^\infty \sqrt{\sigma_p^2 - \sigma_g^2\frac{\hat{x}}{T}} f(\hat{x})d\hat{x}$$
$$= \frac{1}{2^{(T-N)/2}\Gamma\left(\frac{T-N}{2}\right)} \times \int_0^\infty \sqrt{\sigma_p^2 - \hat{\sigma}_g^2}\hat{x}^{\frac{T-N}{2}-1}e^{\hat{x}}d\hat{x}.$$

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Endnotes

1 See point (17) in Section 2, for example.

2 MacKinlay supports this upper bound with theoretical discussion from representative agent asset pricing models and with supporting empirical calculations. Subsequently, he provides simulation results that show relatively fast convergence of the underlying squared Sharpe ratio to its upper bound through a comparison of risk-based alternatives. His reported Sharpe ratios increase from 0.15 to 0.17 as the number of assets increase from 100 to more than 5,000.

3 The mean–standard deviation investment opportunity set is known from Merton (1972) to be a hyperbola, with two linear asymptotes (one with a positive slope and the second with a negative slope), emanating from the vertex portfolio's mean return on the mean axis. The hyperbola's positively sloped segment is tangent to the positively sloped asymptote, at an infinite standard deviation (volatility). Similarly, the negatively sloped segment is tangent to the negatively sloped asymptote, at an infinite standard deviation.

4 Levy and Roll (2010) demonstrate that only small perturbations in mean–variance parameters are necessary to restore the mean–variance efficiency of market proxy portfolios in a zero-beta CAPM. However, Brière *et al.* (2011) subsequently conclude that the equity market portfolio is inefficient.

[5] Numerous empirical issues arise in these robustness analyses with finite asset proxy sets. For example, the traditional unbiased estimate of the squared risk price is negative when only five or ten industry portfolios are used as risky assets. The 38 and 48 industry portfolios have multiple missing values that limit the available time series sample with little benefit in terms of the resultant efficient set constants. Furthermore, even the broad 48 industry portfolios' asset set requires the inclusion of additional debt securities to yield a plausible unbiased risk price. Inclusion of the debt securities increased the unbiased estimate of the squared risk price by over 90%. Although other alternative asset classes would be interesting to consider, they unfortunately tend to have much shorter time series histories.

[6] We leave the identification of factors and the hedges to future research. See Ghysels *et al*. (2005) and Rossi and Timmermann (2015) for a sample of research supporting the ICAPM.

[7] Early developments of limiting efficient sets or Sharpe ratio bounds include Buser (1977), Korkie and Turtle (1997), and Cochrane and Saa-Requejo (2000).

[8] See Appendix 1 and its discussion of Equation (A1). Korkie and Turtle (1997, 2021) also consider a limiting *IOS*. The emphasis here is on the impact of a limiting *IOS* for asset pricing, and Roll's critique.

[9] In addition to the non-uniqueness of the asset pricing portfolio, p, our subsequent empirical analysis also confirms that the literature's many tests of the CAPM with all-equity asset sets are poorly specified given omission of low risk and fixed income securities.

[10] This upper bound on the total risk price, ψ_u, is also the upper bound on the slope of the marginal *IOS* formed exclusively from self-financing portfolios in any asset set. See Korkie and Turtle (2002).

[11] MacKinlay (1995) provides widely accepted estimates of equity Sharpe ratios. Cochrane and Saa-Requejo (2000) discuss the extensive literature limiting Sharpe ratios from becoming too large. The Sharpe ratio bound is also relevant to stochastic discount factor bounds; see Shiller (1982) and Hansen and Jaganathan (1991), and the related discussion in Campbell and Viceira (2002).

[12] Numerous empirical difficulties arise with estimation in finite samples and as the convergence limit is approached. At the convergence limit where the global minimum volatility is zero, the covariance matrix is singular and thus the traditional limiting efficient set constants are undefined. Also, at the limit, the Moore–Penrose generalized inverse results in more than one riskless return and other tractability issues. Lee and Kim (2017) conclude that obtaining efficient portfolios using the Moore–Penrose inverse is problematic. Empirically, Liu *et al*. (2016) approximate the covariance matrix using its largest eigenvalue and shrink it from the ML estimator to estimate efficient portfolio expected returns. Korkie and Turtle (2021) show that an optimized shrinkage estimator of the market risk price, quadratic loss dominates the square root of the unbiased estimator for the squared slope, which is implausibly negative in many samples.

[13] The 2-month Tbill returns and the 1-, 2-, 5-, and 10-year constant maturity Tbond returns are from CRSP. Long-term corporate bond returns are from the SBBI (Stocks, Bonds, Bills and Inflation) yearbook.

[14] More simply, this can also be determined by solving for the portfolio weights at any two efficient portfolios (e.g., the vertex, g, and $\sigma_p = 5\%$ portfolios) and using the fact that *all* other efficient portfolios are convex linear combinations of two portfolios on the efficient set, Roll (corollary 5). This corollary is true regardless of the magnitude and signs of the two portfolios' weights. As Roll states "We might as well pick two portfolios whose means and variances are easy to compute. One would certainly be the global minimum variance portfolio ...." We chose the $\sigma_p = 5\%$ portfolio simply for large separation from the vertex portfolio.

[15] Our thanks to the referee for this suggestion. See Endnote 16 for a problem determining the tangent portfolio.

[16] In some data periods, the riskless rate exceeds the vertex portfolio's mean return, implying that there is no finite risk tangency on the upper segment of the efficient set hyperbola and the tangency occurs on the lower segment. This problem is avoided in the proxy limiting efficient *IOS* because all efficient portfolios have approximately the same maximum Sharpe ratio.

[17] Details are provided in Appendix 2.

[18] For example, at a volatility of 4%/month, the Table 2 panel B asymptote Sharpe ratio of 0.182 ($= (1.066 - 0.338)/4$) decreases slightly to 0.176 ($= (1.066 - 0.362/4$) when the riskless rate of 0.338% is replaced by the global minimum portfolio mean of 0.362 in the numerator.

[19] Jobson and Korkie (1989, Equation (11)) provide a specification of the Shanken (1985) and Huberman and Kandel (HK, 1986, 1987) test statistics, in terms of the information matrix determinant and correction for the HK (1987) exponent typographic error. This is useful

here because it shows what is tested in terms of our notation. The statistic can be written as

$$\frac{(T-N)}{(N-N_1)} \frac{\left(\sqrt{\hat{P}+\frac{1}{\hat{\sigma}_g^2}}-\sqrt{\hat{P}_1+\frac{1}{\hat{\sigma}_{g1}^2}}\right)}{\sqrt{\hat{P}_1+\frac{1}{\hat{\sigma}_{g1}^2}}},$$

where $\hat{P} = \frac{\hat{\psi}^2}{\hat{\sigma}_g^2}$, $\hat{P}_1 = \frac{\hat{\psi}_1^2}{\hat{\sigma}_{g1}^2}$, and is F-distributed with $2(N - N_1)$ and $2(T - N)$ degrees of freedom, under the null hypothesis, if $(N - N_1) \geq 2$. For rows 2 and 3, the spanning test statistic is F-distributed with 2 and $T - N$ degrees of freedom under the null hypothesis, and is given by,

$$\frac{(T-N)}{(N-N_1)} \frac{\left(\left(\hat{P}+\frac{1}{\hat{\sigma}_g^2}\right)-\left(\hat{P}_1+\frac{1}{\hat{\sigma}_{g1}^2}\right)\right)}{\left(\hat{P}_1+\frac{1}{\hat{\sigma}_{g1}^2}\right)},$$

for $(N - N_1) = 1$ (see Kan and Zhou, 2012 for test statistic development).

[20] Beaulieu *et al*. (2023) derive analytical conditions that link the identification of the zero-beta rate to spanning, in non-limiting *IOS*s.

[21] Roll (1977, abstract).

[22] In an earlier version of this paper, we consider a slightly less expansive set of assets that does not include 1-month holding returns on 2-month TBills. The benefit of not including 2-month TBills is greater data availability from 6/1941 with the cost that the IOS does not converge as quickly to the asymptote for low-risk assets. Interestingly, the inclusion of this earlier period increases the estimated risk price to approximately 0.26 (seemingly due to the strong equity returns over this subsample).

[23] The issue of trading costs, assets differing by position, and spanning is addressed in De Roon *et al.* (2001). An interesting empirical question is whether the short sale restricted IOS, with only positive weight portfolios, spans the unrestricted *IOS*.

[24] Some available tests are Miller and Gehr (1978), Jobson and Korkie (1981), Cadsby (1986), Lo (2002), Memmel (2003), Ledoit and Wolf (2008), and Wright *et al*. (2014).

[25] This is obtainable, for example, by solving Kan and Smith (2008; Equation (30)) for the mean return, μ_p. Bodnar and Schmid (2009) also provide useful results regarding the sample efficient frontier. Korkie and Turtle (2021) provide related development of the limiting asset set to examine estimation of the market risk price.

[26] Hypothesis tests are in Section 1.4.

[27] If s is on the inefficient bottom of the hyperbola, the covariance is negative.

[28] We thank Ray Kan for a note proving that an unbiased estimator does not exist.

[29] See Kan and Smith (2008, Equations (13), (21), and (26)).

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