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ASSET PRICING, ASSET ALLOCATION AND RISK-ADJUSTED PERFORMANCE WITH MULTIPLE GOALS AND AGENCY: THE GOALS AND RISK-BASED ASSET PRICING MODEL

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Investment managers require a consistent asset pricing model, asset allocation recommendations, and risk-adjusted performance measures (or the "three facets of investing") to be effective in managing portfolios. Incorporating three critical realities of investing into these models (i.e., that investors have many stochastic goals, seek to delegate to skillful agents, and maximize risk-adjusted returns) provides recommendations on the three facets that are different from the foundational papers of Modern Portfolio Theory (MPT). This paper briefly surveys the literature on MPT, Goals-based Investing (GBI), and agency before providing a normative Goals- and Risk-Based Asset Pricing Model (GRAPM) that includes these three realities of investing and articulates the three facets. GRAPM exploits a simple idea that a relatively risk-free asset for one stochastic goal is a risky asset for another, and vice versa. These two assets, plus the traditional absolute risk-free rate of MPT, allow us to triangulate to establish returns for all other assets based on the return of any goal-replicating asset and multiple correlations. This approach creates a "pair-wise equilibrium" for all assets (and potentially a general equilibrium)—different from MPT—and also lends itself easily to a new asset pricing model with heterogeneous investors (i.e., each investor has a unique goal). GRAPM incorporates a "risk aversion" parameter that is also easily observable, and appears to explain why seemingly similar investors can have markedly different asset allocations or expected returns.

"When it is obvious that the goals cannot be reached, don't adjust the goals, adjust the action steps." Confucius[1]

1 Introduction: Three Key Facets of Good Theory and Importance of Goals and Agency

Investment managers require a robust and consistent method to price, risk-adjust, and allocate assets, as these are the three key facets of an effective investment practice. Thus, effective theory should provide similarly consistent models and recommendations on asset pricing, risk-adjusted performance measures, and asset allocation. The foundational papers of Modern Portfolio Theory (MPT) provide these three facets, though in a piecemeal fashion. Markowitz (1952) and Tobin (1958) provide for asset allocation recommendations, Sharpe (1964), Lintner (1965), Mossin (1966), and Treynor (1966) provide the asset pricing model, and Treynor (1966), Sharpe (1994), Graham and Harvey (1994, 1997), and Modigliani and Modigliani (1997) provide risk-adjusted performance measures for MPT. While each of these papers tackled these three key facets piecemeal/individually, Perold (2004), following Lintner (1965), demonstrates how MPT integrates all three facets and has a unique duality (under certain assumptions); namely, that the Capital Asset Pricing Model (CAPM) and Two Fund Separation (2FS) can be derived from maximizing the Sharpe Ratio or alternatively, that the maximization of the expected utility of wealth leads to CAPM, 2FS, and the Sharpe Ratio. This duality and completeness of approach on all three facets has yet to be matched by any other theory and therefore sets a very high bar to replace these foundational models.

MPT/CAPM is a normative theory that assumes that investors focus on two key variables: expected returns and risk (defined as the absolute volatility of returns). Since MPT CAPM formulas are widely understood and adopted they are not repeated, but the model assumes that a risk-free asset, F, exists (i.e., one with zero volatility of returns and zero correlation to other assets) in net zero supply, with an exogenously determined expected return. It also assumes that a "market portfolio", M (i.e., a market-capitalization weighted portfolio of all risky assets), can be established and identified. Thereafter, all risky assets can be priced based on F, M and beta, but interestingly, the risky asset being priced is also a constituent of M. The theoretical pros and cons of MPT/CAPM are covered extensively in Levy (2011), but the fact that it is still used in practice globally bear testimony to the enduring power of this theory. However, MPT assumes a world distanced from reality to achieve this impressive result.

Subsequent theory recognizes that MPT overlooks two key realities of investing: first, that investors have multiple stochastic goals for which they are saving money (called Goals-based Investing or GBI as in Amenc *et al.*, 2010; Das *et al.*, 2010), and furthermore, that principals delegate investment decisions to agents (Brennan, 1993). Both nuances of investment practice meaningfully alter the asset pricing and asset allocation models, (and, in turn, the risk-adjusted performance measures to be used). Yet, many investors like pension funds globally, with a demonstrable stochastic pension benefit and multiple levels of delegation, have continued to use the traditional models of MPT and have failed to achieve their goals. Cumbo and Wigglesworth (2019) document how pension funds in the US and Netherlands are struggling in the low interest rate environment. In the last decade, their financial condition has declined dramatically: from having more than adequate assets to pay pensions in the late 1990s, to having to potentially institute pension cuts or take excessive risk (by investing in illiquid assets) to meet their goals.

Merton (2007) predicted these problems arguing that pension funds were mispricing the risk of their portfolios; additionally, the asset allocation recommendations of Sharpe and Tint (1990) were largely ignored. More specifically, Muralidhar (2019a, 2019b) argues that these pension crises have been caused because investors incorrectly used MPT for portfolios with stochastic pension goals and delegation. One could speculate that the reason for the wrong use of MPT to manage portfolios with stochastic goals and delegation to agents, and the potential looming crises, is that most papers that highlighted these nuances of GBI and agency were also piecemeal (e.g., Sharpe and Tint (1990) or Merton (2007) only focused on asset allocation for pension funds; Brennan (1993) only focused on asset pricing with agency). Practitioners, not well versed in the nuances of theory, did not adapt their practices despite theory having moved on from MPT to account for stochastic goals and agency. And theory continued to address these issues in a piecemeal fashion as this survey demonstrates. Hence, there is a pressing need to develop a new comprehensive theory that not only acknowledges the fact that investors attempt to achieve multiple stochastic goals (e.g., retirement, health savings, a child's education), and delegate to agents who they hope are skillful, but also provides the three facets (asset pricing, asset allocation, and risk-adjusted performance) in a single, robust and consistent framework.

This paper will attempt to fill this void by offering another normative approach, the Goals and Risk-based Asset Pricing Model (GRAPM). It takes on this monumental challenge by focusing on a "pair-wise" equilibrium (i.e., using combinations of F, and two goal-replicating assets at a time) as opposed to a "one factor" equilibrium in MPT. Typically, academic theories have influenced investment practice, but this paper attempts to do the reverse and use investment practice to derive an academic theory of asset pricing, asset allocation, and risk-adjusted performance. In this multiple goal and agency model, 2FS is replaced by allocation to three assets, the asset pricing model depends on "relative" and not absolute betas, and the model appears to explain phenomena experienced in real life (e.g., why different investors can have different expected returns for the same asset; why seemingly similar investors—defined benefit pension funds—can have markedly different asset allocations). Unlike MPT, GRAPM is not dependent on (a) an unobserved risk aversion parameter, but rather on specific observed variables specified by institutional investors to both achieve their goal (e.g., absolute volatility targets) and the limits they place on agents (i.e., target tracking error); and (b) an unobserved market portfolio, but rather on goal-replicating assets that are potentially easily observed or created. Furthermore, the return of the absolute risk-free asset (which is risky for all goals) can be established endogenously in such a model. This approach provides an alternative heterogeneous investor model by assuming that each investor has a unique goal, and thereby has a unique asset that serves as the relative safe asset.

The paper is structured as follows: Section 1 provides a brief summary survey of the vast literature on agency and GBI and highlights which of the three facets of investing these key papers address.[2] It also briefly reviews the M-cube risk-adjusted performance measure of Muralidhar (2001) because risk-adjusted performance has had less prominence in the theoretical finance literature than asset pricing and asset allocation. Unlike the Sharpe, GH1, GH2 or M-square, M-cube—which is an extension of M-square—allows for both a stochastic goal and delegation. This is the measure of risk-adjusted performance that will be used to derive the other two facets of investing and hence is critical to the paper. However, two key challenges stymie the search for and derivation of a new practical theory: the absence of "safe" assets for GBI, and the challenges of utility theory to address reality. Section 2 addresses the first issue and examines the unique nature of GBI and reviews recommendations to create goal-replicating assets, as these goal-replicating assets serve as the "safe" asset and benchmark for every goal. Section 3 addresses the second issue, noting that investors do not specify or maximize expected utility as assumed by theory, but rather maximize risk-adjusted returns.[3] Since utility theory has yet to incorporate multiple goals and agency into a single, simple function, this paper will use the Perold's (2004) approach of attempting to derive this model and the three facets of investing by maximizing relative risk-adjusted performance using M-cube. Section 4 lays out GRAPM, which assumes that investors maximize goal relative risk-adjusted returns, subject to clearly articulated absolute and relative risk budgets and the desire to hire skillful agents. It explores the implications for pricing, managing and evaluating assets. GRAPM exploits the fact that a relative safe asset for one goal is risky for another (and vice versa), but every asset can have just one price/return. The equilibrium articulated in this section is a "pair-wise" equilibrium and thereby a departure from previous models. Adding more goals just requires more pair-wise equilibria (creating a "lattice" of equations). Section 5 highlights shortcomings and extensions and Section 6 summarizes.

2 Summary of Literature on GBI and Agency and the Three Facets

When one includes GBI and agency, the paradigm changes from a world focused on absolute wealth, to one focused on relative wealth (Muralidhar, 2019b), and all investment aspects are influenced by the goal and/or the agent's benchmark.[4] Table 1 lists[5] major papers in finance that touch on GBI and agency, noting key contributions to theory (column 2), and categorizes them by two key criteria: first, whether the model is focused on absolute (column 3) or relative wealth, further broken down by type of stochastic goal, where applicable (columns 4–6); and second, which of the three facets of investing they provide insights on (columns 7–9). Column 10 provides additional comments, especially whether the papers are based on equilibrium or optimization models. For example, Markowitz (1952) is clearly focused on absolute wealth (with an "X" in column 3) and optimal asset allocation (column 8); examples of models on relative wealth (with an "X" in columns 4–6) include those focused on agency, background risk, peer comparison, liabilities, etc. All "relative models" implicitly or explicitly draw inspiration from Merton (1973), row 6, which while focused ostensibly on absolute wealth, is an equilibrium relative model. The stochastic opportunity set in Merton (1973) requires investors to implement Three Fund Separation (3FS); namely, that the portfolio should be split among the absolute risk-free asset, F, the risky portfolio, M or M' depending on the circumstance, and the "hedging portfolio", L.[6] This crucial insight is the basis of all three facets of investing in the relative world.

The key asset pricing, asset allocation and risk-adjusted performance conclusions of this extensive body of research can be summarized as follows:

Asset Allocation: All relative papers (e.g., rows 7, 8. 9. 13, 14, 15, 16, 21, 22 and 23) follow some form of 3FS as noted earlier. The only difference in the various papers is that the goal-hedging portfolio in goals-based papers (e.g., rows 7, 19, 21, 22, 23) changes to the benchmark replicating portfolio in agency-based papers (e.g., row 9). In other

Table 1 Summary of major finance papers on absolute/relative goals and three facets of investing.

			Focus of Papers - Absolute/Relative				Three Facets of Investing			
			Absolute	Relative						
No.	Paper/author (Year) (1)	Key contribution (2)	One deterministic goal (3)	One stochastic goal (4)	Multiple stochastic goals (5)	Other stochastic benchmark (6)	Asset pricing (7)	Asset allocation (8)	Risk-adjusted performance (9)	Comments (Equilibrium or Optimal Model) (10)
1	Markowitz (1952)	Mean-variance optimization	X					X		Risky market portfolio—Optimal
2	Tobin (1958)	Two Fund Separation						X		Equilibrium
3	Sharpe (1964), Lintner (1965), Mossin (1966)	Asset Pricing Model	X				X			CAPM—Equilibrium
4	Treynor (1966)	Treynor ratio	X				X		X	Rating mutual funds—Equilibrium
5	Black (1972)	Zero beta CAPM		X			X			Non-zero volatility of r(F); Equilibrium
6	Merton (1973)	Intertemporal CAPM	X			Opportunity set	X	X		Continuous Time Model; 3FS; Equilibrium
7	Sharpe and Tint (1990)	Liability focused asset allocation		X				X		Allocation for pensions—Optimal
8	Abel (1990)	Keeping up with Jones utility				Habit	X			Relative utility to single benchmark—Equilibrium
9	Brennan (1993); Cornell and Roll (2005); Brennan and Li (2008)	Agency	X			Agency	X			Agency relative asset pricing—Equilibrium
10	Sharpe (1994)	Sharpe and differential Sharpe ratio	X			Agency			X	Differential Sharpe = Agency—Equilibrium
11	Graham and Harvey (1994, 1997)	GH1 and GH2	X			Agency		X	X	Related to Black (1972)—Optimal
12	Modigliani and Modigliani (1997)	*M*-square risk-adjusted performance	X			Agency		X	X	Performance of mutual funds—Optimal
13	Heaton and Lucas (2000)	Background risk				Background risk	X			Alternative stochastic benchmark—Equilibrium
14	Muralidhar (2001)	*M*-cube risk-adjusted performance		X		Agency		X	X	Ranks agents based on skill—Optimal
15	Lauterbach and Riesman (2002, 2004)	Relative preferences—KUJ				Peers	X			Relative (to peers) asset pricing—Equilibrium
16	Gómez and Zapatero (2003)	Two-factor CAPM				Benchmarking	X			KUJ Utility usage—Equilibrium
17	Perold (2004)	Duality of approach	X				X	X	X	Leverages Linter (1965)—Equilibrium
18	Merton (2007)	Focus on funded status		X				X		Safe MPT asset is risky for GBI
19	Waring and Whitney (2009)	Asset liability asset pricing		X			X			Black (1972) with stochastic liability—Equilibrium
20	Das *et al.* (2010)	Mental accounts			X			X		Optimal allocation with multiple goals—Optimal
21	Muralidhar and Shin (2013)	Using M-cube with a goal and agency		X		Agency		X	X	Relative Asset Pricing Model v 0.1—Optimal
22	Muralidhar *et al.* (2014a, 2014b)	Relative asset pricing model		X			X			Single stochastic goal—Equilibrium
23	Deguest *et al.* (2015)	Goals-based investing		X			X			Asset allocation for GBI—Optimal
24	Das *et al.* (2018)	Goals-based investing			X			X		Joint optimization of goals—Optimal
25	This paper (2019)	Goals and Risk-Based Asset Pricing Model			X	Agency	X	X	X	Following Perold (2004)—Equilibrium

words, for a specific goal like retirement, Sharpe and Tint (1990), row 7, note that the hedging portfolio would be the portfolio that replicates the cash flows of the pension fund; for agency on the other hand, Brennan (1993), row 9, notes that it would be the benchmark (without further description of what the optimal benchmark should be). When one combines GBI and agency, Muralidhar and Shin (2013), row 21, argue that in this situation common among institutional pension funds, the optimal benchmark asset for agents should be the same as the goal-replicating asset. While this survey uses a broad "relative" umbrella, there are some interesting differences between asset allocation recommendations for GBI and agency.

Merton (2007), row 18, argues that retirement or GBI investors should focus on maximizing (expected utility of) funded status and not wealth. As a result, if the investor is fully funded or overfunded (i.e., has sufficient assets to cover their goal), then the goal is ideally completely hedged by allocating to the goal-hedging portfolio. Interestingly, in the late 1990s, many US public pension funds were overfunded (Aubry *et al.*, 2018), but continued to use MPT to set allocations. Investing in assets deemed very risky relative to the goal, these pension funds face major challenges after two major equity market corrections. Merton (2007) characterized this investment behavior as "mispriced risk". If investors are underfunded as assumed in Sharpe and Tint (1990) and Muralidhar *et al.* (2014a), and is widely the case globally in 2020, then the allocation to the three assets, F, L and the risky portfolio will depend on the degree of underfunding and risk aversion. Typically, the greater the underfunding, the greater the allocation to risky assets.[7] This pattern of behavior has been documented in IMF (2013) and Aubry and Crawford (2019)—post-2008, even though equity markets have rallied, US public pensions have increased their allocation to risky assets as funded status declined.

In the case of pure agency, the allocation to the risky asset is determined by the relative risk permitted by the principal. In agency papers, because agents want to maximize relative return to get compensated, they seek as much relative risk as possible. The principal fears they are not skillful (and would gamble the money) and hence seeks to limit relative risk. However, in both GBI and agency, the relative risk taken/permitted reflects the risk tolerance of the investors and is easily observed, either in an investment policy statement or the actual portfolio positions (discussed in Section 3).

Asset Pricing: In all the relative models noted in Table 1 (e.g., rows 7, 8. 9. 13, 14, 15, 16, 21, 22 and 23), F exists with zero net supply, and has an exogenous return. The "hedging portfolio", L, because of its hedging property, is the "relative" risk-free asset and will typically earn a return below that of the risky (relative) market portfolio. Muralidhar *et al*. (2014a), row 22, make an interesting distinction regarding the "market" portfolio. They argue that in a single stochastic goal model, F is risky relative to L. As a result, the third portfolio, M' (and not the traditional M in CAPM or even Merton, 1973), is a unique, risky relative market portfolio. M' includes F and excludes L and, like CAPM, also includes all risky assets being priced. This approach creates complications in a multiple goal model as each goal has its own relative risky market portfolio, making an overall equilibrium extremely hard to solve.

Consider a Relative Asset Pricing Model (RAPM), which can be seen as a proxy for all "relative" models. Equation (1) highlights the basic asset pricing model in this relative paradigm. The intuition and the basic setup of the RAPM is provided in Appendix 1. Since it follows many of the steps used to derive CAPM, its form is similar to the CAPM model with a few variations.

$$\begin{aligned} E[r(i) - r(F)] &= \frac{\text{cov}[r(M') - r(L), r(i)]}{\text{var}[r(M') - r(L)]} \\ &\quad \times\ E[r(M') - r(L)] \\ &= \Gamma_{M'-L,i} \text{x} E[r(M') - r(L)] \qquad (1) \end{aligned}$$

where $\Gamma_{\text{M}'-\text{L},\text{i}}$ = is a "relative beta", which is a relative covariance term divided by a relative variance term. $[r(M') - r(L)]$ is the relative (to goal) market risk premium. Gómez and Zapatero (2003), row 16, term this relative asset pricing model (in their specific example) a "Two-Factor CAPM," because the pricing model depends on M' and L. For simplicity, we use RAPM as a generic reference for all relative models in Table 1 as they all largely follow Equation (1). "L" could easily be the peer benchmark (row 8), the benchmark to which an agent is measured by the principal (row 9), a background risk process (row 13), the home bias benchmark (row 15), or any other relative variant listed above (or even Campbell and Cochrane (1999) focused on habit). In this model, much like CAPM, asset I is a constituent of M'.

MPT can be seen as a very specific case of RAPM—with an implicit assumption that the goal is deterministic. If liabilities are deterministic (i.e., $r(L) = r(F)$), CAPM is retrieved. Under this assumption, maximizing absolute and relative wealth are identical objectives.[8] RAPM, similar to CAPM, makes a strong assumption and assumes that all investors have the same identical stochastic goal or benchmark. The challenge with RAPM is that "L" is not clearly specified in practice, as each investor has a unique L. Hence, it is not easy (and probably impossible) to test RAPM empirically, though the 3FS recommendations for asset allocation are optimal for individual investors and often observed in innovative/advanced pension, insurance, and endowment portfolios.

F and L are critical elements of asset pricing, asset allocation, and risk-adjusted performance in a relative world because: (i) F has zero volatility and zero correlation to all other assets including L and M'; and (ii) L has zero relative risk relative to itself (or perfectly correlated to the goal), but is otherwise like any other asset. This unique assumption of zero absolute volatility, zero correlation, and zero relative volatility embedded in F and L is critical to establishing risk-adjusted performance measures in a goals- and agency-based world.

Risk-Adjusted Performance: The simplest (and zero effort or zero skill) way for an agent to outperform a benchmark is to borrow through F and lever the benchmark to which they are measured. This delegated portfolio will outperform the benchmark on average, and be rewarded unless the principal appropriately risk-adjusts performance for the higher volatility of the levered portfolio. Modigliani and Modigliani (1997) normalize the performance and volatility of an asset (or agent) by leveraging/deleveraging this risky asset, using F, to achieve a target volatility (typically of the benchmark). The M-square risk-adjusted performance clearly nullifies naïve leverage used by an agent to beat a benchmark.

Assume that a principal has a goal that is replicated by L and hires an agent, who creates an active portfolio P. The principal cannot observe the true effort of the agent and can only monitor P's returns. The principal wants to maximize the relative risk-adjusted return of the portfolio (through actions of their own and the agent's portfolio), but does not want to pay the agent for zero "intelligence" activities like leverage or "beta". The M-square performance of the agent can be established by the principal using two conditions shown in Equations (2) and (3). $r(*)$ measures the return of an asset or portfolio.

$$r(A) = d \times r(P) + (1 - d) \times r(F) \quad (2)$$

$$\text{subject to } \sigma(A) = \sigma(L) \quad (3)$$

The principal first calculates the "d" for each agent, where $d = \sigma(L)/\sigma(P)$, σ is the volatility, and d measures the implicit or explicit leverage in portfolio P. d greater than 1 indicates the agent levered the portfolio; d less than 1 indicates deleverage.[9] The principal then calculates risk-adjusted return, $r(A)$, for each agent to rank them after levering or delevering portfolio P to have the same volatility as the benchmark, L. Since all agents have the same absolute risk, their M-square returns (or $r(A)$s) of all agents are now comparable. M-square ranks agents identical to the Sharpe ratio, but is expressed in terms of returns (as opposed to a ratio) and provides insight into how much of the return is derived from naively levering/delevering the benchmark. One unique M-square insight is than an agent with a portfolio that underperforms L (i.e., $r(P) < r(L)$), might actually be quite talented and should be hired if their $r(A)$ exceeds $r(L)$. In the traditional world of investing, this agent might be seen as an underperformer and not selected. But M-square includes allocation advice; namely, lever this manager's returns to have the same volatility of the benchmark as that combination outperforms the benchmark. This is a non-equilibrium model so it is possible for $r(A)$s generated by portfolios of different agents to have different returns, even though they all have the same volatility (because correlation to the benchmark has not been normalized).

Muralidhar (2001) extends this approach by arguing that in a "relative" paradigm, especially agency, there is a dual measure of risk: both absolute and relative risk (or tracking error—TE hereafter), and not just absolute risk as in M-square or MPT. Therefore, an agent's performance P must be risk-adjusted, by levering/delevering using both F and the benchmark hedging asset, L, to create a new portfolio with both a target absolute volatility ($\sigma(L)$), as in M-square, and a target correlation to the benchmark (or *de facto*, target relative risk). The agency literature has ignored this second constraint of normalizing correlations as well and hence typical measures cannot rank agents identical to ranking based on measures of confidence in skill (Ambarish and Seigel, 1996; Muralidhar, 2001). This is a very important point as principals would ideally want only skillful agents. M-cube ensures that all agents' risk-adjusted performance ($r(A)$s) have the same relative risk (TE), but also normalized to have the same volatility ($\sigma(L)$) and target correlation. This is also a non-equilibrium model like M-square.

Much like M-square, M-cube is estimated by assuming that principals maximize the goal relative risk-adjusted return, $r(A) - r(L)$, subject to three constraints highlighted in Equations (4)–(6).

$$r(A) = a \times r(P) + l \times r(L) + (1 - a - l) \times r(F) \quad (4)$$

$$\sigma(A) = \sigma(L) \quad (5)$$

$$\text{and } \mathrm{TE}(A) = \mathrm{TE}(\text{Target}) \quad (6)$$

where a is the allocation to the risky (agent) portfolio, P, and l is the allocation to the goal-hedging portfolio, L (and measures what is *de facto* invested in the low-cost passive benchmark or "beta" in industry parlance). The balance of the assets is invested in the traditional risk-free asset, F (and measure leverage). The term "$1 - a - l$" is the corollary to $1 - d$ in M-square.

This approach is described in more detail in Appendix II as the derivation involves a few more steps than M-square. To summarize, if $\rho(P,L)$ is the correlation of returns of L and P of each agent (which is easily calculated), and $\rho(T, L)$ as the target correlation,[10] which is specified by the principal, we can solve for "a" as in Equation (7), and "l" in Equation (8). The super-script denotes the asset being allocated to; the subscript indicates the target relative risk (T) and the goal-replicating asset (L).

$$a^P_{T/L} = \frac{\sigma(L)}{\sigma(P)}\left[\sqrt{\frac{[1-\rho(T,L)^2]}{[1-\rho(P,L)^2]}}\right] = \frac{\sigma(L)}{\sigma(P)}\left[\frac{\varphi(T,L)}{\varphi(P,L)}\right], \quad \text{where } \varphi(I,J) = \sqrt{[1-\rho(I,J)^2]} \tag{7}$$

$$l^L_{T/L} = \rho(T,L) - \rho(P,L) \times \left[\sqrt{\frac{[1-\rho(T,L)^2]}{[1-\rho(P,L)^2]}}\right] = \rho(T,L) - \rho(P,L) \times \left[\frac{\varphi(T,L)}{\varphi(P,L)}\right] \tag{8}$$

"a" and "l" provide the allocation to risky and the relative risk-free assets in a non-equilibrium but optimal model, respectively, for a given T, L and P. If the investor sets the target correlation ($\rho(T, L)$) to the correlation of the agent's portfolio to the goal ($\rho(P, L)$), then the solution to "a" is "d" in M-square and "l" = 0. In other words, because MPT assumes that the investor is principal (and ignores goals), it tolerates whatever relative risk is actually taken in the portfolio and is insensitive to relative risk. Equations (7) and (8) also highlight the factors that will drive the demand for/allocation to these assets (discussed in Section 3). Very simply, the allocation to risky assets is positively impacted by the volatility of the goal and (non-linearly) by the target TE. The allocation to the goal-hedging asset is invariant to volatilities, but is inversely impacted by the target TE—lower the target TE, the more likely the investor is to increase the hedge (see also Appendix III). By contrast, in traditional MPT, the allocation to an asset is determined by either its market capitalization and linearly to the allocation to risky assets because of risk aversion (in the equilibrium CAPM model because of 2FS) or by an opaque calculation based on whether it is included in an efficient portfolio or not (Markowitz, 1952). It is unaffected by potential substitutes and complementary assets. Section 3 will demonstrate how Equations (7) and (8) appear to provide allocations not very different from actual allocations of a large institutional investor, with similarly articulated risk targets.

Muralidhar (2001) shows how M-cube ranks agents consistent with measures based on confidence in skill (Ambarish and Seigel, 1996) and hence is preferable to Sharpe, GH1/GH2 and M-square in agency arrangements. This is the case because Sharpe, GH1/GH2, and M-square only focus on normalizing volatility (i.e., absolute risk) and ignore correlation (which impacts relative risk).[11] In short, agents with portfolios naively levered or highly correlated to the benchmark are shown to have lower risk-adjusted performance (and lower confidence in skill) than agents with truly unique portfolios. After all, generating excess performance (over the benchmark) by naïve leverage using F or allocating to L does not require skill and should not be rewarded.

Table 2 encapsulates the research reviewed and complements it with examples from practice to summarize the implications of these findings for a new theory and practice. It highlights the differences in behaviors when one compares Traditional MPT (absolute wealth and principals) with Investment Reality (multiple goals and agency). MPT assumes certain behaviors; behaviors in reality are easily observed and hence this

Table 2 Traditional MPT vs Investment Reality—Implications for theory and practice.

	Traditional MPT	**Investment reality**
Assumed vs observed behavior	*Assumed*	*Observed*
Objective	1. Maximize expected utility of wealth by generating highest return per unit of risk	1. Invest to service many goals (retirement; college expenditures; health). 2. Investors (should) care about funded status or relative wealth
Decision maker	1. Individual (principal)	1. Institutional boards (principals) delegate to investment staff (agents), who then delegate to external managers (agents). Includes pension funds, endowments, insurance companies, sovereign wealth funds with different goals and governance structures.
	2. Single investor type	2. Retail investors (principals) delegate to advisors (agents), who then delegate to mutual fund or other managers (agents)
Risk parameter	1. Volatility of portfolio	1. Absolute risk of the goal often stated in Investment Policy Statements (IPS) 2. Relative risk of portfolio delegated to agents; often stated clearly by institutional investors in IPS 3. Drawdown should matter more than volatility because it impacts compounding, but few investors use it explicitly in IPS statements
Risk aversion parameter	1. Critical to the model	1. Correlation of the active portfolio to the benchmark;
	2. Exogenous variable never observed in practice	2. Easily observed and measured;
Skill	Not a concern because principal does not delegate or pay fees	A huge concern because principals do not want to pay fees for luck

contrast is useful. Given the substantial differences between Traditional MPT and Investment Reality, it is critical to establish an effective theory based on reality.

3 Demand and Supply of Absolute and Relative Risk-Free Assets

While the thorny CAPM question has been the identification of M, the key question for GRAPM that must be addressed before deriving or even suggesting a new model is whether assets F and L even exist? After all, L in RAPM (Equation (1)) was not clearly identified and articulated, and hence while potentially theoretically interesting, it is an impractical model to estimate asset prices. The principal challenge to adopting GBI (and as a result 3FS) in reality or in articulating a new theory is that there are many goals—each goal is unique, and often the goal-replicating asset does not exist. Sharpe and Tint (1990) demonstrate how one might proxy the cash flows of a defined benefit pension fund (which aggregates the liabilities of many individuals) or an insurance company. Das *et al*. (2010), among the first to examine a multiple goals approach, develop a (mental accounts) asset allocation recommendation by assuming that every goal has a target expected return. However, this assumption also potentially misses the complexity of current GBI. For example, a 25-year old wishing to save for retirement or a new mother wishing to save for their newly born daughter's college fund cannot buy a liquid, traded instrument today to hedge the cash flows of these very realistic goals. Clearly, one should not derive asset allocation recommendations or asset pricing models for a host of goals based on assets that do not exist. Fortunately, recent work has sought to address this challenge and potentially clears this obstacle from the path to a new theory.

Many investment goals, including retirement, college savings, or buying a house typically, have one feature in common—namely, they can be specified as a set of forward-starting real cash flows (certain or uncertain), indexed to some appropriate measure of inflation, which then last for a (certain or uncertain) period of time once they start. They are clearly not deterministic. Muralidhar *et al*. (2016), and Merton and Muralidhar (2017a), suggest that the goal of retirement is to receive a target, guaranteed, real income, from retirement, say age 65, to death at say 85. In short, individuals save for approximately 40 years, and invest these savings in order to receive these specific cash flows to allow them to have an adequate retirement lifestyle.[12] Similarly, we make the assumption that the goal of saving for a child's college education is to be able to receive a targeted, guaranteed, real income (to pay for tuition and other costs), starting when the child is 18 years old, for a minimum period of 4 years. Here the link would be to tuition inflation.

The key contribution of Muralidhar *et al*. (2016) and Merton and Muralidhar (2017a) is to argue for the creation of a new bond by governments, or even private institutions like insurance companies, that mimics the desired retirement cash flows—Muralidhar *et al*. (2016) call these Bonds for Financial Security (BFFS) and Merton and Muralidhar (2017a) call them Standard-of-Living Forward-starting Income-only Securities (SeLFIES) and they differ only in the inflation index to which they are linked.[13] Similarly, Muralidhar (2016a, 2016b) argue for the creation of Bonds for Education and Student Tuition (BEST) that would mimic the college cash flows, and would be issued by colleges and universities. These goal-appropriate bonds would serve as the relative risk-free asset for these specific goals, greatly simplifying GBI (Muralidhar, 2016b, 2019a, 2019b), and there is an easily identified issuer who would benefit from issuing these bonds (governments for SeLFIES; colleges for BEST), thereby ensuring supply.

Proceeding in a similar fashion, we can make the assumption that every stochastic asset (or combination of assets) that exists potentially meets a unique goal, and obviously there has been a natural issuer who has sought to provide the instrument. Where instruments do not exist to hedge current stochastic goals for say health savings accounts (i.e., where there is demand), the hope is that finance science could potentially lead to the issuance of the new instrument by a supplier as noted in the case of two major goals—retirement and saving for college.[14] In summary, we assume that for each goal (or L) today, either the instrument exists and is currently traded, or finance science needs to be used to create it as there clearly is demand.

But what about F? In MPT/CAPM, two interesting assumptions are made about F, the "safe" or (absolute) risk-free asset to make the model work: (a) that F exists with net zero supply; and (b) that the rate of return is exogenously determined. These are clearly strong assumptions. In practice, investors use the Treasury-bill as a proxy for F, but clearly, the rate of return of T-bills is determined by market forces and it has positive supply. But in a multiple goals-based world, it may be reasonable to assume that there is a unique goal, for some individual, with the cash flow profile of the absolute risk-free asset, F (i.e., there is a traditional MPT investor in this world). The goal replicated by F would be deterministic and would be for an investor that is absolute wealth focused. And similarly, there exists a natural supplier of such instruments (e.g., governments) who benefit from issuing such bonds. In short, this (absolute) risk-free asset, along with all other assets, could exist with positive net supply and an effective model could establish its return endogenously based on market-clearing conditions. Interestingly, this approach of assuming that: (a) every existing asset could possibly satisfy a unique goal of some individual in the economy, or (b) assets could be created for goals without current replicating assets by potentially a natural issuer, is another approach to developing a "heterogeneous investor" model for asset pricing.[15] Section 4 will show how the multiple goal setting could potentially dispense with the need to identify and establish M as long as F and at least two goal-replicating assets exist.

4 Maximize Expected Utility or Risk-Adjusted Performance?

The second obstacle to using the traditional approach to derive a new theory that captures multiple goals and agency is that finance theory is largely based on the assumption that investors maximize expected utility (of wealth). While relative utility functions have been used to address a single goal or agency as shown in Table 1, there is no easily tractable (or currently specified) utility function that could handle (multiple) goals and agency within a single construct. Moreover, Equation (1) for RAPM suggests that each goal has its own unique relative market portfolio; hence an equilibrium with many goals would make for a very complex set of allocation formulas across these many relative market portfolios. More critically, as Table 2 has noted, investors do not maximize expected utility—instead they maximize relative to goal, risk-adjusted, expected returns.

For example, two large and innovative pension funds (the Los Angeles County Employees' Retirement Association—LACERA—and the New Mexico Public Employees' Retirement Association—NMPERA) have very explicit statements of objectives that are worth reviewing. The LACERA Investment Policy (IPS) states: "The Fund's long-term performance objective is to generate risk-adjusted returns that meet or exceed its defined actuarial target as well as its policy benchmark, net of fees, over the Fund's designated investment time horizon." NMPERA's IPS explicitly states that the Board established a 10.5% annualized target volatility for the strategic asset allocation (or L) and a 1.5% annualized tracking error (or TE(Target)) for all delegated decisions.[16] LACERA, like NMPERA, also articulates a relative risk budget. These are not random examples, but reflective of the general practice among many sophisticated investors. More importantly, when one applies Equations (7) and (8) for these given absolute (10.5%) and relative risk parameters (1.5%), M-cube suggests a 63% allocation to risky assets and a 37% allocation to the relative risk-free asset—which is not very different from the actual NMPERA allocations.[17]

Muralidhar (2019b) suggests that finance theory should restate the original six-word phrase, "individuals maximize the expected utility of wealth" as "individuals/institutions delegate to maximize risk-adjusted relative-returns". The latter approach is much more reflective of how investors actually behave (Table 2); namely, they maximize wealth relative to their goals (i.e., relative wealth matters and these goals are very different from one another), and they delegate to agents (i.e., relative risk matters). More importantly, investors do not specify utility functions (Markowitz, 1990), but rather seek to maximize risk-adjusted relative returns as clearly shown in the LACERA IPS. As a result, the Perold's (2004) approach is adopted hereafter, to derive the three facets in a multiple goals and agency world. The M-cube measure lends itself to this approach because it can be used for both goals (i.e., L) and agency (i.e., TE(Target)) in one formula.

5 The Simple Goals- and Risk-Based Asset Pricing Model

With two goal-specific relative risk-free assets and the traditional (absolute) risk-free asset, and investors attempting to maximize risk-adjusted returns based on M-cube, it appears that all assets can be priced. This model does not require a "market portfolio" because the approach uses a "pair-wise equilibrium," and a generic risky asset can be priced with any goal-replicating asset (which is risky for another goal). This section is in the spirit of Debreu (1959), Arrow (1964), Breeden and Litzenberger (1978), and Cochrane (2005), but based on goals, risk budgets, and goal-replicating assets as opposed to Arrow–Debreu securities or options prices on aggregate consumption or stochastic discount factors.[18] The intuition behind this model is very simple: a relatively risk-free asset for one goal (e.g., SeLFIES for retirement) is a risky asset for another goal (college savings) and vice versa. Equation (7) shows that the goal-replicating asset, F and any risky asset can be used to create risk-adjusted portfolios with specific target volatility and correlation characteristics to the goals. If such portfolios can be created for the first goal (first using a generic risky asset and then the second goal-replicating asset), the equilibrium condition for two such portfolios with identical risk characteristics for that goal gives one key equation to price a generic risky asset. But the same is true for the second goal giving a second pricing equation for the generic risky asset. Free trading across investors with different goals forces a "pair-wise" equilibrium as an asset can have only one price/return despite having different attributes (e.g., substitute or complement) for each goal. Adding more goals and goal-replicating assets just adds more "pair-wise" equilibria.

Assumptions: The following assumptions are required:

(i) Assume a one-period world

(ii) Investors/principals have a goal and delegate to a single agent to achieve that goal
(iii) Principals seek the highest goal relative expected risk-adjusted return for their goal
(iv) Agents take risk relative to the relative risk-free asset (either because the principal is underfunded,[19] is incapable of managing assets, or claim to have skill in outperforming the benchmark). Principals have limited relative risk budgets as they are uncertain about the skill of agents and can only observe past returns, not effort or skill. They extrapolate from past returns to forecast expected returns.
(v) Principals specify risk targets as shown in Section 3—with a specific absolute risk level[20] equal to the risk (or volatility) of the goal, and a relative risk budget for their agents (or TE(Target))
(vi) There are two classes of investors pursuing two different goals, G1 and G2. We assume a single representative investor for each goal. Initially, we assume that they are isolated and cannot trade, but then relax that assumption to ensure an equilibrium.
(vii) Both these investors have unique goal-replicating assets, G1 and G2, respectively
(viii) Assume F exists. Also, assume a generic risky asset I, that we are seeking to price and different from G1, G2, and F, exists.
(ix) In other words, assume that there is sufficient supply of all assets (and ignore supply-side issues)
(x) Assume agents can invest in just a single risky asset to create the delegated portfolio relative to the goal– either I or the second goal-replicating asset
(xi) Principals can combine risky assets, with F and a goal-replicating portfolio to create risk-adjusted portfolios, A
(xii) For both investors, the target correlation or $\rho(T,L)$ is identical (not the TE(Target) as that also depends on the volatility of the goal). This just simplifies the formulas, not the end result
(xiii) There are no transaction costs and unrestricted trading ensures equilibrium
(xiv) The investors have equal assets under management so that the initial simple model is free of the fraction of investors for each goal

Solution. Assumptions (i)–(iii) allow us to use the M-cube measure to derive the model. Thereafter, eight major steps are needed to establish the "pairwise" asset pricing equations.[21]

(1) Recall Equations (7) and (8) establish optimal allocations to risky, goal-replicating and risk-free (F) assets for investors maximizing goal-relative risk-adjusted returns, given target absolute and relative risk levels. At this stage, this is an optimal, but non-equilibrium allocation.
(2) First consider the G1 world in isolation. Use Equation (4) to establish the risk-adjusted return of portfolio A, ($E[r(A)_{I,\mathrm{G1}}]$), in the G1 world with I as risky asset, and goal-replicating asset G1.
(3) Similarly, establish the risk-adjusted return of portfolio A, ($E[r(A)_{\mathrm{G2,G1}}]$), in the G1 world with G2 as risky asset.
(4) This is the most important step and claim in this paper. If portfolio A has the same volatility as the goal (G1) and the same target correlation ($\rho(T,\mathrm{G1})$), then it cannot have two values based on which asset was used to create A. Therefore, set $E[r(A)_{I,\mathrm{G1}}] = E[r(A)_{\mathrm{G2,G1}}]$ to derive $E[r(I)_{\mathrm{G1}}]$ as function of G1, G2, and F.[22]
(5) Similarly, in the G2 world, follow Steps 1–3 to derive $E[r(A)_{I,\mathrm{G2}}]$ and $E[r(A)_{\mathrm{G1,G2}}]$. Once again, as in Step 4, set $E[r(A)_{I,\mathrm{G2}}] = E[r(A)_{\mathrm{G1,G2}}]$ to derive $E[r(I)_{\mathrm{G2}}]$ as function of G1, G2, and F.

(6) Now assume the two investors can trade freely. Asset I cannot have two different returns unless there is a market inefficiency. Therefore, set $E[r(I)_{\text{G1}}] = E[r(I)_{\text{G2}}]$ to express $E[r(I)]$ as function of $E[r(\text{G1})]$, $r(F)$, volatilities and correlations (assuming G2 is the goal). This is shown in Equation (9).

$$E[r(I) - r(F)] = \text{Z}_{I,\text{G1/G2}} \times E[r(\text{G1}) - r(F)] \quad (9)$$

where $\text{Z}_{I,\text{G1/G2}}$

$$= \frac{\sigma(I)}{\sigma(\text{G1})}\left[\left[\frac{\varphi(I,\text{G2})}{\varphi(\text{G1},\text{G2})}\right] + \frac{\text{Y}_{I,\text{G1,G2}}}{\text{X}_{I,\text{G1.G2}}}\rho(I,\text{G2}) - \frac{\text{Y}_{I,\text{G1,G2}}}{\text{X}_{I,\text{G1,G2}}}\rho(\text{G1},\text{G2}) \times \left[\frac{\varphi(I,\text{G2})}{\varphi(\text{G1},\text{G2})}\right]\right] \quad (10)$$

and where $\text{X}_{I,\text{G1,G2}}$

$$= \varphi(I,\text{G1}) - \rho(I,\text{G2}) \times \varphi(I,\text{G1}) + \rho(\text{G1},\text{G2})\varphi(I,\text{G2}) \quad (11)$$

$Y_{I,\text{G1,G2}}$

$$= \varphi(I,\text{G2}) - \rho(I,\text{G1}) \times \varphi(I,\text{G2}) + \rho(\text{G1},\text{G2}) \times \varphi(I,\text{G1}) \quad (12)$$

Zeta can be restated in terms of "a" and "l" using Equations (7) and (8) as follows:

$$\text{Z}_{I,\text{G1/G2}} = \frac{a^{\text{G1}}_{T/\text{G2}}}{a^{I}_{T/\text{G2}}} + \frac{\sigma(I)}{\sigma(\text{G1})} \times \left(\frac{Y_{\text{I,G1,G2}}}{X_{\text{I,G1,G2}}}\right) l^{\text{G2}}_{I/\text{G2}} \quad (13)$$

(7) Similarly, express $E[r(I)]$ as function of $E[r(\text{G2})]$, $r(F)$, volatilities, and correlations (assuming G1 is the goal) as in Equation (14).

$$E[r(I) - r(F)] = \text{Z}_{I,\text{G2/G1}} \times E[r(\text{G2}) - r(F)] \quad (14)$$

(8) F has an equilibrium value because of the unique role it plays in risk-adjusted portfolios and can be calculated as

$$r(F)_{\text{G1,G2}} = \frac{\left\{E[r(\text{G1}) - \frac{\sigma(\text{G1})}{\sigma(\text{G2})} \times (\frac{X_{\text{I,G1,G2}}}{Y_{\text{I,G1,G2}}}) \times E[r(\text{G2})]\right\}}{[1 - \frac{\sigma(\text{G1})}{\sigma(\text{G2})} \times (\frac{X_{\text{I,G1,G2}}}{Y_{\text{I,G1,G2}}})]} \quad (15)$$

(9) Additionally, there is a unique relationship between $E[r(\text{G1})]$ and $E[r(\text{G2})]$ as in Equation (16).

$$E[r(\text{G1}) - r(F)] = \frac{\sigma(\text{G1})}{\sigma(\text{G2})} \times \left(\frac{X_{\text{I,G1,G2}}}{Y_{\text{I,G1,G2}}}\right) \times E[r(\text{G2}) - r(F)] \quad (16)$$

Implications of GRAPM. There are many interesting implications from this approach and model.

(i) GRAPM incorporates the three facets into one model (an asset pricing model, asset allocation and M-cube as the risk-adjusted performance measure), while incorporating the realities of investing: multiple goals, delegation and precise, observable risk specification.
(ii) A generic asset I can be priced with just a goal-replicating asset G1 (Equation (9)) or G2 (Equation (14)). These assets, G1 and G2, can be observed as opposed to an unobserved absolute (M) or relative market portfolio (M').

(iii) There are potentially multiple equations to price a particular asset depending on the number of goal-replicating assets. This happens because of the pair-wise equilibria. This is not a shortcoming of the model, but rather a feature and is visualized in Figure 1. While CAPM and RAPM provide a single equation linked to equilibrium market portfolios (which include the generic risky asset) and goal-replicating portfolios, GRAPM provides multiple pair-wise equilibrium formulas based on the number of goal-replicating assets. With two assets, G1 and G2, I can be priced with G1 as the goal and G2 as risky, or vice versa. With three assets, I can be priced using six combinations of G1, G2, and G3. The three-goal case yields some additional intriguing equilibrium conditions that will be addressed in future research (Muralidhar, 2020).

(iv) Alternatively, if there are other conditions that prevent an equilibrium, one can expect that we can derive a range of expected returns for each asset—a more reasonable assumption than having a single global point estimate (assuming everyone has the same expectation). Interestingly, when one compares the forecasts of say consultants who advise institutional or retail clients, there can be wide dispersion in their forecasts and one explanation is that they are only looking at the forecast from the bias of their goals and not from a global equilibrium that Equations (9) and (14) capture. Furthermore, forecasts of expected returns have been poor both in terms of level and direction (Housel, 2015; The Economist, 2017) and hence potentially having a range of forecasts may be more valuable for practitioners.[23]

(v) F cannot take an arbitrary, unspecified value but rather as a very specific value as shown in Equation (15). The equilibria force this result.

(vi) $X_{I,G1,G2}$ (Equation (11)) and $Y_{I,G1,G2}$ (Equation (12)) are only functions of the correlation parameters. This is also the equilibrium condition between r(G1) and r(G2); namely, that in equilibrium, given the unique relationship of these goal-replicating assets, their values will also be determined endogenously in the market and cannot take any exogenous value.

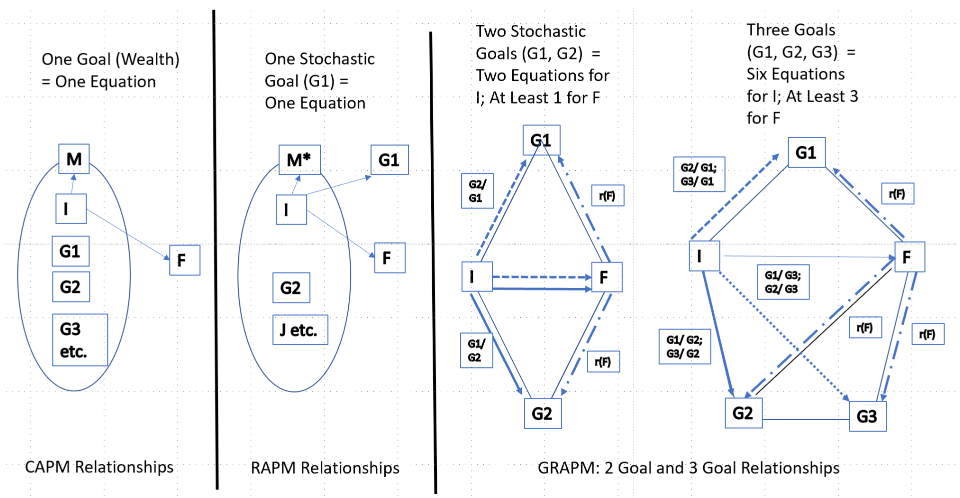

Figure 1 Asset pricing relationships for CAPM, RAPM, and GRAPM (with two and three goals).

(vii) The Zeta term is in the spirit of the beta term of MPT, and relative beta of RAPM. It is a "covariance" term with many additional correlation terms to capture the equilibrium conditions described above.[24] Zeta captures the value of an asset in not only hedging the goals, but also in serving as a risky asset (that potentially earns a higher return than the goal in order to reduce the underfunding). Coqueret *et al*. (2017) have recently argued for examining the use of equities (other than market-cap weighted indices) in hedging bond-like liabilities, effectively capturing the dual value of equities. This explains why Zeta is a unique function of asset allocation to risky assets and the goal-hedging asset. The first term in Equation (13) is the ratio of allocations to the two risky assets for each goal given the target risk, and the second term has the hedging component.

(viii) In short, the expected returns of an asset is impacted by all assets—whether they are substitutes or complements—much like one experiences in financial markets.

(ix) The final asset pricing model is independent of $\rho(T, L)$. As Figure 1 demonstrates, the correlations are the variables that ensure these pair-wise equilibria in large part. With more goal-replicating assets, more correlation terms will be important.

(x) Allocation recommendations are intuitive as shown in Equations (7) and (8) (and in Appendix III). Allocations to risky assets depend on the ratio of relative volatilities of the goal-replicating asset to the risky asset multiplied by a ratio of correlation terms that capture the risk tolerance of the investor (an easily identified and measured correlation term as opposed to MPT's "risk aversion" parameter). The allocation to the goal-replicating asset is solely a function of correlation terms and the risk budget. These have been discussed in Section 1.

(xi) GRAPM also shows why two pension funds like LACERA and NMPERA could have markedly different asset allocations, especially if their goal-replicating portfolios are different, but also because their target relative risk levels could be different. In MPT, the only difference in allocations is in the proportions allocated to M and F—not observed in practice.

(xii) In other words, the "risk aversion" parameter of GRAPM is either the stated $\rho(T, L)$, as in the NMPERA or LACERA example, or the implied $\rho(T, L)$ from measuring permitted tracking error. The volatility of the goal is also a key risk parameter and this is also a stated parameter in the IPS or observed parameter (*ex-post* volatility of the SAA). However, to gather this data for all investors across the globe is probably impossible making this a difficult model to test empirically despite its attractive features and new approach.

In summary, using the traditional absolute risk-free asset, F, and two goal-specific risk-free assets (say G1 and G2) allows one to triangulate to establish the return of any other asset. G1 and G2 individuals who are maximizing risk-adjusted returns subject to a clearly articulated risk budget will each have a unique demand for the risky asset, but the equilibrium conditions across all goals ensure that expected returns of the risky asset will stabilize at a level compatible with both goals. This equilibrium removes the need for non-observable facets of MPT like the "risk aversion" parameter (needed for optimal asset allocation decisions) or the "market portfolio" (needed for the asset pricing model and the asset allocation decision).

In a similar fashion, having heterogeneous investors (e.g., one focused on absolute wealth, one focused on retirement, and one focused on saving for a child's college education) who are supplied their goal-specific risk-free asset and seek to maximize the relative risk-adjusted performance of their portfolios is sufficient to solve out the asset pricing model. In a heterogeneous investors model, G1 individuals are one type of investor, and G2 individuals are a different type of individual and it is clear that risky assets can be priced based on either goal.[25]

6 Extensions/Shortcomings

In this section, we consider some extensions or shortcomings to the model because GRAPM was developed using very simple assumptions.

6.1 Extensions

The simple GRAPM modeled here ignores the weights of each type of investor in the economy and hence is independent of the weight terms.[26] In effect, each world can be seen as a different class of investors and the merging of the worlds essentially ensures that asset prices are determined by the interaction of these different classes of investors, but weighted by the proportion of individuals in each goal. If we include this assumption, Zeta becomes a function of the weights of each market (say w_{G1} and w_{G2}); more explicitly, $X_{I,G1,G2}$ and $Y_{I,G1,G2}$ in Equations (11) and (12) include w_{G1} and w_{G2}, but the allocation formulas are unchanged.[27]

The next simple extension is to assume that portfolio I is made up of multiple assets as opposed to it being a single risky (or alternatively, that an investor hires multiple agents). This just makes the model a bit more complicated, but because the correlations of a portfolio of assets to a goal-specific asset is nothing more than a weighted sum of each asset correlation to the goal-specific asset, the model is easily solved. Muralidhar (2001) demonstrates how this is achieved (in the context of hiring multiple agents) and further how assets that might have been considered valuable from a diversification perspective in an absolute return–risk world may be sub-optimal in a relative risk world.[28] If M is the portfolio of j risky assets, such that

$$r(M) = \sum_j w_j r(j), \quad \text{then}$$

$$\rho(M, \mathrm{G1}) = \frac{\sum_j w_j \rho(j, \mathrm{G1})\sigma(j)}{\sigma(M)} \tag{17}$$

In this multi-asset (or multi-agent) setting, the allocation to each risky asset j that is in this portfolio $= a * w_j$ and can be solved iteratively. This leads to potentially an interesting new research avenue as to how the optimal risky portfolio in G1 relates to the optimal risky portfolio in G2. This offers a new twist to the notion of diversification and also has some ability to explain why many pension funds globally experienced declines in funded status in the equity crises of 2000–2002 and 2008. In short, these pensions used mean–variance optimization in an absolute return–risk space to establish optimal portfolios, while ignoring that they had a goal that essentially resembled a long-duration bond asset. Hence, assets that looked attractive from a diversification perspective in MPT (e.g., equities) proved to be highly risky and even negatively correlated to the goal during crisis events. These assets might not have been included in a GRAPM portfolio for a limited relative risk budget. Coqueret *et al*. (2017) attempt to address this challenge. In a related vein, a more complex extension would be to consider a joint simultaneous optimization of all goals in line with Das *et al*. (2018) and see how that affects the choice of optimal risky portfolios. However, this approach will have to be complemented with the target relative risk for the combined multi-goal optimization, which may make this quite complex.

Third, it is possible to have different relative risk budgets for different goals as the asset pricing equations are independent of $\rho(T, L)$, while the asset allocations are impacted by this variable. Fourth, the model could be extended to a multi-period model as opposed to a single period model much like Merton (1973) does for CAPM.

6.2 *Shortcomings*

The biggest shortcoming, for those academically inclined, is the fact that this approach is not based on a utility function. That choice was explained earlier and could prove an interesting avenue for future research if an appropriate utility function could be specified. More broadly, like any economic model, GRAPM can be faulted for a number of the simplifying assumptions made to arrive at the result. First, the danger of pricing assets based on one or two goal-replicating assets that may be relatively illiquid could pose severe problems as opposed to CAPM where the pricing formula depends on a highly diversified market portfolio.[29] In a related vein, as with all these finance models, one of the important assumptions is that we require F as a key asset with zero volatility and zero correlation. These assumptions greatly simplify the calculations even though this asset does not truly exist. There is no easy way out of this conundrum except to use the method suggested in Black (1972), but then the formulas can get more complicated.

The next big assumption was that investors maximize goal-specific risk-adjusted performance—some may disagree with this starting assumption. However, the examples of LACERA and NMPERA were used to show that such an objective might not be unreasonable in a GBI setting. Moreover, even if target relative risk is not specified, permissible relative risk is extracted by way of revealed preference (by examining the historical tracking error of agents relative to the goal of the fund).

GRAPM, like MPT or RAPM, also requires forecasts of expected returns of the goal-specific relative risk-free assets and the multiple correlations and hence is subject to the same criticism as MPT/CAPM/RAPM. Housel (2015) highlights our inability to forecast these variables and the fact that they are dynamic may require more than a one-period model. GRAPM does provide additional equations to ensure that there are pair-wise equilibria, but there is no guarantee that these can be forecast or tested empirically. Furthermore, this model ignored all supply-side issues and waved them away. Interestingly, since M-cube gives demand equations for assets, even with multiple goals (and different weights on each goal), one could derive a demand–supply equilibrium if a supply curve is specified. Finally, this model assumed that free/friction-less trading between the G1 and G2 worlds will equalize the returns of assets that are risky in both worlds. There are potential challenges to this assumption that are ignored in this simplistic model.

7 Conclusions

GBI is slowly becoming the norm for investors and investors are challenged in achieving their goals because finance theory has not provided an integrated model that incorporates multiple stochastic goals and agency, while still providing investors with the three facets of investing—asset pricing, asset allocation, and risk-adjusted performance measures. Individuals save for a range of goals (e.g., retirement, a child's college expenses) and each has a unique set of cash flows. Researchers have argued for the creation of a new class of relative risk-free securities for each goal. Furthermore, investors delegate to agents and are keen to ensure that the agents are skillful and hence restrict the amount of relative risk they can take relative to the goal. As a result, institutional investors seek to maximize goal relative risk-adjusted returns and specify clear absolute volatility and relative risk targets.

This paper demonstrates that, with this assumption about investor behavior, two such goals/instruments and the traditional risk-free asset, all risky assets can be priced, without requiring a traditional utility function, risk aversion parameter, or market portfolio (none of which are easily identified in real life). Risky assets are easily priced using goal-replicating assets (which are risky for another goal). The asset pricing model is derived from the simple idea that a relatively risk-free asset for one goal is a risky asset for another, and hence these two assets, plus the absolute risk-free rate allow us to triangulate to establish returns for all other assets. In short, assuming such behavior on the part of investors creates a "pair-wise equilibrium." Adding more assets just creates a lattice of "pair-wise" equilibria. This approach also lends itself easily to an asset pricing model with heterogeneous investors and provides useful advice on asset allocation and risk-adjusted performance. It also offers new avenues for research as this is a new approach that is easily extended by relaxing some of the assumptions made to arrive at this model. However, the novelty of this approach is to derive a theoretical model based on empirical observation as to how investors actually behave and how finance science is being used to improve GBI. This might be the most important contribution of this normative model and hopefully help investors achieve their goals.

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Appendix I—Intuition behind the relative asset pricing model

Assume the unique objective function as in the standard CAPM with mean–variance preferences for individual S.

$$U_S(\sigma, E) = -\sigma^2 + 2k_S E_s \qquad \text{(A.1.1)}$$

The investor is interested in maximizing the mean–variance objective function (A.1.1) with respect to the relative wealth at the end of period 1, where assets (A) are divided by the goal (L).

$$\begin{aligned} R_1 - R_0 &= \frac{A_1}{L_1} - \frac{A_0}{L_0} \\ &= \frac{A_0}{L_0} * \frac{(1 + r(A_S))}{(1 + r(L_S))} - \frac{A_0}{L_0} \\ &= \frac{A_0}{L_0} \times \left[\frac{(1 + r(A_S))}{(1 + r(L_S))} - 1 \right] \qquad \text{(A.1.2)} \end{aligned}$$

For relatively small $r(A_S)$ and $r(L_S)$, Lauterbach and Reisman (2004) use the approximation that

$$\begin{aligned} &(1 + r(A_S)/(1 + r(L_S)) - 1 \\ &\quad \approx r(A_S) - r(L_S) \qquad \text{(A.1.3)} \end{aligned}$$

Since A_0 and L_0 are given, we can assume that you want to maximize $\{r(A_S) - r(L_S)\}$ subject to variance of $\{r(A_S) - r(L_S)\}$. This then translates into the following elements of the objective function

$$E_{S,A} = E[r(A_S) - r(L_S)] \quad \text{(A.1.4)}$$

$$\sigma_{S,A} = \sigma_{[r(AS)-r(LS)]} \quad \text{(A.1.5)}$$

$$k_S = \text{risk aversion of investor } S$$

Assume that the investor hedges against $r(L_S)$ by borrowing at $r(F)$ and simultaneously chooses the non-hedging portfolio to obtain $r(NL_S)$. Then, the asset portfolio return is expressed as

$$r(A_S) = r(L_S) + r(NL_S) - r(F) \quad \text{(A.1.6)}$$

Proceeding much like one would in a Sharpe (1964) model, one can solve for RAPM as

$$E[r(i)] - r(F) = \frac{\text{cov}[r(M') - r(L), r(i)]}{\text{var}[r(M') - r(L)]} \times E[r(M') - r(L)] \quad \text{(A.1.7)}$$

Where M' is the relative market portfolio on the relative mean–variance frontier and excludes L and includes F.

Appendix II—Derivation of *M*-Cube Risk-Adjusted Performance Measure

In an agency situation, the principal worries about the relative return of the portfolio (relative mean or $r(P) - r(L)$) created by the agent, where the agent's portfolio is P relative to the benchmark L. In this paradigm, the relative risk of P to L can be defined as below.

$$\text{Tracking Error}(P) = TE(P) = \sqrt{[\sigma^2(P) - 2 * \rho(P, L) * \sigma(P) * \sigma(L) + \sigma^2(L))]} \quad \text{(A.2.1)}$$

Where σ is the volatility and ρ is the correlation parameter.

An intelligent principal interested to examine the risk-adjusted performance of P, especially if they are evaluating multiple agents with the same benchmark, must lever/delever this portfolio with both F and L to have a specific target risk (as in Modigliani and Modigliani, 1997), and a specific relative risk as well. In other words, they can create a new asset portfolio A, which blends P, F, and L to calculate the risk-adjusted performance of P. If they do this for all agents, then they can compare the return of A of all agents to establish truly risk-adjusted performance (as all other aspects like volatility and correlation have been normalized).

Assume that individuals maximize the goal relative risk-adjusted return, $r(A) - r(L)$, subject to 3 constraints highlighted in Equations (A.2.2), (A.2.3), and (A.2.4).

$$r(A) = a \times r(P) + (1 - a - l) \times r(F) + l \times r(L) \quad \text{(A.2.2)}$$

$$\sigma(A) = \sigma(L) \quad \text{(A.2.3)}$$

$$\text{and} \quad \text{TE}(A) = \text{TE(Target)} \quad \text{(A.2.4)}$$

where a is the allocation to the risky portfolio, P, and l is the allocation to the goal-hedging portfolio, L, and the balance of the assets is invested in the traditional risk-free asset, F.

We do not constrain the sign of any of these parameters. Equations (A.2.2) and (A.2.3) are the same assumptions made by Modigliani and Modigliani (1997) and Graham and Harvey (1994, 1997). In an agency paradigm, Equation (A.2.4) is an additional constraint imposed because the principal is concerned about the skill of the agent and does not want to allow unconstrainted risk (Muralidhar, 2001). In other words, it is the level of relative risk the principal is comfortable taking, given its minimum confidence-in-skill threshold for a given time period (Ambarish and Seigel, 1996). Note that this problem can be restated for every level of TE(Target) and $\sigma(L)$, thereby giving a continuum of optimal allocations for various risk levels to create the relative equivalent of the mean–variance frontier.

From the constraint on tracking error (A.2.4), a unique target correlation between portfolio A and liability L, $\rho(T, L)$, is identified. Using the squares of equations on tracking error and Equation (A.2.4), and substituting for $\sigma(A) = \sigma(L)$ obtains

$$\rho(T, L) = 1 - \frac{TE(target)^2}{2 \times \sigma^2(L)} \quad \text{(A.2.5)}$$

which can then be used to solve for a, l, and *1-a-l*.

$$a = \frac{\sigma(L)}{\sigma(P)} \left[\sqrt{\frac{[1 - \rho(T, L)^2]}{[1 - \rho(P, L)^2]}} \right] \quad \text{(A.2.6)}$$

$$l = \rho(T, L) - \rho(P, L) \times \left[\sqrt{\frac{[1 - \rho(T, L)^2]}{[1 - \rho(P, L)^2]}} \right] \quad \text{(A.2.7)}$$

Notice that a (allocation to risky asset) is a function of relative standard deviations and the realized correlation relative to the target correlation. But what is most interesting is that the allocation to the various investment options is completely independent of returns, though one could argue that the TE(Target), and hence $\rho(T, L)$, is a function of the expected return per unit of risk, and this is where the expected return of A enters the allocation formula. The analysis recommends the optimal liability hedge given a risk budget (allocation to l), the amount of leverage or deleverage (*1-a-l*) and the optimal allocation to the risky portfolio (allocation to a).

Appendix III—demonstrating how *M*-cube provides a demand function for various assets assuming a range of target risk

Assume $\sigma(L) = 10.5\%$, $\sigma(P) = 10.5\%$, and $\rho(P, L) = 0.975$. Then a TE(Target) $= 1.5\%$, implies that $\rho(T, L) \approx 0.995$. Figure A.3.1. provides a range of tracking error targets around this TE(Target) by just varying $\rho(T, L)$. For low levels of relative risk (i.e., funded status or relative risk aversion is high), allocation to the liability hedge is high (dashed line) and declines thereafter as the allocation to risky assets (dotted line) rises. In this setting, the non-linearity is not as obvious as in the example in Muralidhar and Shin (2013) that examines likely allocations for a corporate pension fund. The double-lined arrow provides the target correlation ($\rho(T,L)$) for these TE(Target) levels. The allocation to F ranges from -1% to 0% for these settings but again are substantially different from zero in Muralidhar and Shin (2013).

Appendix IV—Deriving GRAPM

STEP 1: Use *M*-cube to establish optimal allocations to risky, goal-replicating, and risk-free assets

$$a^P_{T/L} = \frac{\sigma(L)}{\sigma(P)} \left[\frac{\varphi(T, L)}{\varphi(P, L)} \right], \quad \text{where}$$

$$\varphi(I, J) = \sqrt{[1 - \rho(I, J)^2]} \quad \text{(A.4.1)}$$

$$l^L_{T/L} = \rho(T, L) - \rho(P, L) \left[\frac{\varphi(T, L)}{\varphi(P, L)} \right] \quad \text{(A.4.2)}$$

STEP 2: Establish the risk-adjusted return of portfolio A in G1 world with/as risky asset ($E[r(A)_{\mathbf{I,G1}}]$)

$$E[r(A)_{I,\text{G1}}] = \left\{ \frac{\sigma(\text{G1})}{\sigma(I)} \left[\frac{\varphi(T, \text{G1})}{\varphi(I, \text{G1})} \right] \right\}$$

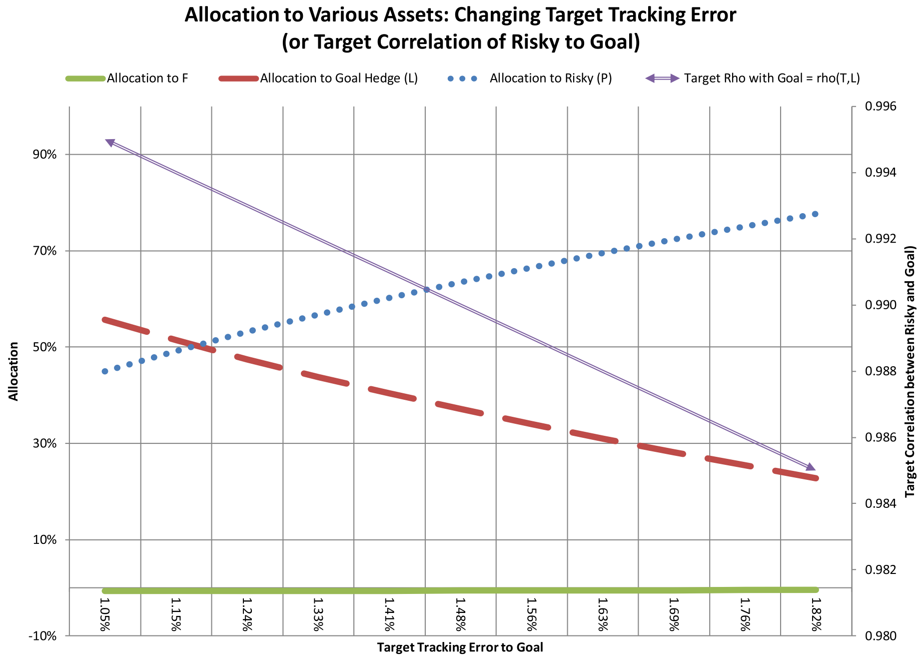

Figure A.3.1 Allocation to risky assets, goal-replicating asset and absolute risk-free asset for different target tracking errors.

$$\times E[r(I)] + \left\{\rho(T, \text{G1}) - \rho(I, \text{G1}) \times \left[\frac{\varphi(T, \text{G1})}{\varphi(I, \text{G1})}\right]\right\} \times E[r(\text{G1})] + \left(1 - \frac{\sigma(\text{G1})}{\sigma(I)}\left[\frac{\varphi(T, \text{G1})}{\varphi(I, \text{G1})}\right] - \left\{\rho(T, \text{G1}) - \rho(I, \text{G1}) \times \left[\frac{\varphi(T, \text{G1})}{\varphi(I, \text{G1})}\right]\right\}\right) \times r(F) \tag{A.4.3}$$

STEP 3: Establish the risk-adjusted return of portfolio A in G1 world with G2 as risky asset ($E[r(A)_{\mathbf{G2,G1}}]$)

$$E[r(A)_{\text{G2,G1}}] = \left\{\frac{\sigma(\text{G1})}{\sigma(\text{G2})}\left[\frac{\varphi(T, \text{G1})}{\varphi(\text{G2}, \text{G1})}\right]\right\} \times E[r(\text{G2})] + \left\{\rho(T, \text{G1}) - \rho(\text{G2}, \text{G1}) \times \left[\frac{\varphi(T, \text{G1})}{\varphi(\text{G2}, \text{G1})}\right]\right\} \times E[r(\text{G1})] + \left(1 - \frac{\sigma(\text{G1})}{\sigma(\text{G2})} \times \left[\frac{\varphi(T, \text{G1})}{\varphi(\text{G2}, \text{G1})}\right] - \left\{\rho(T, \text{G1}) - \rho(\text{G2}, \text{G1}) \times \left[\frac{\varphi(T, \text{G1})}{\varphi(\text{G2}, \text{G1})}\right]\right\}\right) \times r(F) \tag{A.4.4}$$

STEP 4: Set $E[r(A)_{\mathbf{I,G1}}] = E[r(A)_{\mathbf{G2,G1}}]$ to derive $E[r(I)_{\mathbf{G1}}]$ as function of G1, G2, and F

$$\text{If } a'_{\text{G1}} = \frac{\sigma(I)}{\sigma(\text{G2})}\left[\frac{\varphi(I, \text{G1})}{\varphi(\text{G2}, \text{G1})}\right] \tag{A.4.5}$$

$$\text{And } l'_{G1} = \frac{\sigma(I)}{\sigma(G1)} \times \rho(I, G1) - \frac{\sigma(I)}{\sigma(G1)} \times \rho(G2, G1) \times \left[\frac{\varphi(I, G1)}{\varphi(G2, G1)}\right] \quad (A.4.6)$$

Define $E[r(I)_{G1}]$ as the return of asset I in G1.

$$\text{Then } E[r(I)_{G1}] = a'_{G1} \times E[r(G2)] + l'_{G1} \times E[r(G1)] + (1 - a'_{G1} - l'_{G1}) \times r(F) \quad (A.4.7)$$

$$\text{Alternatively, } E[r(I)_{G1} - r(F)] = a'_{G1} \times E[r(G2) - r(F)] + l'_{G1} \times E[r(G1) - r(F)] \quad (A.4.8)$$

STEP 5: Set $E[r(A)_{I,G2}] = E[r(A)_{G1,G2}]$ to derive $E[r(I)_{G2}]$ as function of G1, G2, and F

$$\text{If } a''_{G2} = \frac{\sigma(I)}{\sigma(G1)} \left[\frac{\varphi(I, G2)}{\varphi(G1, G2)}\right] \quad (A.4.9)$$

$$\text{And } l''_{G2} = \frac{\sigma(I)}{\sigma(G2)} \times \rho(I, G2) - \frac{\sigma(I)}{\sigma(G2)} \times \rho(G1, G2) \times \left[\frac{\varphi(I, G2)}{\varphi(G1, G2)}\right] \quad (A.4.10)$$

$$\text{Then } E[r(I)_{G2}] = a''_{G2} \times E[r(G1)] + l''_{G2} \times E[r(G2)] + (1 - a''_{G2} - l''_{G2}) \times r(F) \quad (A.4.11)$$

$$\text{Or alternatively, } E[r(I)_{G2} - r(F)] = a''_{G2} \times E[r(G1) - r(F)] + l''_{G2} \times E[r(G2) - r(F)] \quad (A.4.12)$$

STEP 6: Set $E[r(I)_{G1}] = E[r(I)_{G2}]$ to derive $r(F)$ as a function of G1 and G2

If we define

$$X_{I,G1,G2} = \varphi(I, G1) - \rho(I, G2) \times \varphi(I, G1) + \rho(G1, G2) \times \varphi(I, G2) \quad (A.4.13)$$

$$Y_{I,G1,G2} = \varphi(I, G2) - \rho(I, G1) \times \varphi(I, G2) + \rho(G1, G2) \times \varphi(I, G1) \quad (A.4.14)$$

Then

$$r(F)_{G1,G2} = \frac{\left\{ E[r(G1) - \frac{\sigma(G1)}{\sigma(G2)} \times (\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}) \times E[r(G2)] \right\}}{[1 - \frac{\sigma(G1)}{\sigma(G2)} \times (\frac{X_{I,G1,G2}}{Y_{I,G1,G2}})]} \quad (A.4.15)$$

Also,

$$E[r(G1) - r(F)] = \frac{\sigma(I)}{\sigma(G1)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times E[r(G2) - r(F)] \quad (A.4.16)$$

STEP 7: Express $E[r(I)]$ as function of $E[r(G1)]$, $r(F)$, volatilities, and correlations

$$E[r(I) - r(F)] = Z_{I,G1/G2} \times E[r(G1) - r(F)] \quad (A.4.17)$$

where

$$Z_{I,G1/G2} = \frac{\sigma(I)}{\sigma(G1)} \left[\left[\frac{\varphi(I, G2)}{\varphi(G1, G2)}\right]\right.$$

$$+ \left(\frac{Y_{\mathrm{I,G1,G2}}}{X_{\mathrm{I,G1,G2}}} \right) \times \rho(I, \mathrm{G2})$$
$$- \left(\frac{Y_{\mathrm{I,G1,G2}}}{X_{\mathrm{I,G1,G2}}} \right) \times \rho(\mathrm{G1, G2})$$
$$\times \left[\frac{\varphi(I, \mathrm{G2})}{\varphi(\mathrm{G1, G2})} \right] \Bigg] \tag{A.4.18}$$

STEP 8: Express r(I) as function of r(G2), r(F), volatilities and correlations

$$E[r(I) - r(F)] = \mathrm{Z}_{I,\mathrm{G2/G1}} \times E[r(\mathrm{G2}) - r(F)] \tag{A.4.19}$$

Also,

$$\mathrm{Z}_{I,\mathrm{G2/G1}} = \left(\frac{Y_{\mathrm{I,G1,G2}}}{X_{\mathrm{I,G1,G2}}} \right) \times \left(\frac{\sigma(\mathrm{G2})}{\sigma(\mathrm{G1})} \right) \times \mathrm{Z}_{I,\mathrm{G1/G2}} \tag{A.4.20}$$

Notes

1 https://www.brainyquote.com/quotes/quotes/c/confucius140548.html?src=t_goals

2 In the interests of brevity, the survey focuses on key papers that are relevant to this discussion and potentially ignores related contributions.

3 Muralidhar (2019b) goes further as to argue that the assumption that "investors maximize the expected utility of wealth" be replaced with "investors delegate to maximize relative risk-adjusted returns."

4 In some cases these are identical, which is what we will assume for this paper. In other cases, they can be different and this just makes the problem a bit more complex.

5 In chronological order for simplicity.

6 In Merton (1973), L hedged the stochastic opportunity set.

7 Assuming that the investor cannot contribute/save more and because of the nature of the contract or poor regulation, bears no personal risk if the funded status continues to decline. This is not an unreasonable assumption and is observed in US-defined benefit pension funds.

8 In a deterministic world, the liability should grow at the risk-free rate to prevent arbitrage opportunities.

9 Notice further that M-square can only be used for all benchmarks other than F.

10 Which is an exogenous relative risk target value specified by the principal and derived from TE(Target) once $\sigma(L)$ is known.

11 The differential Sharpe normalizes for tracking error, but Muralidhar (2001) shows how two delegated portfolios with identical tracking errors can imply different confidence in skill, given their correlation and volatility.

12 These papers, for simplicity, assume that the date of death is known.

13 Muralidhar *et al.* (2016a, 2016b) discuss generic inflation indices to which BFFS could be linked. SeLFIES are linked to a very specific index—per capita consumption—so that the standard-of-living of retirees is protected.

14 Interestingly, an instrument that has been proposed with a natural supplier, GDP-linked bonds, has not had much success getting launched because there is possibly no natural goal to ensure sufficient demand. See Shiller (2012).

15 There is an extensive literature on heterogeneous investors (e.g., with different risk aversions, portfolio constraints) and asset pricing that was not covered in Table 1.

16 http://www.nmpera.org/assets/uploads/downloads/RIO/RFP/RFP-NO.-NM-INV-001-FY19-Total-Fund-Overlay-Services.pdf

17 Appendix III shows a continuum of optimal allocations to the three key assets, F, L and M', for various TE Target levels providing a demand function for the key assets.

18 I thank Prof. Campbell Harvey for this point.

19 In just the retirement world, Social Security systems, corporate and public defined benefit pension plans are underfunded and there is evidence that even individuals are not saving enough in their defined contribution plans (Mitchell, 2017). Hence, this is a very reasonable assumption.

20 See also Jorion (2003).

21 The detailed solution is provided in Muralidhar (2016a, 2016b) and summarized in Appendix IV.

22 Some may argue that $E[r(A)_{I,\mathrm{G1}}] = E[r(A)_{\mathrm{G2,G1}}]$ is not guaranteed when the two portfolios have the same absolute and relative risk, even with an identical correlation to the goal. One possibility suggested by a reader is that if $r(I)$, $r(\mathrm{G2})$, $r(\mathrm{G1})$ have the same variance, but are independent, then this may not be true. However, if we plug in $\sigma(I) = \sigma(\mathrm{G2}) = \sigma(\mathrm{G1})$ and if $\rho(\mathrm{G2, G1}) = \rho(I, G1) = \rho(\mathrm{G2}, I) = 0$, then it results

in $r(\mathrm{G2}) = r(\mathrm{G1})$, which violates our assumption (vi) that goal-replicating assets are unique, and G2 cannot be risky. As a result, this is more than likely an acceptable assumption. Additionally, note that $a^{I}_{T/\mathrm{G1}} \neq a^{\mathrm{G2}}_{T/\mathrm{G1}}$ and similarly, for the allocation to G1, because I and G2 are assumed to be distinct assets. Thanks to Prof. Kazuhiko Ohashi for this insight. Alternatively, if one portfolio has a higher return for the same target volatility and tracking error, the investor will never hold the second portfolio (asset), thereby driving demand to zero.

23 Thanks to Prof. Merton for stimulating this insight.

24 Thanks to Prof. Dimitri Vayanos for this suggestion.

25 This model is much simpler than models of investors with heterogeneous expectations/information as those models typically need stylized utility functions (CARA) and assumptions about complete markets.

26 Thanks to Prof. Merton for stimulating this insight.

27 If w_{G1} is the weight of G1 and w_{G2} is the weight of G2 (such that the sum is 1), $X = w_{\mathrm{G1}}\varphi(I, \mathrm{G1}) - w_{\mathrm{G2}}\rho(I, \mathrm{G2}) \times \varphi(I, \mathrm{G1}) + w_{\mathrm{G2}}\rho(\mathrm{G1}, \mathrm{G2}) \times \varphi(I, \mathrm{G2})$ and $Y = w_{\mathrm{G2}}\varphi(I, \mathrm{G2}) - w_{\mathrm{G1}}\rho(I, \mathrm{G1}) \times \varphi(I, \mathrm{G2}) + w_{\mathrm{G1}}\rho(\mathrm{G1}, \mathrm{G2}) \times \varphi(I, \mathrm{G1})$

28 Muralidhar (2001) goes further to demonstrate how this approach is superior to naïve maximization of differential Sharpe ratios (also known as Information ratios). This follows because the M-cube measure of risk-adjusted performance, unlike the Sharpe ratio or the M-square measure of Modigliani and Modigliani (1997), is the only measure that ranks portfolios consistent with a ranking based on confidence in skill.

29 Thanks to Prof. Merton for stimulating this insight.

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