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ATTRIBUTION OF EX-POST REALIZED SHARPE RATIO TO THE PREDICTABILITY OF THE EX-ANTE FORECAST RETURN AND RISK

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We propose to use an attribution formula that enables the ex-post realized Sharpe ratio to be decomposed into realized market conditions, ex-ante predictability of the returns, risk magnitude, and risk factors. We compare the predictability of the ex-ante return and ex-ante risk directly, quantitatively identifying the main source of the reduction of the Sharpe ratio using the attribution. Furthermore, we use excess Sharpe ratio attribution analysis to simultaneously evaluate the qualities of the portfolio and benchmark. We additionally provide numerical examples of the attributions using sector indices.

1 Introduction

Improving the Sharpe ratio is an important goal of sponsors and portfolio managers. This ratio is based on portfolio performance under an ex-post realized return distribution. However, investors construct portfolios based on ex-ante forecast return distributions for investable assets at the beginning of an investment period by using the mean–variance approach. As such, it is natural that the ex-ante forecast return distribution differs from the ex-post realized future return distributions. Investors recognize that attribution analysis for past investments is important for their future investments. Therefore, we need to attribute the Sharpe ratio to the ex-ante forecast return distribution after the investment period.

In this study, we consider a one-time investment in n risky assets, where the expected returns are distributed according to an n-dimensional normal distribution. The investment maximizes the Sharpe ratio using the mean–variance approach.

First, we define the *predictability*, which is a measure of the ex-ante forecast return distribution that ranges from -1 to 1. The realized Sharpe ratio increases with predictability, which is distinct from realized market conditions. Second, we decompose the realized Sharpe ratio into market conditions, predictability of the returns, and risk

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predictability, which we further decompose into the predictability of the risk magnitude and risk factors.

For evaluating the forecast, Grinold and Kahn (1999) introduce the *information coefficient* (IC), defining it as the correlation of each forecast with its actual outcome. If the IC is the same for all forecasts, Grinold and Kahn (1999) propose the *fundamental law of active management* to show the information ratio is the product of the IC and breadth of the square root of the strategy. The breadth is defined as the number of independent forecasts that generate exceptional returns per year for an investor. They then analyze the relationship between the risk-adjusted performance measurement and measurements during the investment process. Even if an investor has an ex-ante forecast return that is close to the ex-post one, her or his portfolio not necessarily achieves a high Sharpe ratio. An ex-ante forecast risk is also needed, which has an important role in portfolio construction processes using the mean–variance approach.

Regarding Sharpe ratio attribution analysis, Steiner (2011) decomposes the Sharpe ratio of a portfolio into risk weights, diversification effects, and asset Sharpe ratios. The risk weight is defined as $w_i \rho_{i,p} \sigma_i / \sigma_p$, where w_i is the weight of the ith asset in portfolio p, $\rho_{i,p}$ the asset covariance with the overall portfolio, σ_i the volatility of the asset, and σ_p the volatility of the portfolio. However, ex-ante forecast returns and risk lead to the weight of ith asset of the portfolio in the mean–variance approach. It is important to analyze the sources of the input that leads the portfolio. As such, we need a Sharpe ratio decomposition that directly uses the forecast returns and risk instead of weights.

The first contribution of this paper is proposing an ex-post realized Sharpe ratio attribution to ex-ante forecast returns and risk. The second contribution is providing the formula for the attribution. To this end, we decompose the risk predictability into the magnitude of the risk and risk factors. This attribution formula enables us to compare return and risk predictabilities, in a quantitative way. The third contribution is identifying the excess Sharpe ratio attribution against a given benchmark using the benchmark portfolio's predictability of the implied returns forecast. As a result, we find the relative strength of the optimal portfolio against the benchmark, as well as the absolute strengths of the portfolio and benchmark. Consequently, our attribution analysis can be applied to diverse situations to derive rich information.

We here illustrate a simple example of the Sharpe ratio attribution, which we discuss details in the subsequent sections. We assume three investable assets: A, B, and C. We assume Asset A has ex-post returns of $r_A = 6\%$, Asset B of $r_B = 4\%$, and Asset C of $r_C = 1\%$. Regarding their ex-post risks of returns, the three assets have the ex-post standard deviations $\sigma_A = 8\%$, $\sigma_B = 8\%$, and $\sigma_C = 2\%$, the returns of Assets A and B have the correlation $\rho_{AB} = 0.6$, the returns of Assets A and C are independent, and the returns of Assets B and C are also independent. Given the above return distribution, we construct the optimal portfolio, which is 25% long on Asset A, 3% long on Asset B, and 72% long on Asset C. The optimal portfolio has a maximum Sharpe ratio of 0.90. Assume an investor who has a perfect ex-ante return forecast and an imperfect ex-ante risk forecast. The investor forecasts that Asset A has the ex-ante return of $r_A = 6\%$, Asset B of $r_B = 4\%$, and Asset C of $r_c = 1\%$. Additionally, the investor forecasts the ex-ante standard deviations of $\sigma_A = 10\%$, $\sigma_B = 6\%$, and $\sigma_C = 2\%$ while pairs of returns are independent. However, the investor must construct the optimal portfolio from her or his ex-ante forecast return distribution using the mean–variance approach as 14% long on Asset A, 26% long on Asset B, and 59% long on Asset C. This optimal portfolio achieves

a Sharpe ratio of 0.79 under the ex-post realized return distribution. The optimal portfolio delivers a Sharpe ratio 87% of the maximum Sharpe ratio of the optimal portfolio from the ex-post realized distribution. We first decompose the realized Sharpe ratio of the optimal portfolio (0.79) into the product of the realized market condition of 0.90, which is the achievable maximum Sharpe ratio by an investor and into the predictability of the ex-ante forecast for 0.87 in this case, based on Shimizu (2017). Additionally, we decompose the predictability of the ex-ante forecast into a predictability of the ex-ante return forecast of 1.0 and into a predictability of the ex-ante risk forecast of 0.87. It is natural that the predictability of the ex-ante return forecast is 1.0 because the investor has perfect return forecasts. Furthermore, we decompose the risk predictability of 0.87 into a predictability of the magnitude of the risk of almost 1.0 and into a predictability of the risk factors of 0.82. From the above example, the realized Sharpe ratio reduction is caused by the weak risk predictability despite the perfect return predictability.

This remainder of article is organized as follows. In Section 2, we review the formulation of our framework in accordance with Shimizu (2017). We first calculate the Sharpe ratio for which the optimal portfolio, based on the ex-ante forecast returns and risk using mean–variance approach, achieves under the ex-post realized returns and risk. To this end, we investigate what returns under the given risk lead the optimal portfolio, which is identical to the optimal portfolio based on the given returns and another given risk, using the mean–variance approach (Theorem 1). Consequently, we determine the formula that expresses the Sharpe ratio that the optimal portfolio, based on the ex-ante forecast returns and risk, achieves under ex-post realized returns and risk. From these results, in Section 3, we determine that the ex-post realized Sharpe ratio is attributed to the realized market conditions, predictability of returns, predictability of risk magnitude, and predictability of risk factors. We provide numerical examples of the attribution analysis using an investment of 10 weekly S&P sector total return indices in Section 4. In Section 5, we discuss the excess Sharpe ratio attribution against a given benchmark. Assuming the implied return forecast of the benchmark portfolio, we decompose the excess Sharpe ratio into the effect of the predictability of the return and that of the predictability of the risk. Additionally, we give numerical examples for the excess Sharpe ratio attribution analysis, assuming that the optimal portfolio of the investor is the equal weighted portfolio and the benchmark portfolio is the market weight one. We thus obtain rich information from the attribution analysis. Finally, we discuss the implications for practical investment actions and the need for further research in Section 6. Appendixes showing the detailed derivations are also provided.

2 Formulation

We here present the formulation of our framework in accordance with Shimizu (2017) by considering a one-period investment in a nonleveraged long/short portfolio. We have n risky assets and construct the portfolio using the mean–variance approach without leverage to maximize the expected Sharpe ratio. For simplicity, we assume a constituent with infinitely divisible units in the portfolio and trade without friction. "Returns" refer to the excess returns with respect to the risk-free one. We assume the returns of the n risky assets behave according to an n-dimensional normal distribution, $N(r, \Sigma)$, where r is an n-dimensional vector of the returns of risky assets and Σ is an $n \times n$ covariance matrix of the returns. We also assume the covariance matrix of the n-dimensional normal distribution has a full rank.

Suppose that an investor forecasts that the n risky assets achieve the n-dimensional return vector r_F and the returns have an $n \times n$ covariance matrix Σ_F in the future investment period. This means the investor takes the ex-ante forecast return distribution $N(r_F, \Sigma_F)$ as an n-dimensional normal distribution at the beginning of the investment period, where r_F is an n-dimensional vector of the ex-ante forecast returns of risky assets and Σ_F the $n \times n$ covariance matrix of the ex-ante forecast risk of returns. Additionally, assume that the n risky assets achieve the n-dimensional return vector r_R and returns have the $n \times n$ covariance matrix Σ_R during the period. This means the n risky assets behave according to an ex-post realized return distribution $N(r_R, \Sigma_R)$ as a n-dimensional normal distribution during the period, where r_R is an n-dimensional vector of the ex-post realized returns of the risky assets and Σ_R is the $n \times n$ covariance matrix of the ex-post realized risk of the returns. While, we do not have perfect information about the future, it is natural that r_F and Σ_F are not identical to r_R and Σ_R, respectively.

From the assumption of the ranks of covariance matrices Σ_F and Σ_R, the properties of linear algebra means we can find the orthonormal $n \times n$ matrices K_F and K_R such that they diagonalize Σ_F and Σ_R respectively based on Shores (2018):

$$K_F^{-1}\Sigma_F K_F = \sigma_F^2, \quad K_R^{-1}\Sigma_R K_R = \sigma_R^2, \quad (1)$$

where σ_F and σ_R are the $n \times n$ diagonal matrices:

$$\sigma_F \equiv \begin{pmatrix} \sigma_F(1) & & 0 \\ & \ddots & \\ 0 & & \sigma_F(n) \end{pmatrix}, \qquad \sigma_R \equiv \begin{pmatrix} \sigma_R(1) & & 0 \\ & \ddots & \\ 0 & & \sigma_R(n) \end{pmatrix}, \quad (2)$$

$\sigma_F(i)$ and $\sigma_R(i)$ $i = 1, \ldots, n$ are the diagonal elements of the matrices, and X^{-1} denotes the inverse matrix of X. K_F is the orthonormal matrix and contains n column vectors $k_F(i), i = 1, \ldots, n$ that are n-dimensional and the ex-ante forecast risk factors. Similarly, K_R is the orthonormal matrix and contains n column vectors $k_R(i)$, $i = 1, \ldots, n$ that are n-dimensional and the ex-post realized risk factors. $\sigma_F(i)$ expresses the ex-ante forecast risk of the i-th ex-ante forecast risk factor $k_F(i)$ and σ_F is the diagonal matrix of the ex-ante forecast risk. Similarly, $\sigma_R(i)$ expresses the ex-post realized risk of the i-th ex-post realized risk factor $k_R(i)$ and σ_R is the diagonal matrix of the ex-post realized risk.

Let $w = (w_1, \ldots, w_n)'$ be an n-dimensional vector of the portfolio weights of the risky assets, where x' denotes the transposition of x. For a given return distribution $N(r, \Sigma)$, we find the weight vector w_0 of the optimal portfolio as follows, which achieves the maximum Sharpe ratio if $e'\Sigma^{-1}r$ is positive, where $e \equiv (1, \ldots, 1)'$ (see Appendix 1):

$$w_0 = \frac{\Sigma^{-1}r}{e'\Sigma^{-1}r}. \quad (3)$$

From (3), n-dimensional vector of the portfolio weights of the risky assets $w_x = \Sigma_x^{-1}r_x / e'\Sigma_x^{-1}r_x$ is the optimal portfolio under return distribution $N(r_x, \Sigma_x)$ or the optimal portfolio, which is based on return distribution $N(r_x, \Sigma_x)$, where r_x is the n-dimensional return vector and $\sum_x$ the $n \times n$ covariance matrix of the returns.

For a discussion on the relationship between an optimal portfolio under ex-ante forecast distribution $N(r_F, \Sigma_F)$ and the optimal portfolio under ex-post realized distribution $N(r_R, \Sigma_R)$, we introduce an $n \times n$ matrix Φ and call it risk-basis transformation matrix:

$$\Phi \equiv \sigma_R K_R^{-1} K_F \sigma_F^{-1}, \quad (4)$$

where Φ has a function of coordinate conversion by term $K_R^{-1}K_F$, which transforms a vector expressed in the space that spans ex-ante forecast

risk factor vectors $k_F(1), \ldots, k_F(n)$ to a vector expressed in the space that spans ex-post realized risk factor vectors $k_R(1), \ldots, k_R(n)$. Additionally, Φ converts given values to other values by σ_F^{-1} and σ_R, where the former values per the ex-ante forecast risks are identical to the latter values per the ex-post realized risks. Therefore, we call the n-dimensional vector $K_R\sigma_R\Phi\sigma_F^{-1}K_F^{-1}r_F$ "the forecast return vector transformed by the risk factors." Note that $K_F^{-1}r_F$ is the ex-ante forecast factor return vector and $\sigma_F^{-1}K_F^{-1}r_F$ is the ex-ante forecast factor Sharpe ratio vector. From (3), we obtain optimal portfolio $w_F = \Sigma_F^{-1}r_F/e'\Sigma_F^{-1}r_F$, which achieves the maximum Sharpe ratio under the ex-ante forecast return distribution $N(r_F, \Sigma_F)$. Therefore, we posit Theorem 1 (see Appendix 2).

Theorem 1:

Optimal portfolio w_F, which is based on ex-ante forecast return distribution $N(r_F, \Sigma_F)$, is identical to optimal portfolio w_1, which based on return distribution $N(K_R\sigma_R\Phi\sigma_F^{-1}K_F^{-1}r_F, \Sigma_R)$ and whose parameters are the forecast return vector transformed by risk factors $K_R\sigma_R\Phi\sigma_F^{-1}K_F^{-1}r_F$ and ex-post realized covariance matrix Σ_R.

We denote $SR(r_F, \Sigma_F \mid r_R, \Sigma_R)$ as the ex-post realized Sharpe ratio that optimal portfolio from ex-ante forecast return distribution $N(r_F, \Sigma_F)$ achieves under the ex-post realized return distribution $N(r_R, \Sigma_R)$. From Theorem 1, $SR(r_F, \Sigma_F \mid r_R, \Sigma_R)$ is identical to the Sharpe ratio that the optimal portfolio under return distribution $N(K_R\sigma_R\Phi\sigma_F^{-1}K_F^{-1}r_F, \Sigma_R)$ achieves under return distribution $N(r_R, \Sigma_R)$. Then, we have the Property 1 (see Appendix 3).

Property 1:

$$SR(r_F, \Sigma_F \mid r_R, \Sigma_R) = \|\sigma_R^{-1}K_R^{-1}r_R\| \left(\frac{\Phi\sigma_F^{-1}K_F^{-1}r_F}{\|\Phi\sigma_F^{-1}K_F^{-1}r_F\|}, g \right), \quad (5)$$

where

$$\Phi \equiv \sigma_R K_R^{-1} K_F \sigma_F^{-1}, \quad g \equiv \frac{\sigma_R^{-1}K_R^{-1}r_R}{\|\sigma_R^{-1}K_R^{-1}r_R\|}, \quad (6)$$

where (x, y) is the inner product of vectors x and y and $\|x\| = \sqrt{(x, x)}$ the norm of x.

Note that $K_R^{-1}r_R$ is the ex-post realized factor return vector and $\sigma_R^{-1}K_R^{-1}r_R$ the ex-post realized factor Sharpe ratio vector. Therefore, quantity $\|\sigma_R^{-1}K_R^{-1}r_R\|$ can be seen as a magnitude of the ex-post realized factor Sharpe ratios. Indeed, it represents the absolute value of the market condition over the investment period. Additionally, $g \equiv \sigma_R^{-1}K_R^{-1}r_R/\|\sigma_R^{-1}K_R^{-1}r_R\|$ is the normalized ex-post realized factor Sharpe ratio vector. Conversely, $\sigma_F^{-1}K_F^{-1}r_F$ is the ex-ante forecast factor Sharpe ratio vector. Therefore, $\Phi\sigma_F^{-1}K_F^{-1}r_F$ is the transformed ex-ante forecast factor Sharpe ratio vector by Φ.

From (5), the ex-post realized Sharpe ratio $SR(r_F, \Sigma_F \mid r_R, \Sigma_R)$ that the optimal portfolio from the ex-ante forecast return distribution achieves is the product of the magnitude of ex-post realized factor Sharpe ratios and the inner product. Additionally, both vectors in the inner product are normalized, so the value of the inner product is a number between -1 and 1. Therefore, the ex-post realized Sharpe ratio that the optimal portfolio from the ex-ante forecast return distribution achieves increases with the value of the inner product. It is natural to assume that an investor achieves a higher Sharpe ratio when a forecast has higher predictability. This means that the inner product is a measure of the predictability of the ex-ante forecast return and risk and also that the maximum realized Sharpe ratio is $\sigma_R^{-1}K_R^{-1}r_R$. The vectors in the inner product are both normalized and $\Phi\sigma_F^{-1}K_F^{-1}r_F/\|\Phi\sigma_F^{-1}K_F^{-1}r_F\|$ does not include information on the ex-post realized return.

Therefore, the inner product does not include information about the absolute value of the market Sharpe ratio over the investment period. Therefore, the inner product evaluates the predictability of the ex-ante forecast separately from the market conditions over the period.

3 Attribution to predictability

Recall that inner product $\left(\frac{\Phi\sigma_F^{-1}K_F^{-1}r_F}{\|\Phi\sigma_F^{-1}K_F^{-1}r_F\|}, g\right)$ is a measure of the predictability of the ex-ante forecast returns and risk. We express the inner product by $P(r_F, \sigma_F, K_F)$, where r_F represents the ex-ante forecast returns, σ_F are the ex-ante forecast risks, and K_F the ex-ante forecast risk factors:

$$P(r_F, \sigma_F, K_F) \equiv \left(\frac{\Phi\sigma_F^{-1}K_F^{-1}r_F}{\|\Phi\sigma_F^{-1}K_F^{-1}r_F\|}, g\right). \quad (7)$$

Indeed, $P(r_F, \sigma_F, K_F)$ measures the predictability of the overall ex-ante forecast return and risk. Substituting σ_F and K_F with σ_R and K_R, respectively, into (7), we get:

$$\begin{aligned} P(r_F, \sigma_R, K_R) &= \left(\frac{\Phi\sigma_R^{-1}K_R^{-1}r_F}{\|\Phi\sigma_R^{-1}K_R^{-1}r_F\|}, g\right) \\ &= \left(\frac{\sigma_R^{-1}K_R^{-1}r_F}{\|\sigma_R^{-1}K_R^{-1}r_F\|}, g\right). \end{aligned} \quad (8)$$

Note that $\Phi = \mathrm{I}$ when $\sigma_F = \sigma_R$ and $K_F = K_R$, where I is the $n \times n$ identity matrix.

$P(r_F, \sigma_R, K_R)$ is the predictability of a forecast, as long as the risk forecast is perfect, that is, $\Sigma_F = \Sigma_R$. Indeed, the forecast of the risk magnitude is perfect, $\sigma_F = \sigma_R$, and so is that of the risk factors, $K_F = K_R$. In other words, it is the predictability when the ex-ante forecast returns is not identical to the ex-post realized returns. In a similar way, we define the predictability of the ex-ante forecast risk, which is not identical to the ex- post realized risk:

$$P(r_R, \sigma_F, K_F) = \left(\frac{\Phi\sigma_F^{-1}K_F^{-1}r_R}{\|\Phi\sigma_F^{-1}K_F^{-1}r_R\|}, g\right). \quad (9)$$

$P(r_R, \sigma_F, K_F)$ is the predictability of a forecast as long as the return forecast is perfect, $r_F = r_R$. Additionally, we define the predictability of the ex-ante forecast magnitude of the risk, which is not identical to the ex-post realized magnitude of the risk:

$$P(r_R, \sigma_F, K_R) = \left(\frac{\sigma_R\sigma_F^{-2}K_R^{-1}r_R}{\|\sigma_R\sigma_F^{-2}K_R^{-1}r_R\|}, g\right), \quad (10)$$

and the predictability of the ex-ante forecast risk factors, which is not identical to the ex-post realized risk factors:

$$P(r_R, \sigma_R, K_F) = \left(\frac{\sigma_R K_R^{-1}K_F\sigma_R^{-2}K_F^{-1}r_R}{\|\sigma_R K_R^{-1}K_F\sigma_R^{-2}K_F^{-1}r_R\|}, g\right). \quad (11)$$

Again, the vectors in inner products (8), (9), (10), and (11) are normalized, so their values are numbers between -1 and 1. Then, we find that value

$$1 - P(r_F, \sigma_F, K_F) \quad (12)$$

shows the level of reduction of the Sharpe ratio that occurs because the ex-ante forecast distribution is not identical to the ex-post realized one. When $1 - P(r_F, \sigma_F, K_F)$ is 1, there is no reduction. Indeed, the ex-ante forecast distribution is the perfect forecast. Similarly, $1 - P(r_F, \sigma_R, K_R)$ shows the level of reduction of the Sharpe ratio when the ex-ante forecast returns are not identical to the ex-post realized returns. $1 - P(r_R, \sigma_F, K_F)$ shows the level of reduction by the ex-ante forecast risk is not identical to the ex-post risk one. The reduction by overall ex-ante forecast returns and risk is not necessarily equal to the sum of the reduction by ex-ante forecast returns and by

ex-ante forecast risk because there is some synergy effect between ex-ante forecast returns and ex-ante forecast risk.

We define a quantity *DTY* expressing the synergy effect:

$$\begin{aligned} DTY \equiv & (1 - P(r_F, \sigma_F, K_F)) \\ & - \{(1 - P(r_F, \sigma_R, K_R)) \\ & + (1 - P(r_R, \sigma_F, K_F))\}. \end{aligned} \quad (13)$$

We call *DTY* duplicate term *Y*. Therefore, we have

$$\begin{aligned} & 1 - P(r_F, \sigma_F, K_F) \\ & \quad = (1 - P(r_F, \sigma_R, K_R)) \\ & \qquad + (1 - P(r_R, \sigma_F, K_F)) + DTY, \end{aligned} \quad (14)$$

where *DTY* is duplicate term *Y*. From (14), the level of the reduction of overall ex-ante forecast returns and risk attributes to the level of the reduction caused by the fact that the ex-ante forecast returns is not identical to the ex-post realized returns, and the level of reduction caused by the fact that the ex-ante forecast risk is not identical to the ex-post realized risk, and duplicate term *Y*.

Similarly, the level of reduction of overall ex-ante forecast risk $1 - P(r_R, \sigma_F, K_F)$ attributes the three components, while $1 - P(r_R, \sigma_F, K_R)$ shows the level of reduction by the ex-ante forecast risk magnitude is not identical to the ex-post realized risk magnitude and $1 - P(r_R, \sigma_R, K_F)$ shows the level of reduction by the ex-ante forecast risk factors is not identical to the ex-post realized risk factors. The reduction by the overall ex-ante forecast risk is not necessarily equal to the sum of the reduction by the ex-ante forecast risk magnitude and the reduction by the ex-ante forecast risk factors because there is some synergy effect between the ex-ante forecast magnitude of risk and ex-ante forecast risk factors.

We define a quantity *DTX* expressing the synergy effect:

$$\begin{aligned} DTX \equiv & (1 - P(r_R, \sigma_F, K_F)) \\ & - \{(1 - P(r_R, \sigma_F, K_R)) \\ & + (1 - P(r_R, \sigma_R, K_F))\}. \end{aligned} \quad (15)$$

We call *DTX* duplicate term X. Therefore, we have

$$\begin{aligned} & 1 - P(r_R, \sigma_F, K_F) \\ & \quad = (1 - P(r_R, \sigma_F, K_R)) \\ & \qquad + (1 - P(r_R, \sigma_R, K_F)) + DTX, \end{aligned} \quad (16)$$

where *DTX* is duplicate term X. Note that $(1 - P(r_R, \sigma_F, K_F))$ is the level of reduction caused by the ex-ante forecast risk not being identical to the ex-post realized risk. From (16), we calculate the level of reduction attributed to the level of reduction caused by the fact that the ex-ante forecast risk magnitude is not identical to the ex-post realized risk magnitude, the level of reduction caused by the fact that the ex-ante forecast risk factors are not identical to the ex-post realized risk factors, and duplicate term X. From (5), (14), and (16), we derive Property 2.

Property 2:

The ex-post realized Sharpe ratio is attributed to six components. One is the absolute level of the ex-post realized Sharpe ratio, $\|\sigma_R^{-1}, K_R^{-1}, r_R\|$, which represents the market conditions over the investment period. The next three components are the predictabilities of the ex-ante forecast return, $P(r_F, \sigma_R, K_R)$, the ex-ante forecast magnitude of risk, $P(r_R, \sigma_F, K_R)$, and the ex-ante forecast risk factors, $P(r_R, \sigma_R, K_F)$, which an investor uses to construct an optimal portfolio using the mean–variance approach at the beginning of the period. The others are two duplicate terms, *DTX* and *DTY*.

4 Numerical examples of attributions

In this section, we demonstrate our attribution analysis using a historical data sample. We use an investment of weekly S&P 500 10 sector total return indices for the 471 weekends from the last weekend of 2007 until the last weekend of 2017. To this end, we need to prepare sample forecast return distributions, which we use to analyze the attributions. In practical analysis, we analyze the forecast return distribution, which we then use to construct the optimal portfolio at the beginning of the investment period. Here, we evaluate "a test forecast distribution" instead of an arbitrary forecast distribution. Recall that a return distribution has a return vector and a covariance matrix. First, we use the realized covariance matrix of sectoral returns for the preceding 52 weeks of a weekend as the test covariance matrix. Second, we use the realized return vector of the next 52 weeks with some random noise as the test forecast return vector of the weekend. We calculate the next 52 weeks' 10-dimensional test forecast return vector $r_F(t)$ and the 10×10 test forecast covariance matrix $\Sigma_F(t)$ for the next 52 weeks from weekend t as:

$$r_F(t) \equiv r_R(t+52) + cs(t)\varepsilon, \tag{17}$$

$$\Sigma_F(t) \equiv \Sigma_R(t), \tag{18}$$

where $r_R(t)$ is the 52 weeks' 10-dimensional realized return vector until weekend t, $\Sigma_R(t)$ is the 10×10 realized covariance matrix of the sectoral returns for the preceding 52 weeks of weekend t,

$$s(t) \equiv \begin{pmatrix} s_1(t) & & 0 \\ & \ddots & \\ 0 & & s_{10}(t) \end{pmatrix}, \quad \varepsilon \equiv \begin{pmatrix} z_1 \\ \vdots \\ z_{10} \end{pmatrix}, \tag{19}$$

where $s_i(t)$ are the standard deviations of each sector's return for the preceding 52 weeks until weekend t, z_i are the random variables under $N(0, 1)$, each pair z_i and z_j is independent, and c is a scalar coefficient. In our numerical examples, c represents the predictability level of forecast returns.

We calculate test forecast return vectors, test forecast covariance matrices, realized return vectors, and realized covariance matrices from the first weekend of 2009 until the last weekend of 2016, for $c = 0.1, 0.3$, or 0.5. The random variables are given by the random function in Excel. The diagonalizations of the covariance matrices are performed using "eigen" function of R (R Core Team. (2016)). From the diagonalizations, we consider the eigen vectors as the risk factor vectors and the eigen values as the risk of the factors. Furthermore, we perform attribution analysis for the test returns, forecast magnitude of the test risks, and test risk factors using Property 2, respectively.

Table 1 summarizes the attribution analysis. The first row shows the average from the first weekend of 2009 until the last weekend of 2016. The other rows show the averages for each year. The values show the averages of the 53 weeks in 2009 and 2015. The remaining years show the average values of the 52 weeks in those years.

In the first row in Table 1, Column (B) shows that realized Sharpe ratio $SR(r_F, \Sigma_F \mid r_R, \Sigma_R)$ of (5) is the optimal portfolio that the test distribution achieves. The optimal portfolios based on the test distribution achieve 0.276 of the realized Sharpe ratio, on average, for the entire period when $c = 0.1$. Column (C) shows that the market condition of the Sharpe ratio is 0.482, on average, indicating that the maximum Sharpe ratio an investor can obtain is 0.482 through a non-leveraged long/short portfolio. The average predictability of the test distribution is 0.567 in Column (D) when $c = 0.1$. Further the imperfect information about the realized distribution declines by approximately 43% of the achievable maximum

Table 1 Sharpe Ratio Attribution to Predictability of Forecast Return and Risk.

(A) Year	(B) Realized Sharpe Ratio c=			(C) Market Condition	(D) Forecast Predictability c=			(E) Forecast Return c=			(F) Forecast Risk	(G) Magnitude Forecast	(H) Factor Forecast	(X) Duplicate	(Y) Duplicate Term Y c=		
	0.1	0.3	0.5		0.1	0.3	0.5	0.1	0.3	0.5				Term X	0.1	0.3	0.5
2009–2016	0.276	0.163	0.103	0.482	0.567	0.331	0.210	0.743	0.387	0.255	0.750	0.963	0.759	−0.027	−0.075	−0.195	−0.206
2016	0.303	0.176	0.111	0.543	0.552	0.317	0.198	0.867	0.496	0.305	0.655	0.960	0.716	0.021	−0.030	−0.165	−0.238
2015	0.213	0.106	0.072	0.347	0.616	0.307	0.213	0.804	0.427	0.303	0.776	0.980	0.796	0.001	−0.036	−0.104	−0.134
2014	0.276	0.211	0.176	0.434	0.635	0.477	0.403	0.832	0.439	0.311	0.688	0.965	0.675	−0.048	−0.115	−0.350	−0.404
2013	0.391	0.236	0.079	0.545	0.717	0.429	0.145	0.824	0.388	0.160	0.805	0.964	0.811	−0.030	−0.088	−0.236	−0.180
2012	0.405	0.242	0.166	0.572	0.703	0.414	0.268	0.775	0.420	0.294	0.791	0.943	0.834	−0.013	−0.138	−0.204	−0.183
2011	0.259	0.187	0.116	0.455	0.572	0.408	0.252	0.710	0.401	0.260	0.720	0.974	0.692	−0.055	−0.141	−0.287	−0.272
2010	0.180	0.067	0.038	0.503	0.353	0.127	0.073	0.456	0.206	0.124	0.807	0.968	0.765	−0.073	−0.090	−0.114	−0.142
2009	0.185	0.081	0.064	0.461	0.394	0.176	0.131	0.673	0.316	0.278	0.755	0.951	0.783	−0.021	0.033	−0.105	−0.098

Data Source: Ten sector total indices. S&P Dow Jones Indices. 2018. S&P Dow Jones Indices LLC,
(C) Market Condition: $\|\sigma_R^{-1} K_R^{-1} r_R\|$, (D) Forecast Predictability: $P(r_F, \sigma_F, K_F)$,
(E) Forecast Return: $P(r_F, \sigma_R, K_R)$, (F) Forecast Risk: $P(r_R, \sigma_F, K_F)$,
(G) Magnitude Forecast: $P(r_R, \sigma_F, K_R)$, (H) Factor Forecast: $P(r_R, \sigma_R, K_F)$.

Sharpe ratio, on average. We also find the natural outcome that the optimal portfolio tends to achieve a higher Sharpe ratio in Column (B) when the forecast predictability is higher in Column (D). Column (E) shows the predictability of test return $P(r_F, \sigma_R, K_R)$ of (8). Column (F) shows the predictability of test risk $P(r_R, \sigma_F, K_F)$ of (9). Indeed, the predictability of the test distribution in Column (D) is attributed to the predictability of the test returns and risk without duplicate term Y in Column (Y) from (14). The predictability of the test return is 0.743 and the predictability of the test risk is 0.750 when $c = 0.1$. As such, the 43% reduction is, on average, attributed to the 26% caused by an imperfect information about the realized return and to the 25% reduction caused by an imperfect information about the realized risk. We can obtain the quantitative information about the predictabilities of the return and risk, simultaneously. Consequently, we can compare the two predictabilities. The magnitude of the duplicate is 7.5% when $c = 0.1$ from Column (Y). Column (G) shows the predictability of the forecast magnitude of test forecast covariance matrix $P(r_R, \sigma_F, K_R)$ of (10). Column (H) shows the predictability of the forecast risk factors of test forecast covariance matrix $P(r_R, \sigma_R, K_F)$ of (11). The predictability of the test risk in Column (F) is attributed to the predictability of the magnitude of the test risk and test risk factors without duplicate term X in Column (X) from (16). We find that the predictability of the magnitude of the test risk is 0.963 and that of the test risk factors is 0.759. Similarly, the 25% reduction caused by imperfection information on the realized risk is attributed to the 4% reduction caused by imperfect information on the magnitude of the realized risk and the 24% reduction caused the imperfect information on the realized risk factors, on average. As a result, we can identify the main source of the reduction for the predictability of risk. The magnitude of the duplicate is 2.7% from Column (X). The predictability of the forecast risk factors of test forecast covariance matrix $P(r_R, \sigma_R, K_F)$ causes most of the predictability of test forecast covariance matrix $P(r_R, \sigma_F, K_F)$, which is the overall test forecast risk, in Columns (F), (G), and (H) based on historical data. From the attribution analysis, we can thus comparatively identify the main source of the reduction in the realized Sharpe ratio. This information may be useful for improving the next investments.

When comparing the rows of years 2016 and 2013, the predictability of the test return in 2016 is, on average, higher than that in 2013 when $c = 0.1$ and $c = 0.3$. However, the realized Sharpe ratios in 2016 are, on average, lower than those in 2013, although the market conditions in both years have the same level. We identify the reasons for the difference in the predictability of the test risks between 2016 and 2013 in Column (F). The average predictabilities of the test risk are 0.655 in 2016 and 0.805 in 2013. This difference comes from the difference between the predictability of the test risk factors $P(r_R, \sigma_R, K_F)$ in 2016 and in 2013. This shows that risk forecast is important, as is return forecast. In particular, we realize it is important to forecast risk factors.

5 Attribution analysis against a benchmark

In this section, we discuss the attribution analysis against a given benchmark using the above predictability. In the usual ways of analyzing against a benchmark, we do not evaluate the performance of the benchmark. Using the attribution analysis above, we can simultaneously evaluate ex-ante forecasts and benchmark performance. For example, we identify two explanations for the Sharpe ratio of the optimal portfolio being higher than that of the benchmark over the investment period: one is that the performance of the optimal portfolio is superior and the other is that the performance of the benchmark is inferior. We can assess which explanation is more acceptable in this situation using attribution analysis.

From (7), the ex-post realized Sharpe ratio that optimal portfolio from the ex-ante forecast return distribution $N(r_F, \Sigma_F)$ achieves under ex-post realized return distribution $N(r_R, \Sigma_R)$ is $SR(r_F, \Sigma_F | r_R, \Sigma_R)$. From Property 2, we have three predictabilities to which the Sharpe ratio can be attributed, the predictability of the ex-ante forecast returns $P(r_F, \sigma_R, K_R)$, that of the ex-ante forecast magnitude of the risk $P(r_R, \sigma_F, K_R)$, and that of the ex-ante forecast risk factors $P(r_R, \sigma_R, K_F)$.

Next, we discuss the realized Sharpe ratio of a benchmark. Assume we have an n-dimensional vector of the portfolio weights of benchmark w_{BM} at the beginning of the investment period. Additionally, we construct benchmark portfolio w_{BM} based on an assumed n-dimensional vector of ex-ante forecast returns r_{BM} and the assumed $n \times n$ covariance matrix of the ex-ante forecast risk of returns Σ_{BM} using the mean–variance approach at the beginning of the investment period. After the investment period, assume that the benchmark portfolio gains realized Sharpe ratio SR_{BM} over the investment period.

From Property 1, we have

$$SR_{BM} = SR(r_{BM}, \Sigma_{BM} | r_R, \Sigma_R). \quad (20)$$

In fact, we have several pairs (r_{BM}, Σ_{BM}) that hold for the above equation. We measure the performance of the benchmark after the investment period. Therefore, in our attribution analysis, we construct benchmark portfolio w_{BM} based on an n-dimensional vector of ex-ante implied forecast returns r_{BM_I} and $n \times n$ covariance matrix Σ_R, which is covariance matrix of the ex-post realized returns of the risky assets over the investment period. Indeed, we have the following implied forecast return for constructing the benchmark portfolio at the beginning of the investment period (see Appendix 4):

$$r_{BM\_I} = \Sigma_R w_{BM}. \quad (21)$$

From (8), we have a predictability of the implied forecast return of the benchmark portfolio:

$$P(r_{BM\_I}, \sigma_R, K_R) \equiv \left(\frac{\sigma_R^{-1} K_R^{-1} r_{BM\_I}}{\| \sigma_R^{-1} K_R^{-1} r_{BM\_I} \|}, g \right). \quad (22)$$

By comparing $P(r_F, \sigma_F, K_F)$ with $P(r_{BM\_I}, \sigma_R, K_R)$, we can evaluate the ex-ante forecasts that construct the optimal portfolio and evaluate the benchmark performance.

Next, we demonstrate the attribution analysis against the benchmark using the historical data sample, that is the weekly S&P 500 10 sector total return indices. It is natural to use the market weight portfolio as the benchmark. For this purpose, we prepare the sample portfolio, which is the optimal portfolio constructed by a given return and risk forecast. When we analyze the performance of the portfolio we construct based on a given forecast return and risk by the mean variance approach, we directly use forecast return and risk. However, we use test returns and test risk instead of an arbitrary return distribution. Here, we compare the performances of the equal weighted portfolio and of the benchmark, which is the performance achieved by the market weight portfolio. Additionally, we use the test risk, as in the previous section, to construct the equal weighted portfolio using the mean–variance approach. The test risk is the realized covariance matrix of the sectoral returns for the preceding 52 weeks of a weekend.

Then, we also have the n-dimensional vector of ex-ante implied forecast returns r_{Eq_I}, which leads to an equal weighted portfolio serving as optimal portfolio when using the test risk and the mean variance approach:

$$r_{Eq\_I} = \Sigma_F w_{Eq}, \quad (23)$$

Table 2 Sharpe Ratio Attribution against A Benchmark.

(A) Weekend of Year	(B) Realized Sharpe Ratio During the next year			(C) Market Condition	(D) The Predictability			(E) Return Predictability			(F) Risk Predictability of the Equal Weighted Portfolio				(Y)
	Portfolio				Portfolio			Portfolio							
	Equal Weighted	Bench Mark	Excess		Equal Weighted	Bench Mark	Excess	Equal Weighted	Bench Mark	Excess		(G) Magnitude Forecast	(H) Factor Forecast	(X) Duplicate Term X	Duplicate Term Y
2016	0.402	0.483	−0.081	0.562	0.716	0.859	−0.143	0.860	0.859	0.001	0.511	0.979	0.707	0.175	−0.345
2015	0.169	0.129	0.040	0.416	0.407	0.311	0.096	0.519	0.311	0.208	0.841	0.987	0.847	−0.007	−0.047
2014	−0.011	0.010	−0.022	0.400	−0.029	0.026	−0.055	−0.268	0.026	−0.294	0.787	0.970	0.820	0.002	−0.452
2013	0.175	0.177	−0.001	0.597	0.294	0.296	−0.002	0.415	0.296	0.119	0.666	0.881	0.642	−0.142	−0.213
2012	0.378	0.407	−0.029	0.675	0.560	0.603	−0.043	0.275	0.603	−0.328	0.860	0.985	0.839	−0.037	−0.425
2011	0.155	0.154	0.000	0.363	0.427	0.426	0.001	0.356	0.426	−0.070	0.813	0.943	0.835	−0.035	−0.258
2010	0.022	0.011	0.012	0.504	0.044	0.012	0.023	0.119	0.021	0.098	0.690	0.976	0.581	−0.134	−0.235
2009	0.123	0.112	0.010	0.482	0.255	0.233	0.021	−0.037	0.233	−0.270	0.802	0.990	0.765	−0.048	−0.489

Data Source: Ten sector total indices. S&P Dow Jones Indices. 2018. S&P Dow Jones Indices LLC,
(C) Market Condition: $\|\sigma_R^{-1} K_R^{-1} r_R\|$,
(D) Predictability: Equal Weighted $P(r_{Eq\_I}, \sigma_F, K_F)$, Benchmark $P(r_{BM\_I}, \sigma_R, K_R)$
(E) Return Predictability: Equal Weighted $P(r_{Eq\_I}, \sigma_R, K_R)$, Benchmark $P(r_{BM\_I}, \sigma_R, K_R)$
(F) Forecast Risk: $P(r_R, \sigma_F, K_F)$, (G) Magnitude Forecast: $P(r_R, \sigma_F, K_R)$, (H) Factor Forecast: $P(r_R, \sigma_R, K_F)$,
(Y) Duplicate term Y of the equal weighted portfolio.

where $w_{Eq} = (0.1, \ldots, 0.1)'$ because we have the weekly S&P 500 10 sector total return indices as investable assets in this numerical analysis.

We perform the attribution analysis when we compare the predictability of the implied return of the equal weighed portfolio with the test risk and predictability of the benchmark.

Table 2 summarizes the Sharpe ratio attribution analysis against the benchmark for equal weighted portfolios over the last weekends of the eight years from 2009 until 2016. From the first row of Table 2, Column (B), the equal weighted portfolio for the last weekend of year 2016 is 0.402 of the realized Sharpe ratio for 2017. Additionally, the benchmark portfolio, which is the market weight portfolio at the end of 2016, is 0.483 of the realized Sharpe ratio for 2017. The excess Sharpe ratio of the equal weighed portfolio is -0.081 against the benchmark. In other words, the Sharpe ratio of the portfolio is inferior to the Sharpe ratio of the benchmark. The predictabilities of the portfolio and the benchmark are shown in Column (D) as 0.716 and 0.859, respectively. The inferiority of the predictability of the portfolio leads the inferiority of the realized Sharpe ratio. The return predictability of the portfolio and the benchmark in Column (E) are at the same level. Note that the returns are the implied returns, which indicates that the equal weighted and benchmark portfolios are the optimal portfolios when using the mean–variance approach. We illustrate the risk predictability of the portfolio and its decomposition of it in Columns (F), (G), and (H). Note that the risk predictability of the benchmark is 1 because of the definition of the implied returns of the benchmark portfolio. Therefore, the inferiority comes from the lower predictability of the risk, especially the lower predictability of the risk factor.

Additionally, when comparing the portfolio of the last weekend of 2016 to the portfolio of the last weekend of 2015, the equal weighted portfolio of the last weekend of 2015 is 0.169 of the realized Sharpe ratio for 2016. The predictability of the portfolio is 0.407. This shows that the predictability of the portfolio of the end of year 2015 is inferior to that for 2016. Additionally, the predictability of the benchmark of the end of year 2015 is 0.311. Therefore, the Sharpe ratio of the portfolio is superior to the Sharpe ratio of the benchmark for 2016. Further, the Sharpe ratio of the portfolio is inferior to the benchmark for 2017, although the predictability of the portfolio at the end of year 2016 is better than the one at the end of year 2015. We recognize that the predictability of the benchmark at the end of year 2016 is superior to the one of the portfolio at the end of year 2016.

We also determine the importance of risk predictability for the portfolio at the end of 2013 and its benchmark. From Column (E), the return predictability of the portfolio is 0.415, which is superior to the 0.296 one of the benchmark. However, the predictability of the portfolio is 0.294, which is almost at the same level (0.296) as the benchmark from Column (D). The risk predictability of the portfolio is 0.666, this inferior value reducing the superior return predictability. We thus obtain rich information using attribution analysis.

6 Conclusions

In this paper, we propose an attribution method of the realized Sharpe ratio that evaluates ex-ante forecast risk and return. Furthermore, we decompose the realized Sharpe ratio into the absolute value of the market Sharpe ratio, return predictability, and risk predictability, which is in

turn decomposed as the predictability of the magnitude of risk and predictability of risk factors. Additionally, we present numerical examples of the attribution analysis.

We find that the attribution formula quantitatively enables a comparative analysis between return and risk predictabilities. Furthermore, the predictability of the risk has a more detailed predictability, as the magnitudes of the risk and risk factors. Additionally, we simultaneously evaluate the quality of the investor's portfolio and of the benchmark against the benchmark. These attribution analyses provide rich information for investment analysis, thus being a useful analytical tool for portfolio managers and sponsors.

However, additional research on the applications of the properties discussed in this article should be conducted. For example, the attribution analysis against the benchmark shows that the realized excess Sharpe ratio of the investor's portfolio is decomposed into the predictability and the realized market conditions. The attribution thus provides a better understanding of the realized Sharpe ratio, which facilitates a more practical analysis. Finally, further research is required to develop effective ways to use these properties to analyze different aspects of the investment process and thereby improve the realized Sharpe ratio.

Appendix 1: Optimal portfolio to maximize the Sharpe ratio

We calculate the derivations in accordance with Shimizu (2017). Assume a return distribution $N(r, \Sigma)$. First, for a given h, we find the minimum-variance portfolio as the solution of the following quadratic programming problem:

$$\begin{aligned} &minimize \quad w'\Sigma w, \\ &\quad s.t. \quad r'w = h, \quad and \quad e'w = 1, \end{aligned} \tag{A1.1}$$

where $e = (1, \ldots, 1)'$. Problem (A1.1) has the unique solution below (Steinbach, 2001)

$$\bar{w} = \Sigma^{-1}(\lambda e + kr), \tag{A1.2}$$

where

$$\lambda \equiv \frac{\gamma - \beta h}{\delta}, \quad k \equiv \frac{ah - \beta}{\delta}, \tag{A1.3}$$

$$\alpha \equiv e'\Sigma^{-1}e, \quad \beta \equiv e'\Sigma^{-1}r,$$

$$\gamma \equiv r'\Sigma^{-1}r, \quad \delta \equiv \alpha\gamma - \beta^2. \tag{A1.4}$$

Therefore, we obtain the risk of the portfolio:

$$\sigma_{\bar{w}}^2 = \bar{w}'\Sigma\bar{w} = \frac{\gamma - 2\beta h + \alpha h^2}{\delta} \tag{A1.5}$$

The Sharpe ratio of the portfolio is:

$$SR_{\bar{w}} = \frac{h}{\sigma_{\bar{w}}} = \frac{h}{\sqrt{\frac{\gamma - 2\beta h + \alpha h^2}{\delta}}} \tag{A1.6}$$

We can obtain the maximum when $h = \gamma/\beta$ holds under the condition that β is positive. By substituting $h = \gamma/\beta$ with (A1.3), we obtain the optimal portfolio from (A1.2) using

$$w_0 = \frac{\Sigma^{-1}r}{e'\Sigma^{-1}r}. \tag{A1.7}$$

We obtain a higher when h increases under the condition that β is not positive. In this case, we continue by substituting $\tilde{r} \equiv \alpha r_F - \beta e$ for r_F. The condition holds where the sum of the Sharpe ratios of all risk factors becomes negative. However, it is difficult to predict such a forecast from an economic perspective. The optimal portfolio often has highly leveraged positions and large short positions when h has a large value. Investors do not usually hold such extreme portfolios.

Appendix 2: Derivation of theorem 1

We calculate the derivations in accordance with Shimizu (2017). From (1), and (3), we have

$$w_F = \frac{\Sigma_F^{-1} r_F}{e' \Sigma_F^{-1} r_F} = \frac{K_F \sigma_F^{-2} K_F^{-1} r_F}{e' K_F \sigma_F^{-2} K_F^{-1} r_F}. \quad \text{(A2.1)}$$

From (1), (3), and (4) as the definition of Φ, we have

$$\begin{aligned} w_1 &= \frac{\Sigma_R^{-1} K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F}{e' \Sigma_R^{-1} K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F} \\ &= \frac{K_R \sigma_R^{-2} K_R^{-1} K_R \sigma_R \sigma_R K_R^{-1} K_F \sigma_F^{-1} \sigma_F^{-1} K_F^{-1} r_F}{e' K_R \sigma_R^{-2} K_R^{-1} K_R \sigma_R \sigma_R K_R^{-1} K_F \sigma_F^{-1} \sigma_F^{-1} K_F^{-1} r_F}. \end{aligned} \quad \text{(A2.2)}$$

From $K_R \sigma_R^{-2} K_R^{-1} K_R \sigma_R \sigma_R K_R^{-1} = I$ we have

$$w_1 = \frac{K_F \sigma_F^{-1} \sigma_F^{-1} K_F^{-1} r_F}{e' K_F \sigma_F^{-1} \sigma_F^{-1} K_F^{-1} r_F} = w_F \quad \text{(A2.3)}$$

Appendix 3. Derivation of property 1

Assume we have return distribution $N(r_x, \Sigma_R)$. From (3), we have optimal portfolio $w_x = \Sigma_R^{-1} r_x / e' \Sigma_R^{-1} r_x$ under return distribution $N(r_x, \Sigma_R)$. Portfolio w_x achieves the Sharpe ratio under ex-post realized return distribution $N(r_x, \Sigma_R)$:

$$\begin{aligned} \frac{w_x' r_R}{\sqrt{w_x' \Sigma_R w_x}} &= \frac{\left(\frac{\Sigma_R^{-1} r_x}{e' \Sigma_R^{-1} r_x}\right)' r_R}{\sqrt{\left(\frac{\Sigma_R^{-1} r_x}{e' \Sigma_R^{-1} r_x}\right)' \Sigma_R \left(\frac{\Sigma_R^{-1} r_x}{e' \Sigma_R^{-1} r_x}\right)}} \\ &= \frac{r_x' \Sigma_R^{-1} r_R}{\sqrt{r_x' \Sigma_R^{-1} r_x}} \end{aligned} \quad \text{(A3.1)}$$

From Theorem 1, substituting $K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F$ into (A3.1), the optimal portfolio based on ex-ante forecast return distribution $N(r_F, \Sigma_F)$ achieves the Sharpe ratio under ex-post realized return distribution $N(r_R, \Sigma_R)$:

$$\begin{aligned} &\frac{(K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F)' \Sigma_R^{-1} r_R}{\sqrt{(K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F)' \Sigma_R^{-1} (K_R \sigma_R \Phi \sigma_F^{-1} K_F^{-1} r_F)}} \\ &= \frac{(\Phi \sigma_F^{-1} K_F^{-1} r_F)' \sigma_R^{-1} K_R^{-1} r_R}{\sqrt{(\Phi \sigma_F^{-1} K_F^{-1} r_F)' (\Phi \sigma_F^{-1} K_F^{-1} r_F)}} \\ &= \|\sigma_R^{-1} K_R^{-1} r_R\| \left(\frac{\Phi \sigma_F^{-1} K_F^{-1} r_F}{\|\Phi \sigma_F^{-1} K_F^{-1} r_F\|}, g \right), \end{aligned} \quad \text{(A3.2)}$$

where

$$\Phi \equiv \sigma_R K_R^{-1} K_F \sigma_F^{-1}, \quad g \equiv \frac{\sigma_R^{-1} K_R^{-1} r_R}{\|\sigma_R^{-1} K_R^{-1} r_R\|}, \quad \text{(A3.3)}$$

and (x, y) is the inner product of vectors x and y, and $\|x\| = \sqrt{(x, y)}$ is the norm of x.

Appendix 4. Implied forecast return of a given portfolio

Let w denote a given portfolio, Σ_F the forecast covariance matrix, and r the implied forecast return vector:

$$\begin{aligned} \textit{minimize} \quad & \frac{1}{2} w' \Sigma_F w - \mu r_F' w, \\ \textit{s.t.} \quad & e' w = 1. \end{aligned} \quad \text{(A4.1)}$$

We solve the above problem using the Lagrange multiplier, as follows:

$$L(w, l) = \frac{1}{2} w' \Sigma_F w - \mu r_F' w - l(e' w - 1). \quad \text{(A4.2)}$$

Solving

$$\frac{\partial}{\partial w} \left(\frac{1}{2} w' \Sigma_F w - \mu r_F' w - l(e' w - 1) \right) = 0 \quad \text{(A4.3)}$$

$$\frac{d}{dl}\left(\frac{1}{2}w'\Sigma_F w - \mu r_F' w - l(e'w-1)\right) = 0 \tag{A4.4}$$

then,

$$\frac{\partial}{\partial w}\left(\frac{1}{2}w'\Sigma_F w - \mu r_F' w\right) = 0. \tag{A4.5}$$

We have

$$r_F = \frac{1}{\mu}\Sigma_F w. \tag{A4.6}$$

We let $\mu \equiv 1$ because r_F is normalized later.

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