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QUANTIFYING THE SKEWNESS LOSS OF DIVERSIFICATION

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Diversification is widely viewed as the "only free lunch" of finance. Unbeknownst to the free lunch crowd, skewness is typically positive for individual stocks and negative for diversified portfolios and thus diversification is not free. This undesirable move from positive to negative skewness that comes with diversification is the skewness loss of diversification. We quantify the economic value of skewness loss using option pricing models, and show that skewness loss is a meaningful cost for investors with skewness preferences and short horizons.

1 Introduction

The standard theory of portfolio choice developed by Markowitz (1952) leads to the conclusion that investors should hold a diversified portfolio. Diversification almost eliminates idiosyncratic risk and reduces a portfolio's total variance without affecting return; thus it helps to achieve a higher Sharpe ratio and a higher geometric mean. This has been a powerful argument for diversification being a "free lunch."

However, despite these apparent benefits, literature has demonstrated that diversification not only leads to a reduction in variance—a characteristic *liked* by investors—but also leads to skewness loss—a characteristic *disliked* by investors—suggesting a trade-off between variance and skewness that is integrated with the quest for return. These offsetting effects of diversification were documented at least as far back as Simkowitz and Beedles (1978), which showed that skewness decreases (or becomes more negative) with diversification.

A number of empirical studies (Conine and Tamarkin, 1981; Odean, 1999; Mitton and Vorkink, 2007; Kumar, 2007; Goetzman and Kumar, 2008; etc.) show that portfolios held by individual investors are often underdiversified, containing less than five stocks on average. Statman (2004) calls this the "diversification puzzle" because this systematic level

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of "under-diversification" is inconsistent with mean–variance portfolio theory.

The key to understanding this diversification puzzle is investors' preference for positive skewness. Much empirical evidence shows that individual investors prefer positive skewness, as illustrated by buying lottery-type stocks (Barberis and Huang, 2008; Ilmanen, 2012). On the other hand, investors dislike losses, especially large unexpected losses that come with negative skewness, so they tend to buy insurance to hedge potentially large losses. In contrast with traditional mean–variance theory, Conine and Tamarkin (1981) show that under-diversification might be optimal if investors knowingly or unknowingly choose their optimal portfolios not only by considering the first two moments of the distribution of returns (mean and variance), but also the third moment (skewness). To put it differently, there are likely investors whose unobservable utility function is better approximated by simultaneously considering the trade-offs between mean, variance, and skewness.[1] The first attempt to introduce the third moment into portfolio choice was proposed by Kraus and Litzenberger (1976), followed by Harvey and Siddique (2000), which provides empirical support for incorporating skewness.

Skewness plays an important role in asset pricing, option pricing, and asset allocation/portfolio construction. A few theoretical models show that skewness can impact asset pricing as long as investors' preferences (1) are driven by a three-moment model (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000; Mitton and Vorkink, 2007); (2) they obey prospect theory (Barberis and Huang, 2008); and (3) they obey optimal expectations theory (Brunnermeier and Parker, 2005; Brunnermeier *et al*., 2007).

Empirically, both skewness and the forth moment, kurtosis, impact option pricing. Asset return distributions exhibit non-zero skewness and fat tails (kurtosis greater than 3). Equity options often show an implied volatility smile or implied volatility skew, in which the implied volatility on the same security/market with the same expiration date varies by strike price. The reason for this is that skewness and kurtosis lead to higher prices for out-of-the-money options. These observations imply deficiencies in the standard Black–Scholes option pricing model, which assumes constant volatility and lognormal distributions of underlying asset returns.

Turning to the impact of skewness on portfolio selection, Kane (1982) shows that allocation to a positively skewed risky asset is increased in a three-moment utility framework relative to the mean–variance utility, and preference for positive skewness provides a rationale for observed investors' non-diversified portfolios. Patton (2004) suggests that knowledge of both skewness and asymmetric dependence (higher correlations in downside markets) leads to economically significant gains. Briec *et al.* (2007) propose a nonparametric efficiency measurement approach for portfolio optimization in mean–variance–skewness space. Xiong and Idzorek (2011) find that while holding asset class means, volatilities, and correlations constant, incorporating skewness and kurtosis into a mean-conditional value-at-risk (M-CVaR) optimization can lead to substantially different allocations than those from traditional mean–variance optimization. In particular, the combination of a negative skewness and a fat tail (large positive kurtosis) has the greatest impact on the optimal asset allocation weights.

More recently, Bessembinder (2017) shows that when stated in terms of lifetime dollar wealth creation, the best-performing four percent of listed companies explain the net gain for the entire U.S. stock market since 1926, as other stocks collectively matched Treasury bills. These striking results highlight the important role of positive skewness in the distribution of individual stock returns.

Previous practitioner-oriented-diversification-literature is dominated by a focus on the benefit of variance reduction, and to our knowledge, largely ignoring the economic value of skewness loss. By quantifying the economic value of skewness loss in a unique and intuitive manner, we hope to paint a complete picture on the important trade-off between variance and skewness.

2 Description of data

Our data source is the stock universe consisting of all the stocks on the New York Stock Exchange (NYSE), American Stock Exchange[2] (AMEX), and Nasdaq Stock Market (NASDAQ) over the 57-year period from January 1960 through December 2016. Daily and monthly returns are collected from the University of Chicago's Center for Research in Security Prices. We include stocks with an initial price greater than $5 in each period. In addition, we remove the micro-cap stocks in the NYSE/AMEX/NASDAQ universe—specifically, those with a market capitalization below the 20th percentile of the combined NYSE/AMEX/NASDAQ universe in each period. We exclude derivative securities of foreign stocks such as ADRs.

We work with simple returns for the next three sections of the analyses, but we switch to log-returns when we quantify the economic value of skewness loss using an option pricing model that admits skewness and kurtosis, as log-returns are required in the option pricing model. In addition, since we are interested in the behavior of the average single stock, it is more convenient to work with equally-weighted market returns, instead of value-weighted market returns. Using value-weighted market returns does not change the conclusions in a meaningful way.

3 Positive *stock* skewness and negative *market* skewness

Empirical studies have found that the average stock, and individual stocks in general, has return distributions with positive skewness, while the market as whole, and diversified portfolios of stocks, has return distributions with negative skewness. In this paper, we refer to that gap as the "skewness loss" that occurs as one moves from a single-stock portfolio to a well-diversified portfolio consisting of a large number of stocks. One explanation for the positive average single-stock skewness was provided by Albuquerque (2012), which argues that around earnings announcements, sporadic and short-lived periods of high volatility and high mean returns can generate positive skewness for a single stock's returns. On the other hand, negative market skewness can be related to asymmetric correlation (higher correlation in downturned markets) among cross-sectional stocks (e.g., Duffee, 2000). Asymmetric correlation has been documented extensively (Longin and Solnik, 2001; Ang and Chen, 2002), and it is an important economic force and a stylized feature of the equity markets.

Figure 1 replicates and updates Figure 1 of Albuquerque (2012) with an additional 20 years of data. This graph shows that the average individual stock skewness is positive and the equally-weighted market skewness is negative for most of the time. Skewness is measured one year at a time using non-overlapping daily simple returns. To be included, stocks are required to have at least 120 days' returns data in one year. The figure plots the average skewness of individual stocks (blue line) and the skewness in the equally-weighted market return (red line) between 1960 and 2016.

For the majority of the time periods between 1960 and 2016, the skewness is mostly positive for individual stocks and mostly negative for the equally-weighted market. The average skewness

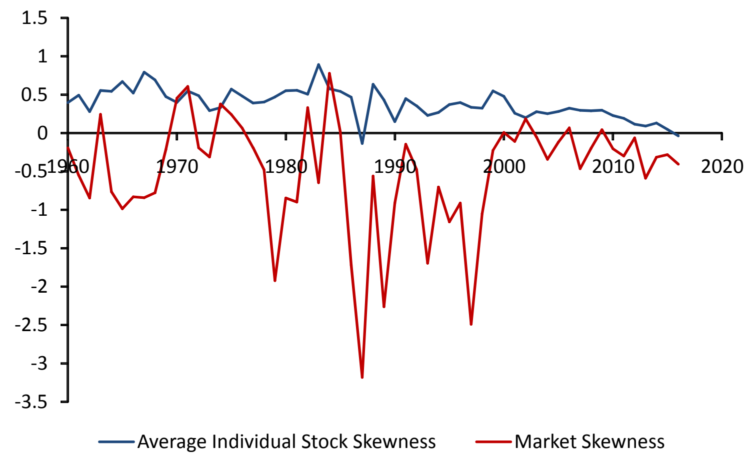

Figure 1 The average skewness of individual stocks and skewness in the equally-weighted market return (from Jan 1960 to Dec 2016).

for individual stocks is a positive 0.39 and the average skewness for the equally-weighted market is a negative 0.51, with an average gap of 0.90. It is interesting to note that the average individual stock skewness seems to be decreasing over the period, and it becomes slightly negative in 2016.[3] As mentioned earlier, Albuquerque (2012) argues that during earnings announcement periods, stocks tend to have high expected returns and high volatility, which causes positive skewness for an individual stock. The declining individual stock skewness might be consistent with declining earnings announcement premia (see Cohen *et al*., 2007). The gap between average stock skewness and market skewness—the skewness loss — is very robust, although it is time-varying—wider from 1980 to 2000 and somewhat narrower since 2000.

4 Diversification lowers variance but results in skewness loss

The relationship between the number of stocks in a portfolio and the total variance (or standard deviation) of the portfolio has been examined extensively as far back as Evans and Archer (1968). The conclusions from these empirical analyses are that adding more stocks will reduce total standard deviation without reducing return. We now apply and extend that type of analysis to both standard deviation and skewness.

We construct equally-weighted portfolios containing a different number (5–40) of randomly selected stocks (without replication), similar to the empirical methodology of Campbell *et al.* (2001). In addition, the market portfolio contains all screened stocks in the universe. Using monthly returns over 60-month periods, we calculate the monthly standard deviation for each portfolio. To create the curve in Figure 2, we then average the monthly standard deviation over 11 non-overlapping five-year periods (from 1961 to 2015). The left-hand vertical axis represents standard deviation.

Also in Figure 2, we calculate skewness for the same randomly selected portfolios and show it on the right-hand vertical axis. On average, an individual stock has positive skewness, but as we move from a portfolio consisting of a single stock to a portfolio consisting of many stocks, the skewness moves from positive to negative. As more stocks are added, the standard deviation is quickly reduced (desirable); however, the skewness decreases (not desirable). Put differently, the

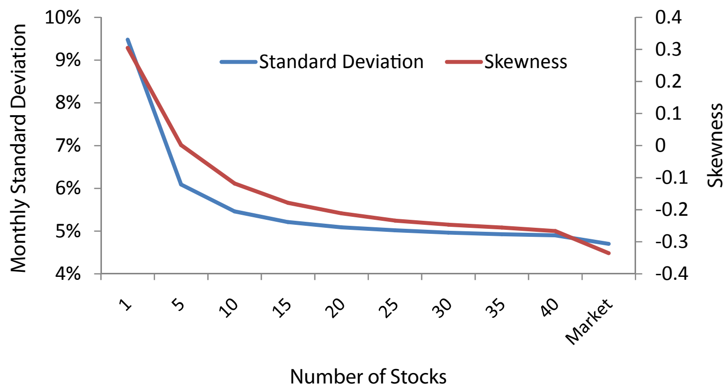

Figure 2 Monthly standard deviation and skewness as a function of number of stocks (from Jan 1961 to Dec 2015).

degree of diversification as represented by the number of stock holdings in a portfolio seems to result in a trade-off between standard deviation and skewness. For the most part, investors have focused exclusively on reducing standard deviation for a given level of return, while unknowingly decreasing the skewness of their portfolio. Moving from a single stock to a diversified portfolio of stocks changes the likely return distribution in two distinct ways that are consistent with Simkowitz and Beedles (1978): 1) standard deviation decreases, and 2) skewness moves from positive to negative.[4]

5 Impact of skewness loss on tail risks

For investors with skewness preferences (i.e., investors dislike negative skewness and prefer positive skewness), left-tail loss and right-tail gain are relevant risk measures, respectively. All else equal, more-negative skewness corresponds to a larger left-tail loss, and more-positive skewness corresponds to a larger right-tail gain. Figure 3 shows that the left- and right-tail risks, measured by conditional value-at-risk (CVaR), as a function of the number of stocks in a portfolio. The *solid curves* correspond to the realized CVaRs, and the *dotted curves* correspond to the CVaRs for a normal distribution. The tail level for CVaR is typically set at 5%. Realized CVaRs are measured on five-year monthly returns and averaged over the 11 non-overlapping five-year periods. The CVaR for a normal distribution at the 5% tail level, denoted by nCVaR, is given by Rockafellar and Uryasev (2000):

$$n\text{CVaR} = \mu \pm 2.06\sigma, \quad (1)$$

where μ and σ are expected/realized average return and standard deviation for an individual stock or portfolio, respectively. The plus and minus signs in Equation (1) correspond to nCVaR for right tail and left tail, respectively.

Figure 3 shows that the normal distribution underestimates both the left-tail loss of the equally-weighted market and the right-tail upside potential gain for an average individual stock; both of these phenomena result mainly from the skewness-loss effect. Specifically, the normal distribution underestimates the *monthly* positive-tail gain by about 2.2 percentage points for average single stock, and underestimates the *monthly* negative-tail loss by about 1 percentage points for market. That is, for a single stock the normal distribution is too pessimistic because it fails to recognize positive skewness, and for a diversified portfolio the normal distribution is

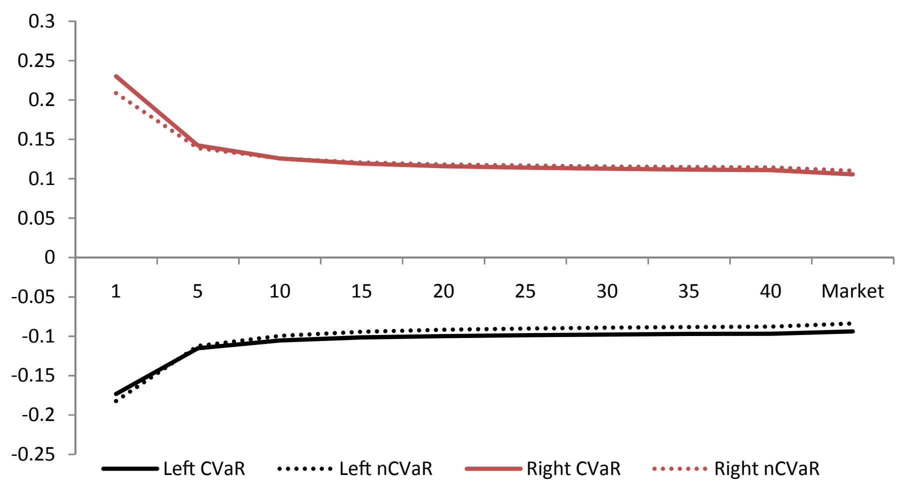

Figure 3 The Averaged left- and right-tail risks, measured by conditional value-at-risk (CVaR), as a function of number of stocks in a portfolio.*

*Left and right CVaRs are measured for randomly selected portfolios. Left and right *n*CVaRs are measured for normal distributions with the same means and standard deviations as the portfolios.

too optimistic because it fails to recognize the negative skewness.

6 Quantifying the economic value of skewness loss

In this section, we quantify the economic value of skewness loss (associated with diversification) for both the single average stock with positive skewness and the equally-weighted market portfolio with negative skewness. As mentioned earlier, we switch to log-returns from now on.

To quantify the economic value of skewness loss, we need to make two choices before we dive into the details. First, we need to choose a reference point to serve as the basis of comparison. A natural choice for this is the lognormal distribution (i.e., log-returns are normally distributed and have zero skewness). More specifically, the reference point is selected so that if both the single average stock and the market log-returns are normally distributed, there is no skewness loss associated with diversification. Using bootstrap simulations, Fama and French (2017) show that log-returns at longer horizons eventually converge to a normal distribution as a result of finite variance due to the central limit theorem. Therefore, it makes sense to choose lognormal distribution as the reference point.

The second choice is to find a framework or model that can price skewness and kurtosis for both the single average stock and the equally-weighted market portfolio. Chen *et al*. (2001) estimate the economic meaning of skewness based on the price differences of relevant options. In the same spirit, in this paper we quantify the economic value of skewness loss by calculating the price of options on both the single average stock and the equally-weighted market portfolio using an option pricing model that specifically accounts for skewness and kurtosis — the Corrado and Su (1996) option pricing model. For the reference point, we use the standard Black–Scholes option pricing model as it assumes returns are lognormally distributed. By cross-comparing the option prices for a non-normal distribution with that for a normal distribution (for log-returns), we can infer the economic value of skewness loss. For absolute clarity, we make the following comparisons to pin point the economic value of skewness loss:

- Single Stock Call (Corrado–Su) vs. Single Stock Call (Black–Scholes).
- Market Put (Corrado–Su) vs. Market Put (Black–Scholes).

The Corrado and Su (1996) model was inspired by Jarrow and Rudd (1982) and adopts a Gram–Charlier series expansion of the normal density function to provide skewness and kurtosis adjustment terms to the Black–Scholes formula. In the Corrado–Su pricing model, the option price is a function of the four moments of the risk-neutral log-return distribution. The first two moments are the same as those of the normal distribution, and the third and fourth moments are introduced as higher-order terms of the density expansion. Brown and Robinson (2002) correct a typographic error in Corrado and Su (1996) and the corrected version of the formula is used in this paper. The Appendix shows the detailed formula. When applying the options pricing models, we assume that both calls and puts are of the European type, the risk-free rate is 4%, and there are no dividends for both single stocks and the equally-weighted market.

For investors concerned about severe downside events (left-tail risk), out-of-the-money *put* options represent an effective method for hedging risk. Conversely, for investors who prefer large upside potential (right-tail benefit), out-of-the-money *call* options represent an effective method of capturing lottery-type gains.[5] Note that all call or put options in this section are the results of applying option pricing models based on averaged historical inputs. We are not evaluating actual traded options.

In previous sections using simple returns, we have shown that skewness is on average positive for single stocks and becomes negative for the equally-weighted market portfolio. Shortly we will see that estimated option prices from the Corrado–Su option pricing model lead to higher estimated prices for both the out-of-the-money call options on the single average stock and the out-of-the-money put options for the equally-weighted market relative to the Black–Scholes option pricing model. More specifically, the *positive* skewness (on log-returns) associated with the single average stock should increase the likelihood that an out-of-the-money *call* option may in fact expire in the money; thus, all else equal, the out-of-the-money call option should be more expensive than it would be for a stock with lognormally distributed returns. Similarly, but conversely, the *negative* skewness (on log-returns) associated with the equally-weighted market portfolio should increase the likelihood that an out-of-the money *put* option may expire in the money; thus, all else equal, the out-of-the-money put option should be more expensive than it would be for the equally-weighted market portfolio with lognormally distributed returns.

As a result, the economic value of skewness loss includes two *positive* option price adjustments relative to the reference point: one from the call option for the single average stock and another from put option for the equally-weighted market. The practical implication is that if one can find two option prices on the same or similar market or individual stock in which one is priced with the Black–Scholes, or BS, formula, and another is priced with the Corrado–Su, or CS formula, the combined arbitrage profits (via buying BS options and selling CS options for both single stock *calls* and market *puts*) are the economic value of skewness loss.

Using the corrected Corrado–Su option pricing formula, Table 1 shows the out-of-the-money call and out-of-the-money put option prices for the average individual stock and the

Table 1 The option costs associated with the skewness loss for average single stock and equally-weighted market.

	Average single stock	Equally-weighted market
A. Daily total log-returns		
Annual volatility	37.21%	14.34%
Skewness	0.12	−0.51
Kurtosis	7.86	6.50
S_0 (Current price)	\$100	\$100
K (Strike price)	\$104.87	\$98.17
T (Time to maturity)	0.004	0.004
P_{CS}	N/A	\$0.0341
P_{BS}	N/A	\$0.0065
$P_{\text{CS}} - P_{\text{BS}}$	N/A	\$0.0276
C_{CS}	\$0.1043	N/A
C_{BS}	\$0.0193	N/A
$C_{\text{CS}} - C_{\text{BS}}$	\$0.085	N/A
B. Monthly total log-returns		
Annual volatility	32.39%	16.36%
Skewness	−0.08	−0.62
Kurtosis	4.14	5.28
S_0 (Current price)	\$100	\$100
K (Strike price)	\$120.11	\$91.08
T (Time to maturity)	0.083	0.083
P_{CS}	N/A	\$0.1411
P_{BS}	N/A	\$0.0338
$P_{\text{CS}} - P_{\text{BS}}$	N/A	\$0.1073
C_{CS}	\$0.1822	N/A
C_{BS}	\$0.1067	N/A
$C_{\text{CS}} - C_{\text{BS}}$	\$0.0755	N/A

equally-weighted market, respectively. To create Table 1, total log-returns are used to calculate the option prices (see Appendix for details). The numbers in the first three rows (standard deviation, skewness, and kurtosis) are averaged over 55 one-year periods from Jan 1961 to Dec 2015 in Table 1A, and averaged over 11 non-overlapping five-year periods from Jan 1961 to Dec 2015 in Table 1B. The costs for put options and call options are calculated on the average standard deviation, skewness, and kurtosis realized over the 55 years using the technique just described (i.e., 55 one-year periods in Table 1A and 11 five-year periods in Table 1B).

As discussed earlier, left-tail loss and right-rail gain, measured by CVaRs, are relevant risk measures for investors with skewness preferences. For individual stock investors who like positive skewness, call options are purchased to capture the lottery-type of gains. For market investors who dislike negative skewness, put options are purchased to hedge the left-tail risk. Therefore, for average individual stock call option, the strike price (K) is selected as the price with right-tail 5% nCVaR gain on current price S_0:

$$K = S_0 + S_0 \cdot |\mu + 2.06\sigma|. \tag{2}$$

For equally-weighted market put option, K is selected as the price with left-tail 5% nCVaR loss on current price S_0:

$$K = S_0 - S_0 \cdot |\mu - 2.06\sigma|, \tag{3}$$

where μ and σ are daily or monthly expected return and volatility for an individual stock or market portfolio with log-returns, respectively. We perform robustness checks for other strike prices a little later. We denote C_{BS} and C_{CS} as the call option prices based on the standard Black–Scholes formula (for the reference point) and the corrected Corrado–Su formula (that accounts for non-normal returns) for the *average stock*, respectively. Conversely, P_{BS} and P_{CS} are put option prices based on the Black–Scholes formula and the corrected Corrado–Su formula for the *equally-weighted market*, respectively.

Focusing on Panel A of Table 1 in which the inputs for the two models are based on daily log-returns, for the average individual stock, the Corrado–Su call option price is \$0.1043 and the Black–Scholes call option price is \$0.0193. The difference ($C_{CS} - C_{BS}$) is \$0.085, which is the additional cost for lottery-type of gains over the starting price of \$100 relative to the reference point. For the equally-weighted market, the Corrado–Su put option price is \$0.0341 and the Black–Scholes put option price is \$0.0065.

The difference ($P_{CS} - P_{BS}$) is \$0.0276, which is the additional hedging cost over the starting price of \$100 relative to the reference point. The combined cross option cost differential is \$0.113(= \$0.085 + \$0.0276), which can be interpreted as an estimate of the economic value of skewness loss of 0.113% based on the starting price of \$100. The annualized economic value of skewness loss in percent is $250 * 0.113\% = 28.25\%$. The economic implication is that one has to pay an additional 28.25% (relative to the reference point) in order to hedge the left-tail risk for the market and capture the lottery-type gains associated with the average individual stock.

Moving to Panel B of Table 1 in which the inputs for the two models are based on monthly log-returns, for the average individual stock, the difference between the two option pricing models for the out-of-the money call ($C_{CS} - C_{BS}$) is \$0.0755.[6] For the equally-weighted market, the difference between the two option pricing models for the out-of-the-money put ($P_{CS} - P_{BS}$) is \$0.1073. The combined cross option cost differential is \$0.183 (= \$0.0755 + \$0.1073), which can be interpreted as the economic value of skewness loss of 0.183% based on the starting price of \$100. The annualized economic value of skewness loss in percentage is $12 * 0.183\% = 2.20\%$, which is still meaningful although much lower than the annualized combined economic value of 28.25% at daily frequency. This suggests that the economic value quickly decreases as time horizon increases from daily to monthly because monthly log-returns are much closer to normal distribution than daily log-returns.

The strike prices in Table 1 are set at the 5% tail level. As a robustness check, we compute the option costs for a range of strike prices, and the results are shown in Figure 4 with monthly frequency. The results for daily frequency are not shown for brevity. The strike prices are set by Equations (2) and (3), but with tail levels ranging from 1% to 25%.[7]

Figure 4 shows estimated option prices for all strike prices with tail levels ranging from 1% to 25%. Each green line links the two different option prices for a given security (blue triangle C_{CS} and red triangle C_{BS} for stock calls, or blue square P_{CS} and red square P_{BS} for market puts) with the same strike price. Note that ($C_{CS} - C_{BS}$) and ($P_{CS} - P_{BS}$) for all strike prices are positive. Next, in almost all cases, the option price differential is greater for options on the equally-weighted market ($P_{CS} - P_{BS}$) than it is for single stocks

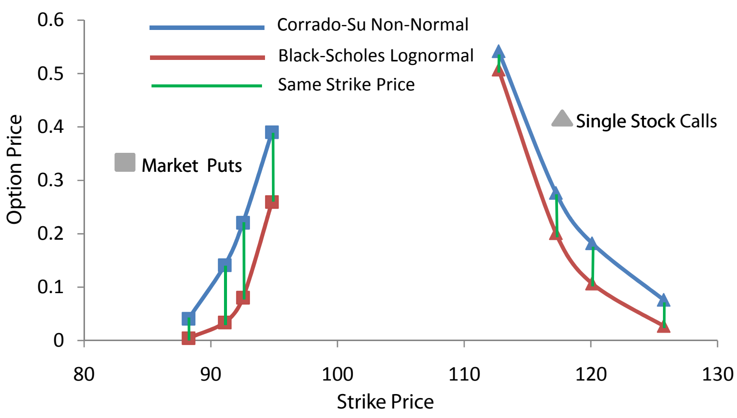

Figure 4 Option prices for average individual stock (call, C_{CS} vs. C_{BS}) and equally-weighted market (Put, P_{CS} vs. P_{BS}) for a range of strike prices at *monthly* frequency.

$(C_{CS} - C_{BS})$, suggesting that the departure of log-returns from the normal distribution is greater for diversified portfolios than it is for single stocks at monthly frequency (see Table 1B).

The corresponding annualized economic value of skewness loss is 1.02%, 2.59%, and 1.98% for the 1%, 10%, and 25% tail level, respectively, for *monthly* frequency. For unreported *daily* frequency, the corresponding annualized economic value of skewness loss is 13.09%, 32.80%, and 24.41% for the 1%, 10%, and 25% tail level, respectively.

Overall, the results are robust and consistent across a range of relevant strike prices. Once again, the economic value for daily frequency is much higher than the value for monthly frequency, indicating that the value is very significant at a short horizon and it decreases as the horizon increases. As the time horizon increases, log-returns eventually converge to the normal distribution, and the economic value of skewness loss should converge to zero as well. With that said, for the time horizons that are of importance to many investors and for the investor who has a skewness preference, the economic value of skewness loss is meaningful.

Our analyses provide useful information for portfolio construction and risk management for investors with skewness preferences and short horizons. Diversification requires investors to give up lottery-type gains and tolerate the market tail risk. For investors with strong skewness preferences and short horizons, they can hold a well-diversified portfolio, and additionally buy put options to hedge the market tail risk and call options to capture individual stock's upside potential gains. The combined excess option costs (priced by the Corrado–Su model) over the Black–Scholes model reveal the economic value of the skewness loss of diversification.

7 Conclusions

Diversification is widely viewed as the "only free lunch" of finance. However, literature has demonstrated that diversification not only leads to a reduction in variance—which is desired by investors—but it also leads to skewness loss — which is not desired by investors—suggesting a trade-off between variance and skewness that is integrated with the quest for return. At shorter time horizons, skewness loss is very robust and it can originate from some important economic forces, such as asymmetric correlation. We contribute to the literature by quantifying the economic value of skewness loss.

Skewness is the key in understanding the under-diversification puzzle observed across individual investor portfolios in which many individual investors hold just a handful of stocks rather than a diversified portfolio. Investors have preferences for skewness, leading them to buy lottery-type stocks (investors seeking positively skewed investments) or call options with upside potential, and insurance or put options to hedge downside risk (investors avoiding negatively skewed investments). We quantify the skewness loss using the corrected Corrado–Su option pricing model that incorporates skewness and kurtosis. The reference point is the lognormal distribution. The economic value of skewness loss is then represented by relevant option price adjustments on the standard Black–Scholes option pricing model.

The economic value of skewness loss is the additional option costs (28.25% for daily options or 2.20% for monthly options) that one must pay for hedging the left-tail risk for the market and capturing the lottery-type of gains for the average individual stock. The combined option costs are higher at daily frequency than those at monthly frequency because monthly log-returns are closer to the normal distribution than daily log-returns. This suggests that diversification is more beneficial at longer horizons where both individual stocks and market log-returns are converged to normal distribution—hence no skewness loss. Diversification requires investors to give up lottery-type gains (positive skewness) and tolerate the market tail risk (negative skewness). For investors with strong skewness preferences and short horizons, skewness loss is a meaningful cost for lunch.

Appendix. Option pricing for skewness and kurtosis

Jarrow and Rudd (1982) proposed a semiparametric option pricing model to account for observed strike price biases in the Black–Scholes (1973) model. They derive an option pricing formula from an expansion of the lognormal probability density function to model the distribution of stock prices. They used an Edgeworth expansion of the lognormal density function to write the option price as a function of the third and fourth movements of the terminal price distribution. The first two moments of the approximating distribution remain the same as that of the lognormal distribution, but third and fourth moments are introduced as the higher-order terms of expansion.

Similar to Jarrow and Rudd (1982), Corrado and Su (1996) extend the Black–Scholes formula by using a Gram–Charlier series expansion of a normal density function to account for non-normal skewness and kurtosis in stock return distributions. Both extensions are equally effective in providing accurate option price adjustment terms, and the difference is that the method developed by Corrado and Su accounts for skewness and kurtosis deviations from normality of stock log prices. It is more convenient to interpret results based on skewness and kurtosis deviations from normal distribution instead of lognormal distributions since normal distribution has a constant skewness of zero and kurtosis of 3, whereas skewness and kurtosis for lognormal distribution vary across different lognormal distributions.

Brown and Robinson (2002) correct a typographic error in Corrado and Su (1996) and the corrected version of skewness and kurtosis adjusted formulas for equity option pricing are as follows:

$$d = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{A.1}$$

$$Q_3 = \frac{1}{3!} S_0 \sigma \sqrt{T}((2\sigma\sqrt{T} - d) \times n(d) + \sigma^2 T N(d)) \tag{A.2}$$

$$Q_4 = \frac{1}{4!} S_0 \sigma \sqrt{T}((d^2 - 1 - 3\sigma\sqrt{T} \times (d - \sigma\sqrt{T}))n(d) + \sigma^3 T^{3/2} N(d)) \tag{A.3}$$

$$C_{BS} = S_0 N(d) - K e^{-rT} N(d - \sigma\sqrt{T}) \tag{A.4}$$

$$C_{CS} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3) Q_4. \tag{A.5}$$

Where C_{BS} is the Black–Scholes call option pricing formula. C_{CS} is the corrected Corrado and Su (1996) call option pricing formula. Q_3 and Q_4 represent the marginal effect of non-normal skewness (μ_3) and kurtosis (μ_4), respectively. The put option prices P_{CS} and P_{BS} can be calculated by using the put–call parity relationship:

$$P_{BS} + S_0 = C_{BS} + K e^{-rT} \tag{A.6}$$

$$P_{CS} + S_0 = C_{CS} + K e^{-rT}. \tag{A.7}$$

The rest parameters are:

- K is the strike price
- S_0 is the current stock price
- r is the risk-free rate
- σ is the volatility of log-returns for single stock or equally-weighted market
- T is the time to maturity

- $n(d)$ and $N(d)$ are the standard normal density function and the cumulative probability distribution function for a standard normal distribution, respectively.

Notes

[1] Kraus and Litzenberger (1976) argue that kurtosis and higher moments can be ignored from the perspective of a positive theory of valuation.

[2] This exchange is now called the NYSE American.

[3] In contrast, the average individual stock skewness for low-priced stocks (price $< \$5$) increases over time. This observation might deserve a separate future research.

[4] The kurtosis results are not reported in Figure 2 for brevity. However, we report partial results in Table 1. In contrast to the monotonic reduction of both standard deviation and skewness in Figure 2, diversification results in a small non-monotonic increase in kurtosis with monthly returns and a small non-monotonic reduction with daily returns. In other words, the impact of diversification on kurtosis is mixed and less significant.

[5] Statman (2004) points out that behavioral portfolio theory suggests "people act as if they are composed of several 'doers,' each with a different goal and attitude toward risk," and that lottery tickets are best for upside-potential doers with high aspiration levels and little money. Upside-potential doers with lower aspiration levels can meet their needs through call options, and those with even lower aspiration levels can buy stocks.

[6] Note that the skewness for an average single stock is 0.3 for simple monthly returns (see Figure 2), and it is lowered to nearly zero (-0.08) for monthly log-returns. Based on Equation (A.5) in Appendix, negative skewness will decrease C_{CS} (both Q_3 and Q_4 are positive at the strike price of \$120.11). However, contribution from excess kurtosis is more positive and thus $(C_{CS} - C_{BS})$ is positive.

[7] Specifically, the 2.06 in Equations (2) and (3) is replaced by 2.67 for the 1% tail level, by 1.76 for the 10% tail level, and by 1.27 for the 25% tail level.

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