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SURVEYS AND CROSSOVER

This section provides surveys of the literature in investment management or short papers exemplifying advances in finance that arise from the confluence with other fields. This section acknowledges current trends in technology, and the cross-disciplinary nature of the investment management business, while directing the reader to interesting and important recent work.

RISK, REWARD, AND BEYOND: ON THE BEHAVIORAL SENSITIVITIES OF MEAN–VARIANCE EFFICIENT PORTFOLIOS

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This paper surveys and extends the literature on the behavioral sensitivities of mean–variance efficient portfolios. We also compare the optimal portfolio allocation that results from either mean–variance or behavioral optimization. Near equivalence is concluded in case of normally distributed returns, based on the analytical expression of a general performance measure. The analysis contributes to a further exploration of the link between the mean–variance framework and insights from behavioral finance, and particularly expands one's capabilities to construct client-centric portfolios. Program code in Python for all the mathematics in the paper is also provided.

1 Introduction

The seminal work of Markowitz (1952) on portfolio construction has shaped an entire business. His mean–variance framework derives the portfolio composition that yields the highest expected return, being the measure of reward, and optimally aligns with the investors' attitude towards variance, as a measure of risk. However, the money management industry currently experiences a content-driven transformation, fueled by insights from behavioral finance, see Lo (2017) for a motivation. With respect to portfolio construction, behavioral finance takes a different look at risk and reward. Shefrin and Statman (2000), for example, represent portfolios as a collection of goals. The measure of reward is then linked to reaching or overshooting the goal, while the measure of risk refers to the extent or the probability of not reaching the goal.

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Exploring the link between a mean/variance approach and a behavioral approach is relevant because it deepens our understanding of how the optimal asset allocation depends on the definition of risk, the definition of reward, or investor preferences. The literature in this area deals with either one of these aspects. For example, Das *et al.* (2010) show the mathematical equivalence between the Markowitz (1952) mean–variance model and an optimization that maximizes the expected return, given a probability of not reaching a return target. Continuing the assumption of normality, Levy and Levy (2004) show semi-equivalence with respect to alternative definitions of investor preferences. Their analysis reveals that the efficiency sets obtained via standard risk averse utility or via behavioral utility almost coincide. Zakamouline (2014) uses the same assumption of behavioral utility and finds that the optimal portfolio allocation maximizes a performance measure that balances upside potential and downside risk. This suggests another equivalence with respect to alternative definitions of both risk and reward.

The present paper derives the analytical expression of the performance measure put forward by Zakamouline (2014), which allows us to extend the work of Das *et al.* (2010). Specifically, we replace expected return by upside potential as a quantification of reward in parallel to the replacement of variance by downside risk as a quantification of risk. The analysis then focuses on the behavioral sensitivities of mean–variance efficient portfolios and on a comparison of the optimal portfolio allocation that results from either mean–variance or behavioral optimization.

2 Mean–variance optimization linked to behavioral portfolios

Das *et al.* (2010) establish the mathematical link between mean–variance optimization and behavioral portfolios. This section briefly recaps their main results, on which we build in the remainder is this paper.

Mean–variance optimization in the tradition of Markowitz (1952) finds the combination of assets that yields the highest expected return, given a stated level of variance. Let $w \in \mathcal{R}^n$ represent the column vector of portfolio weights for n assets, $\sum \in \mathcal{R}^{n\times n}$ the covariance matrix of the n asset returns, and $\mu \in \mathcal{R}^n$ the column vector of n expected returns. Return distributions are entirely described by their mean and variance through the assumption of normality. In addition, investor preferences imply that more wealth is preferred over less and that risk, quantified by variance, is disliked. The mean–variance efficient asset allocation for an investor with risk aversion parameter γ is found by maximizing the following utility function:

$$\max_w w^\top \mu - \frac{\gamma}{2} w^\top \sum w$$

As we can see, the utility here is a function only of the first moments of all the assets embodied in the vector of asset returns μ and the second moments, captured by covariance matrix of returns $\sum$. The trade-off between the first and second moments (i.e., return versus risk) is modulated by the risk preference parameter γ. The solution to this problem is as follows (see Das *et al.*, 2010, p. 316):

$$\begin{aligned} w^* &= \frac{1}{\gamma} \sum\nolimits^{-1} \left[\mu - \left(\frac{a-\gamma}{b}\right)\right] \\ a &= \mathbf{1}^\top \sum\nolimits^{-1} \mu \\ b &= \mathbf{1}^\top \sum\nolimits^{-1} \mathbf{1} \end{aligned} \tag{1}$$

where $\mathbf{1} \in \mathcal{R}^n$ is a unit vector, short selling is allowed, and full investment holds such that $w^\top \mathbf{1} = 1$.

Das *et al.* (2010) show that the solution in Equation (1) for the mean–variance efficient portfolio is also obtained if the optimization problem is alternatively phrased in terms of behavioral finance. The main difference is that risk is defined as the probability of not reaching a target. Put differently, the risk preference of the investor—previously quantified as γ—is now implied by an expression which states that the probability of not reaching a portfolio return target H should at the most equal α:

$$P(r < H) \leq \alpha \tag{2}$$

Equation (2) can also be written as follows if multivariate normality of asset returns is assumed:

$$H \leq w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \sum w} \tag{3}$$

where $\Phi^{-1}(\cdot)$ is the inverse normal function. The inequality in Equations (2) and (3) is an equality when optimality is achieved. For a specific value of risk aversion γ in Equation (1), Equation (3) holds with equality, and this value of γ is the "implied" risk aversion of an investor in the mean–variance world for portfolio preferences (H, α) specified in the behavioral world. We note that this mapping from the behavioral world to the mean–variance world is a special case of the behavioral portfolio theory (BPT) of Shefrin and Statman (2000), where the assumption of quadratic utility is imposed, and therefore the portfolios lie on the efficient frontier. There is no such requirement in BPT, as behavioral effects may lead investors to eschew portfolios that lie on the efficient frontier, for example, as might occur when investors display loss aversion.

Das *et al.* (2010) state three main consequences of their results. First, optimal behavioral portfolios over quadratic utility specifications are also mean–variance efficient. The framing of the problem statement, notably the definition of risk, is hence relative, in this setting. Defining risk as variance or alternatively defining risk as the probability of not reaching a target leads to the same portfolio allocation. This is useful for practitioners as they can introduce behavioral portfolio concepts without relinquishing mean–variance theory. Second, each optimal behavioral portfolio preference set implies a particular level of risk aversion in the mean–variance world. The seminal work of Shefrin and Statman (2000) defines behavioral portfolios as layers that are associated with distinct goals. Das *et al.* (2010) allow the linking of these distinct mental accounts to different levels of risk aversion. The third consequence of their analysis is most relevant in the current context. Das *et al.* (2010) demonstrate how, for a given level of risk, the optimal mean–variance portfolio maps on a multiple of H, α combinations that yield optimal behavioral portfolios. We extend this conclusion beyond the mere definition of risk by linking the H, α combination to a behavioral performance measure. This implies that the equivalence between mean–variance and behavioral portfolio optimization continues to hold if we redefine the trade-off between risk and reward.

3 Upside potential and downside risk

This section starts by presenting a general performance measure. Like any performance measure, the intention is to quantify the balance between risk and reward. As in behavioral portfolio theory, we define risk in terms of downside risk rather than variance. Similarly, reward is defined in terms of doing better than a predefined target, referred to as upside potential. The performance measure exhibits particular properties in case of optimal portfolios obtained with Equation (1) from the previous section.

Assume a distribution of returns r_i with $i = 1, \ldots, N$. Furthermore, let the return level H distinguish between above- and below-target returns. Now define the upside potential of the distribution as

$$UP(r, H) = \frac{1}{N}\sum_{i=1}^{N} \max(0, r_i - H) \tag{4}$$

and the downside risk of the same distribution as

$$DR(r, H) = \frac{1}{N}\sum_{i=1}^{N} \max(0, H - r_i) \qquad (5)$$

The ratio $UP(r, H)/DR(r, H)$ is motivated by Shadwick and Keating (2002) as a universal performance measure. A generalized version that includes risk preferences, potentially different for above- and below-target returns, is provided by Farinelli and Tibiletti (2008). Zakamouline (2014) demonstrated the validity of the performance measure to rank return distributions, regardless of any distributional assumptions.

We will now return to the assumption that the portfolio returns r_i are normally distributed with mean $\mu_p = w^\top \mu$ and standard deviation $\sigma_p = \sqrt{w^\top \sum w}$. In this case the equation for upside potential and downside risk has an analytical solution. To see this, first note how the upside potential defined in Equation (4) can be written as the product of the average above-target return and the probability of an above-target return:

$$UP(r, H) = \frac{\sum_{i=1}^{N} I(r_i > H)}{N} \cdot \frac{\sum_{i=1}^{N} \max(0, r_i - H)}{\sum_{i=1}^{N} I(r_i > H)} \qquad (6)$$

with indicator function $I()$ equal to 1 if the condition between the brackets is met and 0 otherwise. Expression (6) can be written in continuous terms as

$$UP(r, H) = P(r > H) \cdot E(r - H \mid r > H) \qquad (7)$$

A similar reformulation holds for downside risk

$$DR(r, H) = P(r \leq H) \cdot E(H - r \mid r \leq H) \qquad (8)$$

In case of normally distributed returns r_i the expected or average below-target return is calculated as:

$$E(H - r \mid r \leq H) = (H - \mu_p) + \frac{\sigma_p \phi[\Phi^{-1}(\alpha)]}{\alpha} \qquad (9)$$

with $\alpha = P(r \leq H)$ as in the previous section, $\Phi()$ the cumulative standard normal distribution, and $\phi()$ the density function of the standard normal distribution. Substitution of Equation (9) in Expression (8) yields a closed form for the quantification of downside risk

$$DR(r, H) = (H - \mu_p)\alpha + \sigma_p \phi[\Phi^{-1}(\alpha)] \qquad (10)$$

3.1 *Numerically verify the solution for DR*

See that, assuming normality, Equation (5) may be written as

$$DR(r, H) = E[\max(0, H - r)] = \int_{-\alpha}^{H} (H - r)\phi(r)dr \qquad (11)$$

We check this numerically as follows using the following parameter choices: $\mu_p = 0.06$; $\sigma_p = 0.15$; $H = -0.05$; and $\alpha = \Phi(H)$. The code below shows that this is indeed correct!

```
In [1]: %pylab inline
        import pandas as pd
```

```
Populating the interactive namespace from numpy and matplotlib
```

```
In [2]: #Calculations using Python
        from scipy.stats import norm

        mu_p = 0.06
        sigma_p = 0.15
```

```
H = −0.05
alpha = norm.cdf((H-mu_p)/sigma_p)
print('alpha=',alpha)

DR_eqn10 = (H - mu_p)*alpha + sigma_p * norm.pdf
(norm.ppf(alpha))
print ('DR_eqn10=',DR_eqn10)

from scipy.integrate import quad
DR_eqn10_check = quad(lambda x: (H-x)*norm.pdf
(x,mu_p,sigma_p),−10, H)
print('DR_eqn10_check=',DR_eqn10_check[0])
```

```
alpha= 0.23167757463479827
DR_eqn10 = 0.020247905834249767
DR_eqn10_check= 0.02024790583424977
```

Finally, we calculate this expression for an optimized portfolio we know from Equation (3) that $\mu_p - H = \sigma_p \cdot \Phi^{-1}(\alpha)$ and we can write Equation (10) as

$$DR(r, H) = \sigma_P \cdot \Phi^{-1}(\alpha) \cdot \alpha + \sigma_p \cdot \phi[\Phi^{-1}(\alpha)] \quad (12)$$

In [3]: *#Check the equations above*

```
DR_eqn11_check = sigma_p*norm.ppf(alpha)*alpha + sigma_p*norm.pdf
(norm.ppf(alpha))
print('DR_eqn11_check=',DR_eqn11_check)

DR_eqn11_check= 0.020247905834249767
```

3.2 Deriving upside potential

The upside potential is found by exploiting the general property $UP(r, H) - DR(r, H) = \mu_p - H$:

$$UP(r, H) = \sigma_p \cdot \Phi^{-1}(\alpha) \cdot (\alpha - 1) + \sigma_p \cdot \phi[\Phi^{-1}(\alpha)] \quad (13)$$

This corresponds to the integral formulation

$$UP(r, H) = E[\max(0, r - H)] = \int_H^\infty (r - H)\phi(r)dr \quad (14)$$

In [4]: *#Check the equations above from first principles using integration*

```
UP_eqn12 = sigma_p*norm.ppf(alpha)*(alpha-1)
+ sigma_p*norm.pdf(norm.ppf(alpha))
print('UP_eqn12=',UP_eqn12)

from scipy.integrate import quad
UP_eqn12_check = quad(lambda x: (x-H)*norm.pdf(x,mu_p,sigma_p),H,10)
print('UP_eqn12_check=',UP_eqn12_check[0])
```

```
UP_eqn12=0.13024790583424978
UP_eqn12_check=0.13024790583424978
```

4 A performance metric

As a result, the performance measure of the optimized portfolio simplifies into

$$\Omega = \frac{UP(r, H)}{DR(r, H)} = \frac{\Phi^{-1}(\alpha) \cdot (\alpha - 1) + \phi[\Phi^{-1}(\alpha)]}{\Phi^{-1}(\alpha) \cdot \alpha + \phi[\Phi^{-1}(\alpha)]} \tag{15}$$

Note that Equation (15) does not include the target level H. This is an interesting observation, given that Das *et al.* (2010) map any given level of risk aversion on a multiple of (H, α) combinations. What does this mean? Of course, the mathematical relationship of Das *et al.* (2010) between risk aversion coefficient γ and a (H, α) pair continues to hold. However, while any γ maps onto a multiple of (H, α) combinations, the performance measure that corresponds to any of these portfolios only depends on α, not on H. Or alternatively, the performance measure depends on H, not on α. The intuition for this comes from the fact that once a specific portfolio with a given return–risk trade-off, embodied in (μ_p, σ_p) is chosen, then α and H are tied together from the fact that $\Phi[(H - \mu_p)/\sigma_p] = \alpha$. We may in fact write $\alpha(H)$, i.e., probability becomes a function of the threshold H. But note that there are many (μ, σ) pairs that satisfy the BPT conditions stipulated by (H, α).

We note the following properties of Ω:

(1) The performance measure Ω may be computed *ex-ante* or *ex-post*. When the portfolio is established the forecast (μ_p, σ_p) may be used to determine the projected Ω upfront. Here, normality may be assumed, if only for convenience. *Ex-post*, the measure may be computed directly using the series of returns from Equations (4) and (5).

(2) If normality is not to be assumed, then Equations (14) and (11) may be used instead with the PDF $\phi()$ replaced by the density function for any other probability function, or even an empirical density function.

(3) Because of the use of the normal distribution, the measure Ω does not explicitly contain the parameter σ_p. However, this may not be true for other distributions.

Under optimality, as we have seen already, the constraint Equation (3) is exactly satisfied:

$$H = \mu_p + \Phi^{-1}(\alpha)\sigma_p \tag{16}$$

where $\mu_p = w^\top \cdot \mu$ and $\sigma_p = \sqrt{w^\top \cdot \sum \cdot w}$. Therefore, fixing μ_p and σ_p, we get a huge collection of pairs of (H, α) that may be satisfied by the same set of (μ_p, σ_p). This is shown in Figure 1.

Now consider Figure 2, with two different return distributions. The distribution in the upper panel is derived for a risk aversion parameter γ_1, while the distribution in the lower panel is derived for a risk aversion parameter γ_2. The shape and positioning of the distributions reveal that γ_1 yields an optimized portfolio with higher mean and higher variance than γ_2.

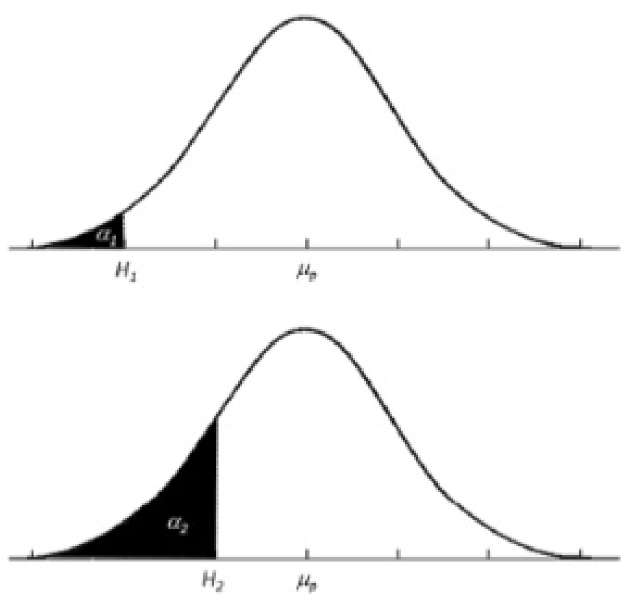

Figure 1 Alternate (H, α) for the same distribution.

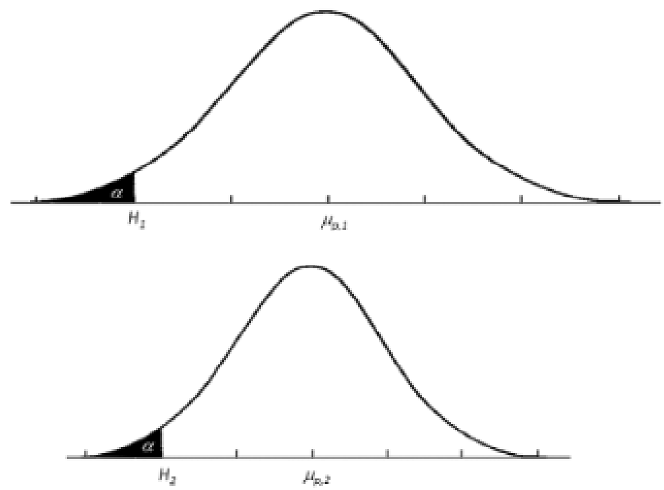

Figure 2 (H, α) combinations with common α for two different return distributions for the same distribution.

Figure 2 highlights in both distributions the (H, α) combination with common α. Equation (15) implies that the two distributions of Figure 2 exhibit the same balance between upside potential and downside risk when evaluated at α, regardless of the difference in mean or variance. This facilitates any comparison between optimized mean–variance and behavioral portfolios. We elaborate on this comparison in the next section.

5 Behavioral sensitivities of mean–variance optimized portfolios

In this section we investigate the sensitivities of mean–variance optimized portfolios with respect to parameters related to behavioral theory. We exploit the result that concluded the previous section and expands the relevance of Das *et al.* (2010). A simplified three-asset example is copied from Das *et al.* (2010) to perform the analysis. The assets have a mean vector and covariance matrix of returns as follows:

$$\mu = \begin{bmatrix} 0.05 \\ 0.10 \\ 0.25 \end{bmatrix} \tag{17}$$

$$\sum = \begin{bmatrix} 0.0025 & 0.0000 & 0.0000 \\ 0.0000 & 0.0400 & 0.0200 \\ 0.0000 & 0.0200 & 0.2500 \end{bmatrix} \tag{18}$$

Figure 3a plots the expected return of the optimal portfolio for alternative pairs of H and α, with H the target level and α the probability of not reaching the target. Figure 3a replicates Figure 4, page 323 of Das *et al.* (2010). The value for H is -5%, -10%, or -15%. Corresponding threshold probabilities are 0.15, 0.05, 0.20, respectively. Note that in all of these cases a return of -4% is considered "above target" and as such contributes to the upside potential. With $H < 0$ the investor attitude implies a true focus on downside risk. Given these values for H, we observe that the optimal portfolio comes with a higher expected return if the investor allows for a higher α. Put differently, the return distribution of the optimal portfolio has a higher expected value (and a higher variance) if the investor is more tolerant with respect to reaching a negative return target. Equivalently, we observe in Figure 3a that the expected return increases if the investor, given the tolerance in terms of α, sets a less ambitious return target $H < 0$. The program code for the example follows.

```
In [5]: #GENERATE AND CHECK SOLUTION HERE
        from scipy.optimize import fsolve, brute, minimize

        MU = matrix([0.05,0.10,0.25]).T
        SIG = matrix([[0.0025,0.0,0.0], [0.0,0.04,0.02],
              [0.0,0.02,0.25]])

        def CONSTRC(gam,H,alpha,MU,SIG):
             wuns = matrix(ones(len(MU))).T
             Sinv = inv(SIG)
             a = float(wuns.T.dot(Sinv).dot(MU))
             b = float(wuns.T.dot(Sinv).dot(wuns))
             cvec = MU - wuns*float((a-gam)/b)
             w = Sinv.dot(cvec)*float(1.0/gam)
             return float(w.T.dot(MU) + norm.ppf(alpha)*sqrt
             (w.T.dot(SIG).dot(w)) - H)

        def GAM(H,alpha,MU,SIG):
             sol = fsolve(C0NSTR,0.1,args=(H,alpha,MU,SIG))
             return sol[0]

        def WTS(H,alpha,MU,SIG):
             gam = GAM(H,alpha,MU,SIG)
             wuns = matrix(ones(len(MU))).T
             Sinv = inv(SIG)
             a = float(wuns.T.dot(Sinv).dot(MU))
             b = float(wuns.T.dot(Sinv).dot(wuns))
             cvec = MU - wuns*float((a-gam)/b)
             w = Sinv.dot(cvec)*float(1.0/gam)
             return w

In [6]: #Case 1: (test)
        H = -0.10; alpha = 0.05
        gam = GAM(H,alpha,MU,SIG)
        print("Risk Aversion (gamma) = ",gam)
        wts = WTS(H,alpha,MU,SIG)
        print("Weights = ")
        print(wts)

Risk Aversion (gamma) = 3.795014902838235
Weights =
[[0.53943223]
 [0.2656202]
 [0.19494757]]

In [7]: #THREE CASES
        H_list =  [-0.10, -0.05, -0.15]
        alpha_list =  [0.05, 0.15, 0.20]
```

Figures that follow are presented in two parts: (i) code to generate the figure, and (ii) the figure itself. The code for the figures is presented first on the following page.

```
CODE FOR FIGURE 3a
In [8]: figure()
        H_list = [−0.10, −0.05, −0.15]
        A = linspace(0.05,0.30,30)
        for H in H_list:
               mu_p = []
               for alpha in A:
                    w = WTS(H,alpha,MU.SIG)
                    mu_p = append(mu_p,float(w.T.dot(MU)))
               plot(A,mu_p); grid(); xlabel ('$P(r<H)$');
               ylabel('Expected portfolio return ($μ_p$)')
        legend(['H = −10%','H = −5%','H = −15%'])
        show()

CODE FOR FIGURE 3b
In [9]: figure()
          H_list = [−0.10, −0.05, −0.15]
          A = linspace(0.05,0.30,30)
          for H in H_list:
               gam_p = []
               for alpha in A:
                    gam_p = append(gam_p,GAM(H,alpha,MU,SIG))
               plot(A,gam_p); grid(); xlabel('$P(r<H)$');
               ylabel('Risk aversion ($γ$)')
          legend(['H = −10%','H = −%5','H = −15%'])
          show()
```

The optimal portfolios that make up Figure 3a all come with a distinct degree of risk aversion. The corresponding level of risk aversion is shown in Figure 3b, again as a function of (H, α) pairs. We logically observe an inverse relation between the expected return of the optimal portfolio in Figure 3a and the degree of risk aversion in

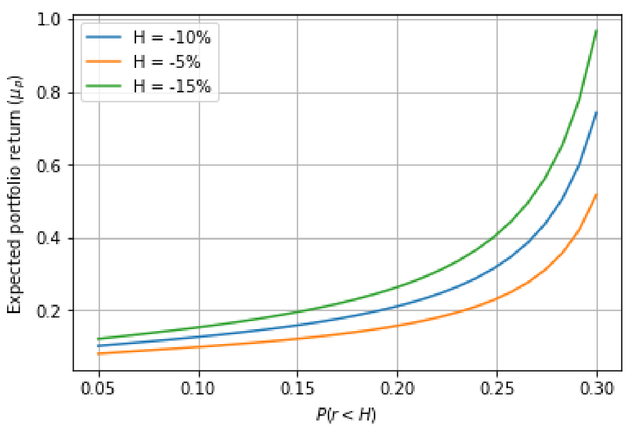

Figure 3a The expected return of optimized portfolios as a function of not reaching a target. The target level that distinguishes between gains and losses equals −5%, −10%, or −15%.

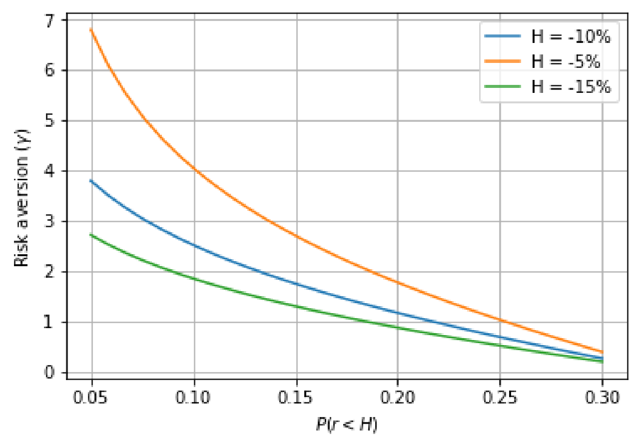

Figure 3b The implied risk aversion parameter of optimized portfolios as a function of not reaching a target. The target level that distinguishes between gains and losses equals −5%, −10%, or −15%.

Figure 3b. For this reason the curves in Figure 3b are downward sloping. In addition, more negative levels of H map on to more risk tolerant investors. For example, the curve for $H = -15\%$ is situated the highest in Figure 3a and the lowest in Figure 3b.

Negative values for the target level H, as shown in Figures 3a and 3b, focus on the tolerance toward negative scenarios. We next focus our attention on positive values for the target level H. This can be equally relevant for any investor. Or, a behavioral portfolio might contain mental accounts that span a range of target levels H.

In addition, an examination of the sensitivities in case of higher target levels for H will sharpen our understanding. This is because it makes sense to consider higher levels for α as H increases. In Figures 3a and 3b the value for α stopped at 30% because investment goals that are permitted to fail a target of −5% or −15% with a probability above 30% make little economic sense. However, it is realistic to consider a 40% probability of not reaching the target in case such target is set at +15%, for example. Note that for $\alpha = 50\%$, Equation (15) implies that the distribution of the optimal portfolio satisfies $UP(r, H) = DR(r, H)$. And since it holds for any portfolio that $UP(r, H) - DR(r, H) = \mu_p - H$, the optimal portfolio additionally has $\mu_p = H$ for $\alpha = 50\%$. Put differently, due to the assumption of normality the expected return of the optimal portfolio equals the target level H if $\alpha = 50\%$.

For $\alpha < 50\%$, the optimal portfolio exhibits upside potential that exceeds downside risk. For this reason we will ignore levels $\alpha > 50\%$ even in case we consider large positive values H. Figures 4a and 4b plot the results for targets of 10%, 15%, and 20%. The tolerance level of not meeting the target now ranges from 35% to 50%.

The curves in Figure 4 exhibit a different slope compared to the curves in Figure 3. Figures 4a and 4b confirm that a more ambitious positive target increases the expected return of the optimal portfolio and implies more risk tolerance, given the probability of not reaching the positive target. The curve that corresponds to a target of 20% is positioned on top in Figure 4a because of the high expected return and at the bottom in Figure 4b because of the low-risk aversion. Finally, note how in Figure 4a the expected return of the optimized portfolio indeed converges to the target level H as α moves toward 50%.

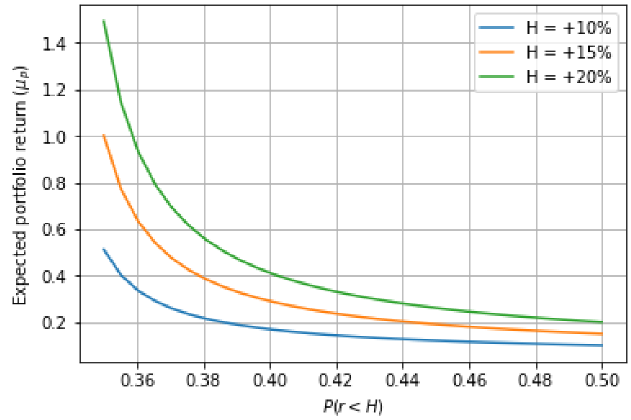

Figure 4a The expected return of optimized portfolios as a function of not reaching a target. The target level that distinguishes between gains and losses equals 10%, 15%, or 20%.

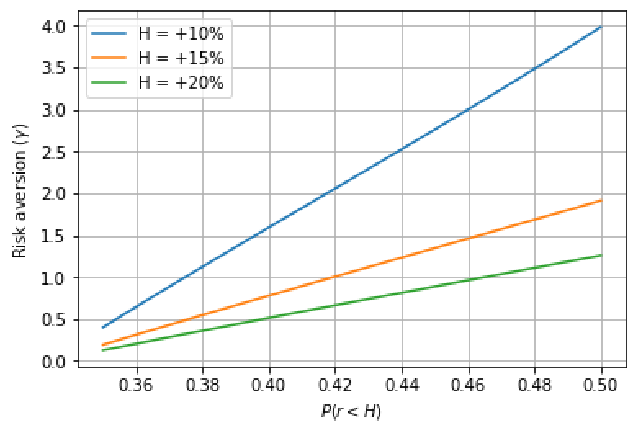

Figure 4b The implied risk aversion parameter of optimized portfolios as a function of not reaching a target. The target level that distinguishes between gains and losses equals 10%, 15%, or 20%.

```
CODE FOR FIGURE 4a
In [10]: figure()
           H_list = [0.10,0.15,0.20]
           A = linspace(0.35,0.50,30)
           for H in H_list:
                mu_p = []
                for alpha in A:
                     w = WTS(H,alpha,MU,SIG)
                     mu_p = append(mu_p,float(w.T.dot(MU)))
                plot(A,mu_p); grid(); xlabel('$P(r<H)$')
                ylabel('Expected portfolio return ($\mu_p$)')
           legend(['H = +10%', 'H = +15%', 'H = +20%'])
           show()

CODE FOR FIGURE 4b
In [11]: figure()
           H_list = [0.10,0.15,0.20]
           for H in H_list:
                gam_p = []
                for alpha in A:
                     gam_p = append(gam_p,GAM(H,alpha,MU,SIG))
                plot(A,gam_p); grid(); xlabel('$P(r<H)$');
                ylabel('Risk aversion ($\gamma$)')
           legend(['H = +10%', 'H = +15%', 'H = +20%'])
           show()
```

6 Redefining the optimization

The previous section summarizes the return distribution of mean–variance efficient portfolios by means of a performance measure that quantifies the upside potential per unit of downside risk. In this section we analyze whether the same optimal portfolio, and hence return distribution, would result if we redefine the optimization directly into a problem that maximizes upside potential, given the same level of downside risk. DeGiorgi (2011) and Cumova and Nawrocki (2014) investigate the link between mean–variance efficiency and behavioral portfolio theory by means of a similar alternative risk–reward formulation. DeGiorgi (2011) showed how behavioral portfolios explain the observed violation of the two-fund separation property of mean–variance efficient portfolios. The analysis of Cumova and Nawrocki (2014) compares the efficient frontier in both settings. We will focus on the stability of the optimal portfolio allocation, or lack thereof.

The analysis proceeds as follows.

- First, we simulate returns for three assets that are considered to be included in the portfolio. The return characteristics of the assets are copied from the example in the previous section.
- Given these simulations, the second step is to seek the asset allocation that maximizes $w^\top \mu - \frac{\gamma}{2}\sqrt{(w^\top \sum w)}$. We find the optimal combination after considering all weighting schemes that exclude short selling, as in Das and Statman (2013). Put differently, we examine all combinations (w_1, w_2, w_3) with $0 \leq w_1 \leq 1, 0 \leq w_2 \leq 1, 0 \leq w_3 \leq 1$ and $w_1 + w_2 + w_3 = 1$. We let w_1, w_2, and w_3 vary in steps of 1%. Obviously this yields the same result as Equation (1) in case the lack of short selling turns out not to be binding.
- Step three calculates the downside risk of the optimal portfolio found in step two. We use Equation (5) to perform the calculation.

- Step four again examines all combinations (w_1, w_2, w_3) but now to detect the portfolio allocation that maximizes the upside potential according to Equation (4) for a downside risk that at the most equals the downside risk of step three.
- The last step uses the downside risk of the mean–variance efficient portfolio as a boundary condition when maximizing the upside potential.

In order to make the analysis more specific we consider three different investment goals, represented by different levels of risk aversion to start the optimization. Similar to Das *et al.* (2010) we assume a retirement portfolio, an education portfolio and a bequest portfolio which show up in Table 1 with a decreasing level of risk aversion. For each of the portfolios we start by using Equation (1) to find the closed-form optimal allocation as a reference. This allocation is indicated in Table 1 as composition (A). Note how the weight of asset 1 falls in parallel with the level of risk aversion. Asset 1 has the lowest expected return and the lowest variance. The opposite can be observed for the most risky asset 3. Composition (B) in Table 1 is for each portfolio found as the mean–variance efficient portfolio based on the simulations. We use 12,500 random drawings and apply the antithetic technique to obtain 25,000 simulations of 1 year returns. Equations (5) and (6) generate respectively the upside potential and downside risk of the return distribution that corresponds to this allocation (B). The downside risk of the mean–variance efficient allocation is indicated in Table 1 as "DR" and serves as a boundary condition for the final step. The exercise ends by seeking the allocation that maximizes the upside potential for a downside risk at the most equal to the boundary level. The result is labeled "UP" in Table 1. Program code for Table 1 follows, and is based on analytical results, with numerical implementations for optimization and integration, achieved without simulation.

```
In [12]: # Program code to generate Table 1
         #NUMERICAL OPTIMIZATION WITH NO SHORT SELLING
         from scipy.optimize import minimize
         def obj_fn(w,cv,mu,gam):
              res = w.T.dot(mu) − (gam/2.0)*(w.T.dot(cv).dot(w))
              return float(−1.0*res[0])

         cons = ({'type': 'eq', 'fun': lambda x: float(sum(x)−1.0)},
                  {'type': 'ineq', 'fun': lambda x: float(min(x)−0.0)}
                )

In [13]: #THREE CASES
         H_list = [-0.10, -0.05, 0.10]
         alpha_list = [0.05, 0.15, 0.41]
         cv = SIG
         mu = MU

         #GAM = [3.795, 2.7063, 1.8249]
         wO = matrix([0.3,0.3,0.4]).T
         GAM = linspace(0.1,10,1000)
         ww = []
         gg = []
         for j in range(3):
              H = H_list [j]; alpha = alpha_list[j]
```

```
          for g in GAM:
               sol = minimize(obj_fn,wO,args = (cv,mu,g),
               method = "SLSQP",constraints = cons)
               w = sol.x
               #BPT Constraint satisfied or not
               chk = abs(w.T.dot(mu) + norm.ppf(alpha)*sqrt
               (w.T.dot(cv).dot(w)) − H)
               if chk < 0.001:
                 wstar = w
                 gstar = g
          ww = append(ww, wstar)
          gg = append (gg, gstar)

In [14]: print("Retirement portfolio wts: ",ww[:3].round(4))
          print("Education portfolio wts: ",ww[3:6].round(4))
          print("Bequest portfolio wts: ",ww[6:9].round(4))
          print("Implied risk aversion (gamma): ",gg)

Retirement portfolio wts: [0.5418 0.2646 0.1936]
Education portfolio wts: [0.3848 0.3467 0.2685]
Bequest portfolio wts: [0.1061 0.4928 0.401]
Implied risk aversion (gamma): [3.81621622 2.73603604 1.82432432]

In [15]: #Generate the table
          results = zeros((9,7))
          for j in range(3):
               w = ww[j*3:(j*3+3)]
               mu_p = w.T.dot(mu)
               sigma_p = sqrt(w.T.dot(cv).dot(w))
               i = 0
               for H in [−0.2, −0.15, −0.1, −0.05, 0.0, 0.05, 0.1, 0.15, 0.2]:
                    results[i,0]=H
                    DR = quad(lambda x: (H-x)*norm.pdf(x,mu_p,sigma_p),
                    −10,H)[0]*100
                    UP = quad(lambda x: (x-H)*norm.pdf(x,mu_p,sigma_p),
                    H,10)[0]*100
                    results [i,j+j+1]=DR; results[i,j+j+2]=UP
                    i = i+1
          results = pd.DataFrame(results)
          results.columns=["H","Retirement_DR","Retirement_UP",
          "Education_DR","Education_UP", "Bequest_DR","Bequest_UP"]
          results
```

Table 1 reports for each portfolio the targeted downside risk and the optimized upside potential in case of target returns H ranging from −20% to 20%. Note how the downside risk increases when reading from top to bottom and when reading from left to right. From top to bottom the downside risk increases because there is an increasing likelihood of realizing a return below the increasing target H. The distributional assumption of the underlying asset returns impacts this conclusion. From left to right the downside risk increases because the optimal return distribution is more dispersed due to the increasing level of risk tolerance.

Table 1 Results of mean–variance and upside potential, downside risk portfolio optimization.

	RETIREMENT portfolio		EDUCATION portfolio		BEQUEST portfolio	
Risk aversion γ	3.795		2.7063		1.8249	
Asset allocation	(A)	(B)	(A)	(B)	(A)	(B)
w_1	53.94%	54%	37.87%	37%	10.82%	13%
w_2	26.56%	26%	34.99%	36%	49.17%	48%
w_3	19.49%	20%	27.14%	27%	40.01%	39%

H	Retirement DR (%)	Retirement UP (%)	Education DR (%)	Education UP (%)	Bequest DR (%)	Bequest UP (%)
−0.20	0.03	30.22	0.16	32.26	0.75	36.23
−0.15	0.09	25.28	0.34	27.44	1.17	31.66
−0.10	0.25	20.45	0.68	22.78	1.79	27.27
−0.05	0.63	15.82	1.26	18.36	2.64	23.12
0.00	1.38	11.58	2.20	14.30	3.78	19.26
0.05	2.71	7.91	3.60	10.70	5.25	15.74
0.10	4.78	4.98	5.55	7.65	7.10	12.59
0.15	7.65	2.85	8.10	5.20	9.36	9.84
0.20	11.27	1.47	11.24	3.34	12.02	7.51

7 Conclusion

This paper contributes to a better understanding of the link between mean–variance and behavioral portfolios. We found optimal portfolio allocations through an optimization that applies a definition of both risk and reward inspired by behavioral theory. We explored the behavioral sensitivities of mean–variance portfolios through the analytical expression of a general performance. The analysis demonstrated "near" identity of mean–variance and upside potential-downside risk optimization in case of normally distributed asset returns.

The optimization exercise is a numerical illustration of the conclusion by Levy and Levy (2014) that in case of normality, behavioral efficiency is "almost" identical to mean–variance efficiency. In many real-life situations, however, the assumption of normality is violated. Possibly, even on purpose, whenever portfolio constituents are selected on the criterion of generating asymmetric return distributions. Behavioral investor preferences favor positively skewed return distributions of, for example, portfolio insurance or options, see Barberis and Huang (2008), Bernard and Ghossoub (2010), or Das and Statman (2013). The article advocates pursuing an upside potential-downside risk optimization, regardless of the distributional properties of the portfolio constituents.

Apart from the methodological arguments outlined above, DeGiorgi and Hens (2009) show that gains of a behavioral approach are additionally to be found in the client advisory process. Upside potential and downside risk are more easily communicated to clients than expected return or variance. In short, upside potential-downside risk optimization contributes to what Meir Statman calls "finance for normal people" (Statman, 2017).

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