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TIME AGGREGATION OF SHARPE RATIO—A BETTER EXTRAPOLATION RULE

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The $\sqrt{T}$ rule extrapolates a one-period Sharpe Ratio to T periods. But the rule ignores compounding. By considering compounding, Levy (1972) and others show that the Sharpe Ratio changes non-monotonically with horizon. We also theoretically and empirically show that the Sharpe Ratio term structure is hump-shaped and not upward sloping as the $\sqrt{T}$ rule suggests. We offer a better extrapolation rule. Using bootstrapped Generalized method of Moments (GMM), we provide robust Sharpe Ratio estimates of several popular test assets. The empirical results reject the $\sqrt{T}$ rule over a long horizon.

Introduction

As popularized by Siegel and Coxe (2002), there is a widely held belief that equity investing is less risky in the long run. This view arises from the concept of "time-diversification" and has significant implications for investors. For example, professional financial planners recommend the 100 minus age rule for portfolio allocation. Under this prescription, a 25-year old should invest 75% of her portfolio in risky assets and a 70-year old should invest 30% of her portfolio in risky assets. The spirit of the 100 minus age rule is automated in the glide path of the "target-date retirement funds".[1] The 100 minus age rule, clearly, highlights the impact of investment horizon on portfolio allocation.

The time diversification concept is reinforced by a faulty understanding of the long-term behavior of the Sharpe Ratio, which remains the most popular index for characterizing a portfolio's reward to risk performance. This popularity lies in its inherent simplicity: it is the ratio of the expected excess simple return of an investment to its volatility. Since both the expected return and volatility depend on the investment horizon, the Sharpe

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Ratio is also horizon dependent. That is, any given asset is associated with a term structure of Sharpe Ratios. To account for the horizon, assuming serially uncorrelated returns, Sharpe (1998) offers the following extrapolation rule: Letting $SR(1)$ and $SR(T)$ denote the Sharpe Ratios for 1 and T periods, respectively, then:

$$SR(T) = \sqrt{T} \times SR(1).$$

This rule is called the $\sqrt{T}$ rule. The $\sqrt{T}$ rule clearly implies that the reward to risk performance, as implied by the Sharpe Ratio, improves with the investment horizon.[2] In this study, we show the shortcomings of the $\sqrt{T}$ rule over the long run. We prove (both theoretically and empirically) that the use of the $\sqrt{T}$ rule may not be appropriate for long investment horizons. We, then, offer a more accurate extrapolation rule that is consistent with the empirical findings of Hodges *et al.* (1997), Lin and Chou (2003), and Levy (2017). These findings categorically reject the upward sloping Sharpe Ratio term structure and in turn they reject the appropriateness of the $\sqrt{T}$ rule.

After selecting the risky portfolio allocation, investors must still select the specific asset to include in their portfolio. Again, Sharpe Ratio comes to the rescue. The mantra is: Buy the fund with the highest Sharpe Ratio. This mantra is academically motivated as well. Sharpe (1966, 1994) shows that choosing the fund with the highest Sharpe Ratio is consistent with expected utility maximization if returns are normally distributed. $\sqrt{T}$ rule implies that there are no ranking reversals over the investment horizon. If Sharpe Ratio of Fund A is higher than the Sharpe Ratio of Fund B over a short horizon, then Sharpe Ratio of Fund A will be higher than the Sharpe Ratio of Fund B for all horizons. To summarize, two implications of the $\sqrt{T}$ rule are:

(1) **Horizon Monotonicity:** The Sharpe Ratio increases with horizon.

(2) **Ranking Reversal:** There can be no ranking reversals.

With these implications in mind, consider the following reported Sharpe Ratios. Figure 1 shows the Morningstar Sharpe Ratio over different horizons for two funds: Vanguard Total Stock Market Index Fund (VTSMX) and Vanguard 500 Index (VFINX). VTSMX is the largest mutual fund in the world with $343 billion in assets under management. VFINX, on the other hand, is the largest passive mutual fund mimicking S&P 500 with $173 billion in assets under management. Both implications of the $\sqrt{T}$ rule are violated according to Figure 1. Specifically, the 15-year Sharpe Ratio is lower than the 3-year Sharpe Ratio for both funds. That is, according to these data, the Sharpe Ratio does not increase with horizon. Moreover, there is a ranking reversal. The 3-year Sharpe Ratio for VFINX is higher than the 3-year Sharpe Ratio for VTSMX. At the 15-year horizon, the relative ranking flips. As we prove in the empirical section, such violations are not specific to these funds.[3] Levy (2017) captures the dilemma of the ranking reversal lucidly:

"*Another central problem of the Sharpe Ratio is that it produces a ranking that is horizon-dependent. If Sharpe Ratios are calculated using monthly returns* (*as is typically done*), *but the investor's horizon is say*, *10 years*, *selecting the fund with the highest Sharpe Ratio may lead to a sub-optimal choice*"

Figure 1 leads to two important questions. First, under what conditions is it appropriate to use the $\sqrt{T}$ rule? Both mathematically and empirically, we show that $\sqrt{T}$ rule is a good approximation when the horizon is short. The rule becomes increasingly worse with longer horizons. Instead of the $\sqrt{T}$ rule, we offer an alternate, more accurate rule that works well over all horizons.

Second, are there always going to be Sharpe Ratio ranking reversals? The answer depends on the return frequency (daily, weekly, or monthly

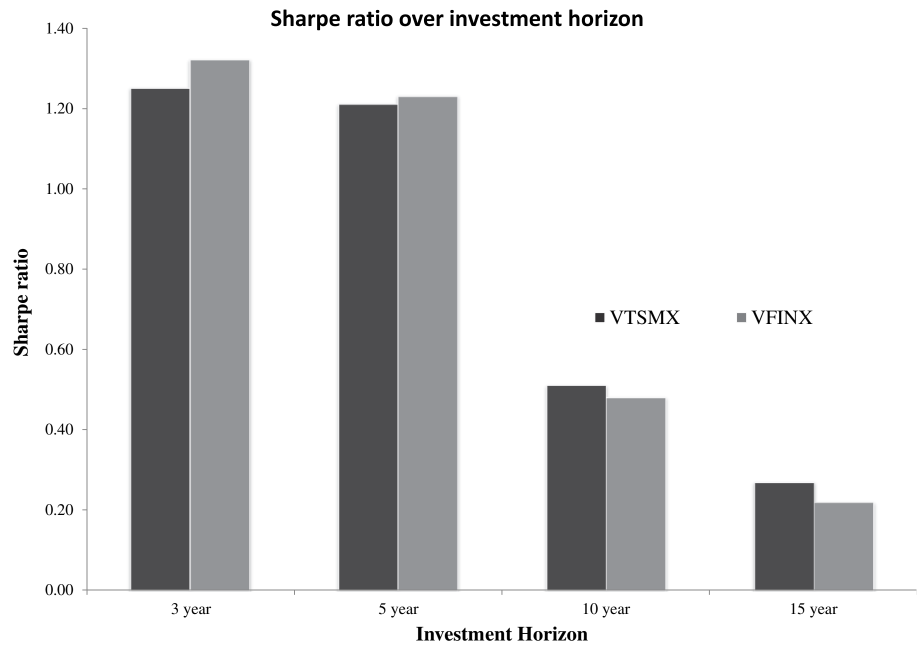

Figure 1 The figure shows the behavior of the Sharpe Ratio for two funds as calculated by Morningstar as of August 12, 2014. VTSMX is the symbol for Vanguard Total Stock Market Index Fund and VFINX is the symbol for Vanguard 500 Index.

returns) and the investment horizon. For example, if one uses daily returns to calculate the Sharpe Ratio, then one may find ranking reversals within a horizon of, say, 2 years. If one uses monthly returns, on the other hand, then one may not necessarily notice a ranking reversal.[4]

We theoretically show that the main culprit behind the $\sqrt{T}$ rule violation is compounding. Both the mean and the volatility of the simple return change non-linearly with the horizon due to compounding.[5] In turn, the Sharpe Ratio non-linearly depends on the horizon, which sometimes causes a switch in relative rankings. Specifically, we show that the Sharpe Ratio is hump-shaped: it initially rises and then falls with horizon.

Levy (1972) was probably the first to theoretically show that the Sharpe Ratio does not increase monotonically with horizon as implied by the $\sqrt{T}$ rule. We expand on Levy (1972) and show how compounding along with higher-order moments determine the behavior of the Sharpe Ratio. It may seem that higher-order moments do not affect the Sharpe Ratio, however, this observation is false. The reason is the fact that when the log-returns are skewed, then it turns out that the mean and variance of the simple return depend on the skewness of the log-return. As a result, Sharpe Ratio, which depends on the first two moments of the simple returns, depends on the skewness also. In the same manner, all the higher moments of the log-return affect the Sharpe Ratio. Indeed, we show that the Sharpe Ratio depends on all the higher-order moments, though the effect of these moments decays with horizon. The role of higher moments is best understood in the context of using options for portfolio management. Due to the non-linear option payoff, Sharpe Ratio can be manipulated to enhance portfolio's risk-adjusted performance (Goetzmann *et al.*, 2007). Bednarek and Patel (2017) show that the use of options changes the skewness of the portfolio, which in turn affects the Sharpe Ratio.

Using the higher moments and the investment horizon, we offer an alternate to the $\sqrt{T}$ rule. For reasonable parameters, the alternate rule is not too different from the $\sqrt{T}$ rule over a short horizon. However, over longer horizons, the proposed extrapolation rule generates the empirically observed hump-shape, which is substantially different from the $\sqrt{T}$ rule.

Lastly, we offer a comprehensive empirical analysis of the Sharpe Ratio of several test assets. Following Lo (2002), we use the Generalized Method of Moments (GMM) procedure, which accounts for heteroskedasticity, serial correlation, and non-normality, to estimate the Sharpe Ratio of the selected test assets across different horizons. The use of GMM is appropriate because it accounts for the fact that both the mean and the volatility of the excess returns are sample estimates, and are measured with error. Using size, book-to-market, and other popular test assets, we show that the $\sqrt{T}$ rule provides inaccurate estimates of the long-horizon Sharpe Ratios. We also document several instances of ranking reversals that occur both at short and long horizons. The implication of the empirical analysis is stark: using the $\sqrt{T}$ rule may lead to sub-optimal portfolio allocations.

This paper is related to several strands of the performance measurement literature. It is widely known that the Sharpe Ratio is inadequate in capturing a fat-tailed risk–reward trade off.[6] Despite these drawbacks, the Sharpe Ratio remains the most popular portfolio performance measure. The popularity, in fact, may be empirically and theoretically justified.[7] Eling and Schuhmacher (2007) and Brown *et al.* (2010) show that the majority of the alternate performance measures suffer from similar shortcomings and provide virtually identical rankings as the Sharpe Ratio. More importantly, other proposed performance measures (except the Stutzer (2000) index and geometric returns of Levy (2017)) are also horizon dependent. For example, the Sortino Ratio (Sortino and Price, 1994), which uses the downside risk instead of volatility, is also horizon dependent.

Finally, our study contributes to the literature on the statistics of the Sharpe Ratio. Departing from the parametric sample moment statistic approach of Jobson and Korkie (1981) and Memmel (2003), Lo (2002) recommends the use of GMM to analyze the statistics of a single Sharpe Ratio. Opdyke (2007) generalizes Lo's (2002) approach for pair-wise comparisons of Sharpe Ratios. Ledoit and Wolf (2008) recommend the use of bootstrapped GMM for such pair-wise comparisons. We offer another innovative use of the bootstrapped GMM—we compare Sharpe Ratios across different investment horizons.

1 Effect of horizon on the Sharpe Ratio

Suppose there are two traded assets. The first asset is the money market account yielding a continuously compounded risk-free return r_f. The second asset is the risky asset or a portfolio with price $S(t)$. Specifically, the asset price at any investment horizon, T, is composed of independent return increments:

$$S(T) = S(0)\exp\{r(T)\}; \quad r(T) = \sum_{i=1}^{i=T} r_i. \qquad (1)$$

Equation (1) states that the asset price evolves as a Geometric Levy process in discrete time. Let $F(x; T) = \Pr[r(T) \leq x]$ be the cumulative distribution function, and

$$\begin{aligned} C(z; T) &\equiv \ln(\mathrm{E}[\exp\{zr(T)\}]) \\ &= \ln\left(\int_{-\infty}^{\infty} \exp\{zx\} dF(x; T)\right) \end{aligned}$$

its cumulant generating function (CGF), and $\mathrm{E}[\cdot]$ is the expectation operator.[8] Since $r(T)$ is composed of independent increments, we have

$$C(z; T) = C(z; 1) \times T.$$

The CGF captures departures from normality in a tractable manner. The properties of CGF can be seen by expanding $C(z; 1)$ as a power series in z (around $z = 0$), where

$$C(z; 1) = \sum_{i=1}^{i=\infty} c_i \frac{z^i}{i!}; \quad c_i \equiv \frac{\partial^i}{\partial z^i} C(z; 1)\Big|_{z=0} \quad (2)$$

is the ith cumulant of the random variable $r(1)$. The first cumulant, c_1, is the mean; the second cumulant, c_2, is the variance; the third scaled cumulant, Skew $\equiv c_3/c_2^{3/2}$, is the excess skewness; and the fourth scaled cumulant, Kurt $\equiv c_4/c_2^2$, is the excess kurtosis. In summary, cumulants encompass all the information about the random variable $r(1)$.[9]

Standard calculation shows that the cumulants of the T-period return, $r(T)$, are

$$c_i(T) \equiv \frac{\partial^i}{\partial z^i} C(z; T)\Big|_{z=0} = T \times c_i.$$

Before moving forward, it is helpful to consider two examples.

Example 1. Normally distributed log-returns

Suppose the asset price follows the Geometric Brownian Motion, then,

$$r_i \sim \mathrm{N}\left(\mu - \frac{1}{2}\sigma^2, \sigma^2\right).$$

By Equation (2)

$$C(z; 1) = z\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{z^2\sigma^2}{2};$$

$$c_1 = \mu - \frac{1}{2}\sigma^2; \quad c_2 = \sigma^2; \quad \text{and}$$
$$c_i = 0 \quad \forall i \geq 3.$$

Since log-returns are normally distributed, skewness, kurtosis, and higher cumulants are zero.

Example 2. Log-returns subject to either disasters or booms or both

Suppose the asset price follows a discrete version of Merton's (1976) jump diffusion model; hence log-return, r_i, is the sum of two random components:

$$r_i = X + Y.$$

As in Example 1, the first component is normally distributed: $X \sim \mathrm{N}\big(\mu - \frac{1}{2}\sigma^2, \sigma^2\big)$. However, the second component, Y, generates jumps leading to either booms or disasters or both. The random variable, Y, is a Poisson mixture of normals, where N representing the number of jumps is Poisson distributed with parameter λ:

$$\Pr[N = n; \lambda] = \frac{\exp\{\lambda\}\lambda^n}{n!}.$$

Conditional on the number of jumps n, Y is normally distributed: $Y \sim \mathrm{N}(n\mu_j, n\sigma_j^2)$. If the jump probability, λ, is small and if the average jump size, μ_j, is large and negative, then r_i is subject to rare disasters and is negatively skewed. In the same spirit, if the average jump size, μ_j, is large and positive, then r_i is subject to rare booms and is positively skewed. Lastly, if the average jump size, μ_j, is zero and if volatility, σ_j, is large, then r_i is subject to fat-tailed risk and it exhibits zero skewness and large kurtosis.

For this return-generating model, Carr and Wu (2004) show that $C(z; T)$ is

$$C(z; T) = T \times \left(z\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{z^2\sigma^2}{2} + \lambda\left(\exp\left\{z\mu_j + \frac{z^2\sigma_j^2}{2}\right\} - 1\right)\right);$$

and the first four cumulants are calculated as

$$\mathrm{E}[r(T)] = c_1(T) = T \times \left(\mu - \frac{1}{2}\sigma^2 + \lambda\mu_j\right);$$

$$\mathrm{Var}[r(T)] = c_2(T) = T \times (\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2));$$

$$c_3(T) = T \times (\lambda(\mu_j^3 + 3\mu_j\sigma_j^2));$$
$$c_4(T) = T \times (\lambda(\mu_j^4 + 6\mu_j^2\sigma_j^2 + 3\sigma_j^4)).$$

The excess skewness of the T-period log-return is

$$\begin{aligned}\text{Skew}(r(T)) &= c_3(T)/c_2(T)^{3/2}\\ &= \frac{1}{\sqrt{T}} \times \frac{\lambda(\mu_j^3 + 3\mu_j\sigma_j^2)}{(\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2))^{3/2}}\\ &= \frac{1}{\sqrt{T}} \times \text{Skew}(r(1)),\end{aligned}$$

and the excess kurtosis of the T-period log-return is

$$\begin{aligned}\text{Kurt}(r(T)) &= c_4(T)/c_2(T)^2\\ &= \frac{1}{T} \times \frac{\lambda(\mu_j^4 + 6\mu_j^2\sigma_j^2 + 3\sigma_j^4)}{(\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2))^2}\\ &= \frac{1}{T} \times \text{Kurt}(r(1)).\end{aligned}$$

Note that both the skewness and the kurtosis of the log-return are decreasing in T. This is the reason the effect of skewness, kurtosis, and higher moments decays with horizon.

1.1 Expression of the Sharpe Ratio

The Sharpe Ratio depends on the first two moments of the simple return—not the log-return. The expressions for the mean and the variance of the T-period simple return, $R(T) \equiv S(T)/S(0) - 1$, are

$$\begin{aligned}\text{E}[R(T)] &= \text{E}[\exp\{r(T)\}] - 1\\ &= \exp\{C(1;1) \times T\} - 1;\\ \text{Var}[R(T)] &= \text{E}[R(T)^2] - \text{E}[R(T)]^2\\ &= \exp\{C(2;1) \times T\}\\ &\quad - \exp\{2C(1;1) \times T\}\end{aligned}$$

With the expressions of the mean and variance at hand, the Sharpe Ratio is

$$\begin{aligned}SR(T) &= \frac{\exp\{C(1;1) \times T\} - \exp\{r_f \times T\}}{\sqrt{\exp\{C(2;1) \times T\} - \exp\{2C(1;1) \times T\}}}\\ &= \frac{1 - \exp\{(r_f - C(1;1)) \times T\}}{\sqrt{\exp\{(C(2;1) - 2C(1;1)) \times T\} - 1}}.\end{aligned} \tag{3}$$

In Equation (3), the Sharpe Ratio depends on the CGF and hence all the cumulants of the log-return, making the expression of the Sharpe Ratio mathematically complicated. Two remarks are in order.

Remark 1. Skewness, kurtosis, and other cumulants of the log-return influence the Sharpe Ratio.

Remark 2. Since skewness, kurtosis, and other cumulants of the log-return decay with horizon, their influence on the Sharpe Ratio also decays with horizon

The implication of Remarks 1 and 2 is slightly nuanced. Typically, for any random fund, daily returns are more skewed and they exhibit more kurtosis than monthly returns. Hence if one calculates the Sharpe Ratio using daily returns, then the Sharpe ratio term structure may be different than when one calculates the Sharpe Ratio term structure using monthly returns.

We can study the effect of horizon, T, by utilizing the approximation, $\exp\{x\} \approx 1 + x$. With this approximation,

$$\begin{aligned}SR(T) &\approx \frac{C(1;1) - r_f}{\sqrt{C(2;1) - 2C(1;1)}} \times \sqrt{T}\\ &= SR(1) \times \sqrt{T}.\end{aligned} \tag{4}$$

We show the derivation in detail in Subsection A of the Appendix. Equation (4)—a first-order approximation—is precisely the $\sqrt{T}$ rule. It means that when the horizon is "short", the complicated expression in Equation (3) is accurately approximated by the $\sqrt{T}$ rule. However, as the numerical examples below show, for longer horizons, the $\sqrt{T}$ rule provides an inaccurate approximation.

From Equation (3), it is clear that the Sharpe Ratio depends on the mean and the variance along with the skewness, kurtosis and higher-order moments of the log-return. It is important to understand their effect by considering different parameters in Example 2. Specifically, consider the set of parameters presented in Table 1, which has four columns corresponding to four special cases. The first column corresponds to the case where the log-return excludes the jump component (Example 1). The second column describes the case where log-return is subject to tail-risk. Note, that tail-risk is symmetric—there is equal probability of disasters and booms which leads to zero skewness.

The third column describes the case where log-return is subject to rare but significantly negative drop in returns—a disaster. The average jump size μ_J is set to -99%, that is, a disaster effectively wipes out the investment in the asset. Since μ_J is large and negative, skewness is also large and negative (-10.61), while kurtosis is large (154.24). In the same spirit, the fourth column describes the case where log-return is subject to rare booms. The average jump size μ_J is set to 99%, effectively a boom doubles the value of the investment. Since μ_J is large and positive, skewness is also large and positive (10.61), while kurtosis remains large (154.24). In all cases, we set the risk-free rate to 2%. Moreover, we set the parameters so that the first two cumulants of the log-return are equal across all cases.

The top panel of Figure 2 shows the skewness for cases 3 and 4. As expected, as the investment horizon grows, the skewness falls. The bottom panel of Figure 2 shows the excess kurtosis for cases 2, 3 and 4. Just like skewness, excess kurtosis also

Table 1 Parameters for four different cases.

	Case 1 No disasters or booms	Case 2 Equal prob of and booms	Case 3 Rare disasters	Case 4 Rare booms
Parameters (annualized)				
μ	9.00%	9.13%	8.72%	9.79%
σ	13.50%	2.49%	5.39%	5.39%
λ	0.00%	3.18%	1.00%	1.00%
μ_J	0.00%	0.00%	−99.00%	99.00%
σ_J	0.00%	74.35%	74.35%	74.35%
Log-return cumulants				
Mean $r(1)$	8.09%	8.09%	8.09%	8.09%
Volatility $r(1)$	13.50%	13.50%	13.50%	13.50%
Skewness $r(1)$	0.00	0.00	−10.61	10.61
Excess Kurtosis $r(1)$	0.00	0.00	154.24	154.24

Case 1 considers log-returns without either disasters or booms. Case 2 considers log-returns with symmetric tail risk. Case 3 considers log-returns with rare disasters. Case 4 considers log-returns with rare booms. In all cases, the parameters are chosen so that the first two cumulants (mean and volatility) of the log-return, $r(1)$, are equal. The risk free rate is assumed to be 2.00% in all cases.

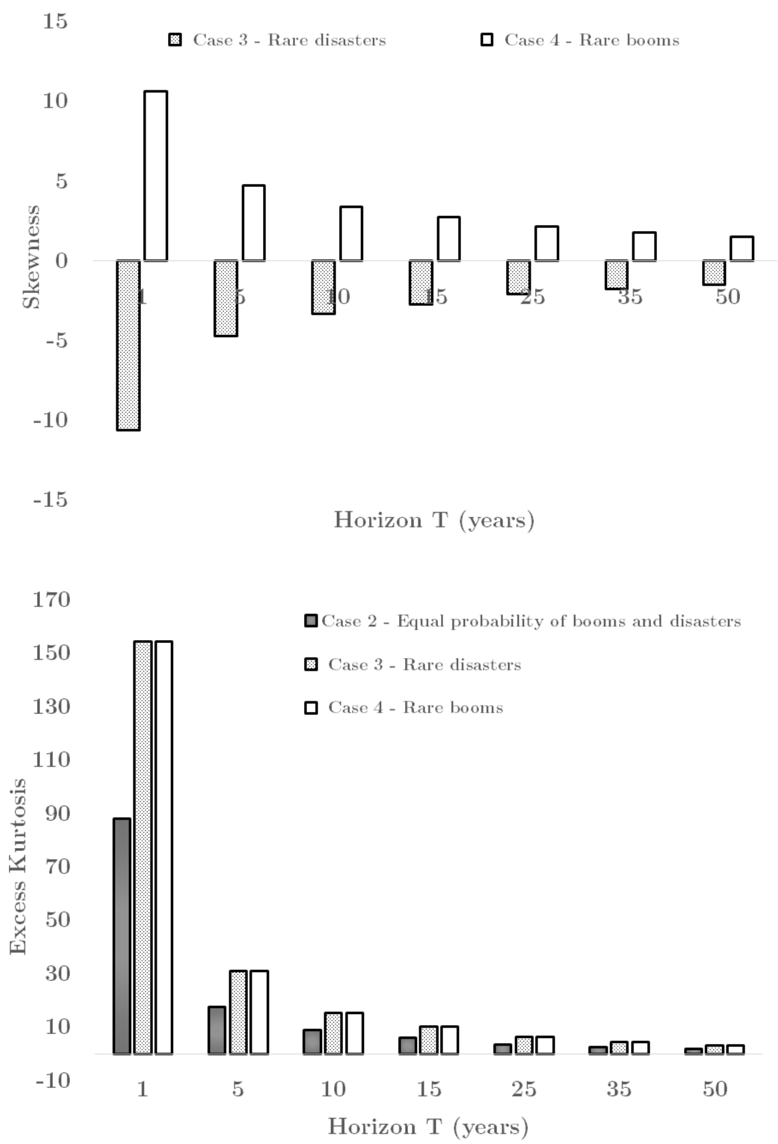

Figure 2 The figure shows the behavior of both skewness (top panel) and excess kurtosis (bottom panel) of the log-return across different horizons.

falls with the length of the investment horizon. More importantly, excess kurtosis decays faster than skewness.

Figure 3 shows the Sharpe Ratio across different investment horizons for the four special cases. Three observations are in order. First, it is clear that the Sharpe Ratio non-monotonically depends on the horizon. Specifically, the Sharpe Ratio is hump-shaped and this feature exists across all cases. Second, while the hump-shape is ubiquitous, the precise location of the inflection point varies across the four cases, depending on the parameters. In case 1, without disasters and in case 3, with rare disasters, Sharpe Ratio is maximal around 15 years. In case 2 where disasters and booms take place with equal probability, Sharpe Ratio is maximal around 10 years. Finally, in case 4 with rare booms, Sharpe Ratio is maximal around 5 years. This is a consequence of

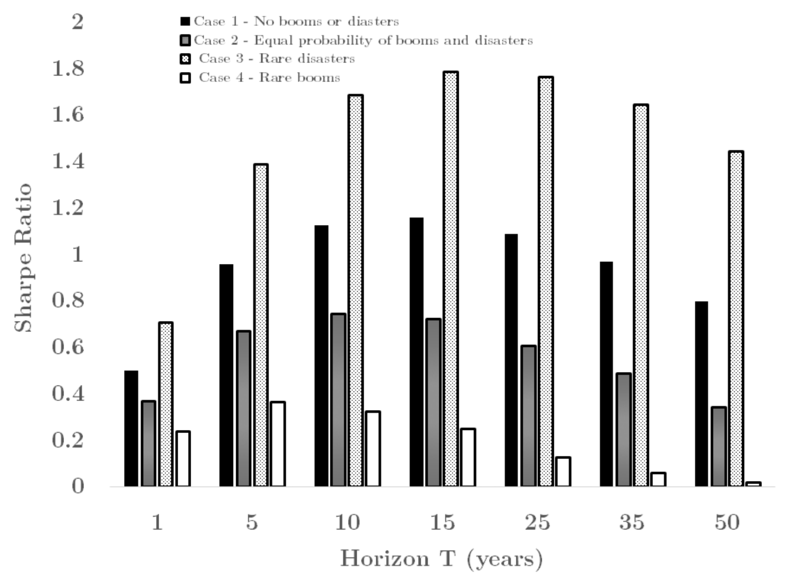

Figure 3 The figure shows the behavior of the Sharpe Ratio across horizon for the four cases presented in Table 1.

higher cumulants of the log-return. Clearly, the Sharpe Ratio decreases with both skewness and kurtosis. In other words, across all horizons, Sharpe Ratio is the highest when skewness is negative and is the lowest when the skewness is positive.

The following two remarks summarize this section:

Remark 3. As long as the log-returns are independent and identically distributed, the Sharpe Ratio will be hump-shaped. That is, the Sharpe Ratio initially rises reaching an inflection point and then falls with investment horizon.

Remark 4. All the cumulants and the investment horizon affect the inflection point of the Sharpe Ratio. Everything else equal, if log-return is negatively skewed (subject to disasters), then the inflection point takes place at a longer horizon. An increase in kurtosis decreases the inflection point.

To summarize, the hump-shape across all four cases implies that the $\sqrt{T}$ rule is invalid for long horizons. As a result, a new rule capable of capturing this feature is needed, a task we turn to next.

1.2 An alternate extrapolation rule

Defining $A \equiv \exp\{r_f - C(1;1)\}$ and $B \equiv \exp\{C(2;1) - 2C(1;1)\}$, the expression for the Sharpe Ratio in Equation (3) can be written as

$$\begin{aligned} SR(T) &= \frac{1 - A^T}{\sqrt{B^T - 1}}; \\ &= \left(\frac{1 - A}{\sqrt{B - 1}}\right) \\ &\quad \times \left(\frac{1 + A + A^2 + \cdots + A^{T-1}}{\sqrt{1 + B + B^2 + \cdots + B^{T-1}}}\right); \\ &= SR(1) \times \text{Adjustment Factor}; \end{aligned} \tag{5}$$

where

$$\text{Adjustment Factor} \equiv \left(\frac{1 + A + A^2 + \cdots + A^{T-1}}{\sqrt{1 + B + B^2 + \cdots + B^{T-1}}}\right).$$

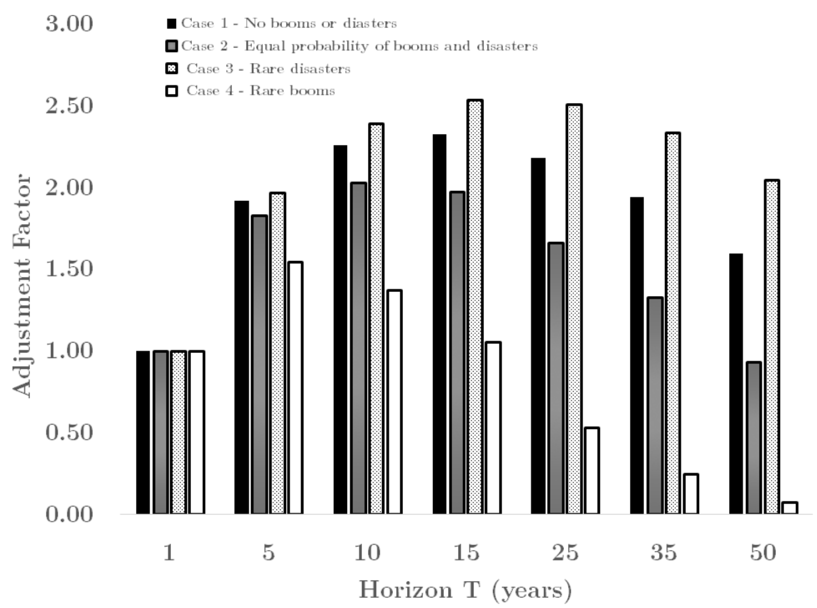

Figure 4 The figure shows the behavior of the Adjustment Factor across horizon for the four cases presented in Table 1.

Equation (5) extrapolates the one-period Sharpe Ratio to T periods using a non-linear Adjustment Factor. This new expression shows how the horizon affects term structure of the Sharpe Ratio while accounting for all the moments of the log-return.

Figure 4 shows the Adjustment Factor across all four cases presented earlier. This Adjustment Factor explicitly accounts for the effect of compounding, higher cumulants and rising horizon. The adjustment factor is a non-linear extrapolation rule that generates the empirically observed humped shape documented in Levy (1972), Hodges *et al.* (1997), Lin and Chou (2003) and Bednarek *et al.* (2016).

2 Empirical analysis

To illustrate the validity of our proposed extrapolation rule, we construct the term structure of the Sharpe Ratios using bootstrapped GMM. Since the GMM procedure does not require strong assumptions regarding the return-generating process, Lo (2002) used this procedure to estimate the Sharpe Ratio. As we explain below, the GMM procedure accounts for the fact that the mean, the volatility and in turn Sharpe Ratio itself are estimated with error. The magnitude of the error depends on the skewness, kurtosis, and the serial correlation of the returns. We extend Lo's (2002) procedure to analyze the effect of the length of the investment horizon. In this sense, we may be the first to use GMM to estimate the Sharpe Ratio term structure. We calculate the Sharpe Ratio of several popular test assets: portfolios sorted by size, book to market, earnings to price ratio, five industries and also a set of mutual funds' returns. We use the monthly returns spanning the period 1927–2014 from French's website. In the Appendix, we provide details about these test assets.

The first two columns of Table 2 show the annualized mean and volatility of simple returns for the selected assets. The mean ranges between 4.0% and 13.0%, while the volatility ranges between 9.0% and 30.0%.[10] The AR(1) coefficient ($\hat{\rho}_1$), in the third column, ranges between 0.3% and

Table 2 Annualized sample mean ($\hat{\mu}$), the standard deviation ($\hat{\sigma}$), the AR(1) coefficient ($\hat{\rho}_1$) (all in %), and the 1-year Sharpe Ratio of several test assets.

Asset	$\hat{\mu}$ (%)	$\hat{\sigma}$ (%)	$\hat{\rho}_1$ (%)	SR 1 year	$SR^{\sqrt{T}}$ 3 years	SR 3 years	SR^A 3 years	$SR^{\sqrt{T}}$ 10 years	SR 10 years	SR^A 10 years
BM-High	12.72	25.1	1.04	0.44	0.76	0.66	0.67	1.39	0.75	0.79
BM-Medium	10.44	19.96	0.83	0.43	0.74	0.67	0.68	1.35	0.86	0.89
BM-Low	9.21	18.68	0.8	0.38	0.66	0.60	0.61	1.21	0.81	0.83
Size-High	9.58	18.09	0.82	0.41	0.72	0.65	0.65	1.31	0.88	0.89
Size-Medium	11.83	23.46	0.97	0.43	0.75	0.66	0.66	1.37	0.78	0.81
Size-Low	11.92	29.31	0.99	0.39	0.67	0.57	0.59	1.22	0.62	0.68
Div/P-High	4.73	20	0.45	0.17	0.3	0.29	0.28	0.55	0.43	0.44
Div/P-Medium	6.02	17.9	0.54	0.24	0.41	0.39	0.39	0.76	0.59	0.60
Div/P-Low	6.51	19.69	0.63	0.25	0.44	0.41	0.41	0.8	0.59	0.61
CF/P-High	10.04	14.95	0.83	0.41	0.70	0.65	0.65	1.28	0.94	0.90
CF/P-Medium	7.88	14.32	0.67	0.29	0.50	0.47	0.47	0.91	0.73	0.69
CF/P-Low	7.33	16	0.64	0.25	0.43	0.41	0.41	0.78	0.62	0.60
E/P-High	10.81	15.56	0.86	0.43	0.75	0.70	0.68	1.37	0.97	0.91
E/P-Medium	8.11	14.09	0.69	0.32	0.55	0.52	0.52	1.01	0.79	0.76
E/P-Low	7.13	15.91	0.63	0.23	0.40	0.38	0.38	0.73	0.59	0.56
Consumer	6.69	18.44	0.59	0.27	0.46	0.43	0.44	0.85	0.63	0.66
Manufacturing	6.04	19.2	0.6	0.24	0.42	0.39	0.39	0.76	0.57	0.60
Hi-tech	6.28	19.42	0.6	0.25	0.43	0.40	0.41	0.79	0.59	0.61
Health	8.33	19.37	0.77	0.34	0.59	0.54	0.55	1.09	0.74	0.77
Other	4.74	22.41	0.49	0.18	0.32	0.30	0.30	0.58	0.43	0.46
FCNTX	11.33	15.84	0.94	0.45	0.78	0.71	0.71	1.42	0.98	0.93
AMECX	8.69	9.13	0.62	0.58	1.00	0.94	0.94	1.83	1.42	1.38
VINIX	9.25	14.32	0.81	0.48	0.83	0.77	0.77	1.52	1.08	1.09
VTSAX	5.80	15.00	0.47	0.35	0.60	0.56	0.58	1.10	0.83	0.91
VFIAX	4.83	14.61	0.42	0.29	0.51	0.48	0.49	0.92	0.73	0.79
VWUSX	6.83	19.25	0.66	0.25	0.44	0.41	0.41	0.80	0.59	0.60
VTRIX	6.17	17.74	0.59	0.22	0.39	0.37	0.36	0.71	0.55	0.55
VGHCX	17.02	13.96	1.32	0.87	1.51	1.33	1.30	2.76	1.59	1.49

The fifth column shows the 3-year Sharpe Ratio ($SR^{\sqrt{T}}$) calculated using the $\sqrt{T}$ rule. The sixth column shows the actual 3-year Sharpe Ratio calculated using the bootstrapped GMM. The seventh column shows the 3-year Sharpe Ratio (SR^A) calculated using our extrapolation rule. The last three columns compare the Sharpe Ratios calculated using $\sqrt{T}$ rule and the extrapolation rule with the actual Sharpe Ratios at a 10-year horizon.

1.4%—all portfolios exhibit little serial correlation (at least at a monthly frequency). Lo (2002) also finds little serial autocorrelation while analyzing similar fund returns. The fourth column shows the Sharpe Ratio at a 1-year horizon. The fifth column shows the 3-year Sharpe Ratio calculated using the $\sqrt{T}$ rule: $SR^{\sqrt{T}}(3) = SR(1) \times \sqrt{3}$. The sixth column shows the actual estimate of the 3-year Sharpe Ratio. The seventh column shows the 3-year Sharpe Ratio calculated using the Adjustment Factor.

For simplicity, we assume that the asset returns are normally distributed. That is, we use the $\hat{\mu}$ and $\hat{\sigma}$ from the first two columns and set the risk-free rate to 2.40%.[11] The 10-year Sharpe Ratios are compared in the last three columns. Lastly, note that since we bootstrap with one-million draws, we do not report the standard errors as the errors decrease with the number of draws.[12]

Comparison of the extrapolated with the actual Sharpe Ratios leads to the following important findings. First, the extrapolation rule is highly accurate at all horizons, whereas the accuracy of the $\sqrt{T}$ rule declines with horizon. Second, the relative accuracy of our proposed rule does not change across the test assets, whereas the same does not hold true for the $\sqrt{T}$ rule. Clearly, the proposed extrapolation rule accurately generates the term structure of Sharpe Ratios documented in Table 2. Moreover, considering these test assets, our results demonstrate that the $\sqrt{T}$ rule results in significant errors in measuring the Sharpe Ratio, subsequently leading to suboptimal asset allocation.

Bootstrapped GMM

Our test asset data consists of 87 years of monthly returns. Denote the sample of returns for each test asset by $\{U_t\}$ and the sample of risk-free returns by $\{U_t^f\}$. Eighty-seven years is not big enough to compute the Sharpe Ratio over a 10, 15, or 25 year horizon. Hence, we use the following bootstrap GMM algorithm:

(1) **Horizon:** Choose a test asset and an investment horizon $T = 1, 2, 3, \ldots, 25$ years.
(2) **Bootstrap:** Randomly pick $12 \times T$ monthly stock returns and the corresponding risk-free returns. Denote this subsample by $\{\tilde{U}_t\}$ and $\{\tilde{U}_t^f\}$. The T-year simple excess return is computed as $U(T) = \prod_{t=1}^{12\times T}(1+\tilde{U}_t) - \prod_{t=1}^{12\times T}(1+\tilde{U}_t^f)$. Repeat this procedure $I = 10^6$ times with replacement. The final output will be a $I \times 1$ vector of T-year excess returns with each element denoted by $U_i(T)$.
(3) **GMM:** Estimate the Sharpe Ratio by minimizing the following function:

$$\hat{\theta} = \arg\min_{\theta} \hat{m}[\theta]'\hat{W}\hat{m}[\theta], \qquad (6)$$

where $\theta = [\mu(T), \sigma(T), SR(T)]'$ is the 3×1 parameter vector comprising of the mean, the volatility, and the Sharpe Ratio of the

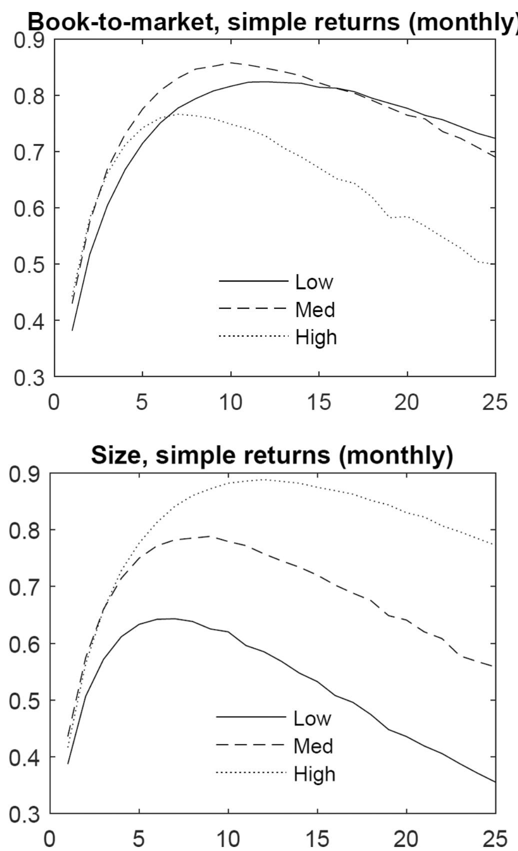

Figure 5 The top panel of the figure shows the Sharpe Ratio estimate of the book-to-market-sorted portfolios at different investment horizons. The bottom panel shows the Sharpe Ratio estimate of size-sorted portfolios.

T-period excess returns;

$$\hat{m}[\theta] = \frac{1}{I}\begin{bmatrix} \sum_{i=1}^{i=I} U_i(T) - \mu(T) \\ \sum_{i=1}^{i=I} (U_i(T) - \mu(T))^2 - \sigma(T)^2 \\ \sum_{i=1}^{i=I} U_i(T) - \sigma(T)SR(T) \end{bmatrix},$$

where $\hat{m}[\theta]$ is a 3×1 sample average vector and $\hat{W}$ is a positive-definite weighting matrix. Effectively, the GMM estimator chooses θ so the sample average $\hat{m}[\theta]$ is close to zero, where closeness depends on the matrix $\hat{W}$. Upon inspection of $\hat{m}[\theta]$, it is clear that the error in the estimation of the mean influences the estimation of the volatility and in turn the Sharpe Ratio. The matrix $\hat{W}$ accounts for various forms of heteroskedasticity and/or serial correlation. We show results for the Newey and West (1987) weighting matrix.[13]

Note that this algorithm implicitly assumes that excess returns are serially uncorrelated. This assumption is consistent with the small autocorrelation coefficients reported in Table 2.

Table 3 Sharpe Ratio estimate for different investment horizons. The Sharpe Ratio is calculated using simple monthly returns.

	Horizon in years						
Asset	1	3	5	7	10	15	25
Div/P-High	0.17	0.28	0.35	0.39	0.43	0.46	0.45
Div/P-Medium	0.24	0.39	0.48	0.54	0.58	0.62	0.61
Div/P-Low	0.26	0.41	0.50	0.55	0.59	0.62	0.58
CF/P-High	0.41	0.66	0.79	0.87	0.94	0.97	0.93
CF/P-Medium	0.29	0.47	0.58	0.66	0.73	0.79	0.81
CF/P-Low	0.25	0.41	0.50	0.56	0.62	0.67	0.68
E/P-High	0.43	0.70	0.83	0.91	0.97	0.99	0.92
E/P-Medium	0.32	0.52	0.64	0.71	0.79	0.85	0.86
E/P-Low	0.23	0.38	0.47	0.53	0.59	0.64	0.65
Consumer	0.27	0.43	0.52	0.58	0.63	0.66	0.63
Manufacturing	0.24	0.39	0.47	0.52	0.57	0.60	0.57
Hi-tech	0.25	0.40	0.49	0.54	0.59	0.61	0.57
Health	0.34	0.54	0.64	0.70	0.74	0.75	0.67
Other	0.18	0.30	0.36	0.40	0.43	0.44	0.42
FCNTX	0.45	0.71	0.85	0.92	0.99	1.00	0.91
AMECX	0.58	0.95	1.15	1.29	1.42	1.53	1.55
VINIX	0.48	0.77	0.92	1.01	1.08	1.10	1.03
VTSAX	0.35	0.56	0.68	0.76	0.83	0.87	0.86
VFIAX	0.29	0.48	0.59	0.66	0.73	0.79	0.79
VWUSX	0.25	0.41	0.50	0.55	0.59	0.62	0.58
VTRIX	0.22	0.36	0.45	0.50	0.55	0.59	0.59
VGHCX	0.88	1.33	1.51	1.58	1.59	1.49	1.23

In this manner, we depart slightly from the block-bootstrap method used in Lin and Chou (2003). Moreover, to ensure robust Sharpe Ratio estimates, we re-sample a million draws with replacement. The large number of draws results in very low standard errors for the estimated Sharpe Ratios.[14]

Figure 5 shows the Sharpe Ratio estimates of the book to market and size-sorted portfolios at different investment horizons. The Sharpe Ratio is hump-shaped for both portfolios, which is consistent with our theory. Upon inspection of the top panel of Figure 5, there is a clear instance of a ranking reversal. At a 10-year horizon, low book-to-market portfolio has a higher Sharpe Ratio than the high book-to-market portfolio. However, at a 5-year horizon, ranking reverses. It is important to emphasize that the ranking reversal is not a ubiquitous phenomenon as shown in the bottom panel of Figure 5. Table 3 shows the Sharpe Ratio of other test assets. The non-monotonic feature is evident across all the assets.

3 Conclusion

For over two decades, finance professionals have relied on Sharpe's $\sqrt{T}$ rule to extrapolate from one to T-period Sharpe Ratios. A number of empirical and theoretical studies have shown that the accuracy of $\sqrt{T}$ rule declines with the investment horizon. Specifically, the term structure of the Sharpe Ratio is hump-shaped. The error in the $\sqrt{T}$ rule is due to overlooking the effect of compounding (Levy, 1972). Over a short horizon, ignoring compounding may be reasonable. However, over longer horizons, ignoring compounding leads to estimation error.

In this paper, we offer an alternate extrapolation rule. Using robust bootstrapped GMM procedures and monthly returns data (1927–2014), we estimate the term structure of the Sharpe Ratios for several popular test assets. We then show that our proposed extrapolation rule accurately matches the term structure of Sharpe Ratios documented in the extant empirical literature.

The effect of horizon on various portfolio performance measures is not well understood and should be studied. The methodology proposed in this paper can be extended to other performance measures such as the Treynor and the Sortino Ratios. Our analysis assumes that the log-returns are independent and identically distributed. It is certainly possible that the returns are time-varying. One direction to enhance our analysis would be to permit for time-varying returns. Extending the paper along the lines of the habit formation model of Campbell and Cochrane (1999) or the long-run risk model of Bansal and Yaron (2004) can provide another fruitful direction for future research.

Appendix

A. Derivation of the first-order approximation of the Sharpe Ratio

In this subsection, we derive the details behind Equation (4). From Equation (3), we have that

$$\begin{aligned} SR(T) &= \frac{\exp\{C(1;1)\times T\} - \exp\{r_f \times T\}}{\sqrt{\exp\{C(2;1)\times T\} - \exp\{2C(1;1)\times T\}}} \\ &= \frac{1-\exp\{(r_f - C(1;1))\times T\}}{\sqrt{\exp\{(C(2;1)-2C(1;1))\times T\} - 1}}. \end{aligned}$$

For a short-horizon T so that $T \approx 0$, the numerator can be written as

$$\begin{aligned} 1 - \exp\{(r_f - C(1;1))\times T\} &\approx 1 - [1 + (r_f - C(1;1))\times T] \\ &= (C(1;1) - r_f)\times T \end{aligned}$$

In the approximation, we use the fact that $\exp\{x\} \approx 1 + x$ for small x. Note that the numerator scales linearly with T. Moreover, the approximation has a simple interpretation. The term $C(1;1) - r_f$ is the excess risk premium.

In the same spirit, the denominator becomes

$$\sqrt{\exp\{(C(2;1) - 2C(1;1)) \times T\} - 1}$$
$$\approx \sqrt{1 + (C(2;1) - 2C(1;1))T - 1}$$
$$= \sqrt{C(2;1) - 2C(1;1)} \times \sqrt{T}$$

Note again that the denominator scales as a square-root function of T. Also, the denominator has a simple interpretation. The term inside the square-root represents the variance of the simple return. Combining both the numerator and denominator gives Equation (4).

B. Data details of the test assets

In this section, we provide additional details of the test assets. The return data is taken from Kenneth French's website.

- **Dividend/price** (D/P): These portfolios are formed on D/P at the end of each June using NYSE breakpoints. The dividend yield used to form portfolios in June of year t is the total dividends paid from July of $t-1$ to June of t per dollar of equity in June of t. Data are monthly spanning 1928 to 2014.
- **Cash flow/price** (CF/P): These portfolios are formed on CF/P at the end of each June using NYSE breakpoints. The cash flow used in June of year t is total earnings before extraordinary items, plus equity's share of depreciation, plus deferred taxes (if available) for the last fiscal year end in $t-1$. Data are monthly spanning 1952 to 2014.
- **Earnings/price** (E/P): These portfolios are formed on E/P at the end of each June using NYSE breakpoints. The earnings used in June of year t are total earnings before extraordinary items for the last fiscal year end in $t-1$. P is price times shares outstanding at the end of December of $t-1$. Data are monthly spanning 1952 to 2014.
- **Five industry portfolios**: There portfolios are formed by assigning each NYSE, AMEX, and NASDAQ stock to an industry portfolio at the end of June of year t based on its four-digit SIC code at that time. Returns calculated from July of t to June of $t+1$. Data are monthly spanning 1928 to 2014.
- **Selected mutual funds:** The data for mutual funds is from CRSP.

  — FCNTX – Fidelity Contrafund with monthly data spanning 1980 to 2015;
  — AMECX – American Funds Income Fund of America with monthly data spanning 1986 to 2015;
  — VINIX – Vanguard Institutional Index I, with monthly data spanning 1990 to 2015;
  — VTSAX – Vanguard Total Stock Market Index with monthly data spanning 2000 to 2015;
  — VFIAX – Vanguard 500 Index Admiral with monthly data spanning 2000 to 2015;
  — VWUSX – Vanguard US Growth Investment with monthly data spanning 1980 to 2015;
  — VTRIX – Vanguard International Value Investment with monthly data spanning 1983 to 2015; and
  — VGHCX – Vanguard Health Care Investment with monthly data spanning 1984 to 2015.

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Notes

[1] Chang *et al.* (2017) analyze the performance of target dated funds.

[2] The purpose of this paper is not to rekindle the time diversification debate. Rather, the goal is focused on the Sharpe Ratio itself.

[3] The assets under management estimate is from Morningstar as of August 13, 2014.

[4] An alternative as suggested by Bednarek *et al.* (2016) and Levy (2017) is to use log or geometric returns to measure the risk-adjusted performance.

[5] In a related note, Korkie (2017) shows that the popular variance ratio test is incapable of determining whether the investment risk increases with horizon.

[6] The set of alternate performance measures is large: Sortino and Price (1994), Keating and Shadwick (2002), Kaplan and Knowles (2004), Dowd (2000), Favre and Galeano (2002), Rachev *et al.* (2007), Aumann and Serrano (2008), Zakamouline and Koekebakker (2009), and Stutzer (2000) have proposed alternate measures for characterizing the risk–reward trade off.

[7] Schuhmacher and Eling (2011, 2012) theoretically justify the popularity of Sharpe Ratio using a decision theoretic analysis.

[8] Karlin (2014) provides an excellent textbook treatment about the cumulant properties; Martin (2012) elegantly uses the cumulants to study asset pricing implications.

[9] In a related matter, CGF is closely related to the moment generating function. In fact, there is a one to one mapping between cumulants and moments. Going forward, we use cumulants and moments interchangeably.

[10] As we show in detail later, the estimates in Table 2 are calculated using bootstrapped GMM. We use one-million draws for the bootstrap. The bootstrap was computed using Matlab's Parallel Computing Toolbox on a 32-core server grade workstation.

[11] If we were to use the skewness and kurtosis parameters, then the adjustment factor would be even more accurate.

[12] Bootstrapped GMM is computationally intensive. Using 5,000 draws, Ledoit and Wolf (2008) use simulations to compute the size and power of their inference statistic. We extend their methodology but use a million draws for our simulation exercise. Consequently, we are able to report robust Sharpe Ratio estimates of several popular test assets over long horizons.

[13] For robustness, we also analyzed our results using the heteroskedasticity-consistent matrix of White (1980), the Parzen kernel of Gallant (1987), the truncated kernel of Hansen (1982) and Hansen and Hodrick (1980), and the Tukey–Hanning kernel of Andrews and Monahan (1992).

[14] More importantly, the number of draws is significantly higher than previous literature (Lin and Chou, 2003, use 5,000 draws).

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