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HORIZON EFFECTS THAT ARE LARGER THAN YOU THINK: DYNAMIC ALLOCATION WITH A REPRESENTATIVE INVESTOR

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This paper illustrates optimal dynamic allocation in a traditional two-fund capital market model. As in previous literature, a mean-reverting market portfolio implies a "horizon effect" in typical investors' allocations. For investors whose risk aversion is higher than the representative investor's, the horizon effect becomes substantially larger in the capital market model than in previous models.

Samuelson (1991) discovered that an investor's optimal allocation between a risky asset and a risk-free asset has a "horizon effect", where the risky asset allocation is higher for a longer horizon, assuming: (1) the risky asset is mean-reverting; (2) the risk-free rate is constant; and (3) the investor has constant relative risk aversion (CRRA) > 1 and rebalances the allocation each period. The horizon effect is due to the intertemporal hedging benefit of the uncertainty in future expected risky asset returns, in the fashion of Merton (1973). Kritzman (1994) presented some binomial illustrations of the horizon effect.

The analysis here extends Kritzman's (1994) illustrations to a traditional two-fund capital market setting where the market portfolio is the risky asset and the representative investor has a multiperiod horizon and a constant CRRA > 1. In contrast to the Samuelson–Kritzman model, the model here implies uncertainty in the future risk-free rates, in addition to the uncertainty in future expected market portfolio returns. Both uncertainties affect the risk-free rate and the market risk premium through intertemporal hedging effects, as in Merton (1973) and Campbell (1996). For investors with CRRA above the representative investor's, the horizon effect gets substantially larger in the capital market model than in the Samuelson–Kritzman model.

In the capital market model an investor who is more risk averse than the representative investor holds a positive allocation in the risk-free asset with a market portfolio position, and thus typifies a traditional long-term investor who allocates between a long fixed income position and a stock portfolio. Therefore, the larger horizon effect

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found in the capital market model may be of practical interest to many traditional long-term investors and investment managers.

1 Framework

The illustrations use power utility functions of the form $U = 1 - W^{1-b}$, where b denotes the degree of CRRA. (Power utility describes empirical data fairly well, e.g., Guo and Whitelaw, 2006.) The illustrations use a two-period binomial tree for the risky asset/market portfolio (dividends reinvested), where each up-state (down-state) is 1.24 (0.94) times the prior period's realized value, as shown in Figure 1.

Investors rebalance their allocations each period to maximize the expected utility of horizon (time-2) wealth. For time-0, and time-1 in the up- and down-state, respectively, let w_0, w_{1u}, and w_{1d} denote the investor's percentage allocations to the risky asset, and let r_0, r_{1u}, and r_{1d} denote the single-period risk-free rates. Initial wealth is 1. The possible outcomes for 2-period horizon wealth, W_2, are shown in Equations (1)–(4):

$$[1.24w_0 + (1 + r_0)(1 - w_0)] \cdot [1.24w_{1u} + (1 + r_{1u})(1 - w_{1u})] \quad (1)$$

$$[1.24w_0 + (1 + r_0)(1 - w_0)] \cdot [0.94w_{1u} + (1 + r_{1u})(1 - w_{1u})] \quad (2)$$

$$[0.94w_0 + (1 + r_0)(1 - w_0)] \cdot [1.24w_{1d} + (1 + r_{1d})(1 - w_{1d})] \quad (3)$$

$$[0.94w_0 + (1 + r_0)(1 - w_0)] \cdot [0.94w_{1d} + (1 + r_{1d})(1 - w_{1d})] \quad (4)$$

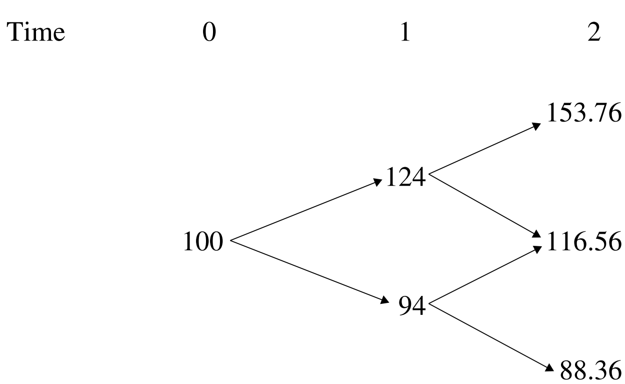

Figure 1 Risky asset/market portfolio.

An investor with a 2-period horizon first determines the optimal allocations for the time-1 up- and down-states, w_{1u} and w_{1d}; then, working backward, the investor determines the optimal time-0 allocation, w_0.

2 Review of the random walk case

Mossin (1968), Samuelson (1969), and Merton (1969) proved that a CRRA rebalancer "myopically" allocates between a risky asset that follows a random walk and a risk-free asset with a constant rate. The initial allocation for a 2-period horizon is "myopic" because it is the same as for a single-period horizon. A review of the random walk case provides a helpful background for the mean-reversion scenarios.

Assume the probability that the risky asset in Figure 1 moves to an up- or a down-state is 0.50 in any time period. Thus, the risky asset's single-period expected rate of return is 9%, with a standard deviation of 15%, regardless of the time period or state of nature. Assume also for now that the short-term risk-free rate is constant and equal to 3.78%; that is, $r_0 = r_{1u} = r_{1d} = 0.0378$. The reason for this particular rate will be apparent later.[1]

First consider an investor with a single-period horizon and the utility function $U = 1 - W^{-2}$, where $b = 3$. The investor's time-0 expected utility of time-1 wealth is $0.50\{1 - [1.24w_0 + 1.0378(1 - w_0)]\}^{-2} + 0.50\{1 - [0.94w_0 + 1.0378(1 - w_0)]\}^{-2}$. For the maximum expected utility of time-1 wealth, the investor's time-0 risky asset allocation is $w_0 = 0.870$, or 87.0%.

Next consider an investor with the same utility function but a 2-period horizon. At the time-1 up-state, the expected utility of time-2 wealth is $0.50\{1 - [1.24w_{1u} + 1.0378(1 - w_{1u})]\}^{-2} + 0.50\{1 - [0.94w_{1u} + 1.0378(1 - w_{1u})]\}^{-2}$. Maximizing this expected utility yields an optimal risky asset allocation of $w_{1u} = 0.870$, or 87.0%, in the time-1 up-state. A similar analysis for the time-1 down-state yields the same allocation. Working backward, using Equations (1)–(4),[2] maximizing the time-0 expected utility of time-2 wealth yields the "myopic" time-0 risky asset allocation of 87.0%.

A similar analysis for an investor with the utility function $U = 1 - W^{-3}$, where $b = 4$, yields a risky asset allocation of 64.7% in each period regardless of the investment horizon. In a mean–variance model, the ratio of investors' risky asset allocations equals the inverse of the ratio of the investors' CRRA levels, e.g., Campbell and Viceira (2002). In the binomial framework, the equality is only an approximation.

Based on the random walk case, Samuelson (1990) stressed that no CRRA rebalancer would make a higher risky asset allocation for a longer horizon, as suggested by the practitioner *time diversification* strategy of gradually decreasing a risky asset's allocation over the time until the investment horizon.[3]

3 Mean reversion and the Samuelson–Kritzman model

After Samuelson (1990) used the random walk case to advocate against the time diversification strategy, Samuelson (1991) discovered a "horizon effect": for a *mean-reverting* risky asset and a constant risk-free rate, a CRRA rebalancer with $b > 1$ will optimally make a higher risky asset allocation for a longer horizon. Given that most scholars agree that the average investor's CRRA is higher than 1 (e.g., Campbell, 1996; Guo and Whitelaw, 2006), and because of empirical support for mean reversion (e.g., the summary in Campbell *et al.*, 2001), Samuelson (1991) noted that the horizon effect vindicated the notion of "business man's risk" and the time diversification strategy. Because Samuelson's (1991) model appeared in a relatively obscure book of readings, Kritzman's (1994) mean-reversion illustrations served the important role of popularizing Samuelson's (1991) horizon effect result.

To illustrate the Samuelson–Kritzman mean-reversion model, assume that for the risky asset, the first period's probabilities are each still 0.50, but the probability of moving up (down) from the time-1 up-state is 0.40 (0.60), and the probability of moving up (down) from the time-1 down-state is 0.60 (0.40). The risky asset's expected single-period rate of return is still 9% at time-0, but is 6% (12%) at the time-1 up-state (down-state). See Figure 2.

Continue to assume a constant risk-free rate of 3.78%. We have already seen that an investor with $b = 3$ and a single-period horizon will allocate 87.0% of wealth to the risky asset. Now consider an investor with $b = 3$ and a 2-period horizon. At the time-1 up-state, the expected utility of time-2 wealth is equal to $0.40\{1 - [1.24w_{1u} + 1.0378(1 - w_{1u})]\}^{-2} + 0.60\{1 - [0.94w_{1u} + 1.0378(1 - w_{1u})]\}^{-2}$, which yields

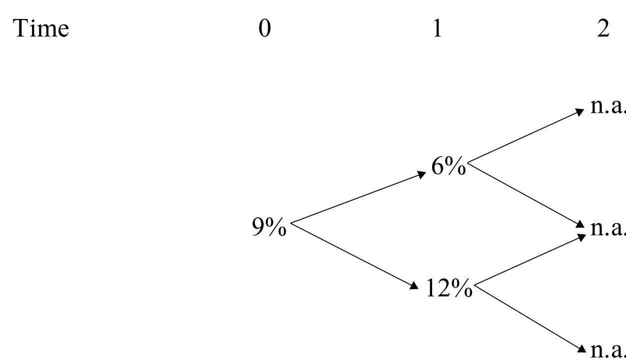

Figure 2 Risky asset's expected single-period rate of return with mean reversion.

an optimal risky asset allocation of $w_{1u} = 0.377$, or 37.7% (with 62.3% in the risk-free asset). A similar analysis for the time-1 down-state yields that a levered risky asset allocation of 137.9% is optimal (with an allocation of −37.9% in the risk-free asset). Working backward, given the time-1 state-dependent allocations, the optimal time-0 risky asset allocation is 98.9%.[4] Figure 3 shows the risky asset allocations for the mean-reversion scenario.

The 98.9% initial risky asset allocation by an investor with $b = 3$ and a 2-period horizon is higher than for a single-period horizon, 87.0%. A similar analysis for an investor with $b = 4$ and a 2-period horizon yields a time-0 risky asset allocation of 73.5%, compared to 64.7% for a single-period horizon.

These examples illustrate the Samuelson–Kritzman horizon effect for a mean-reverting risky asset and a constant risk-free rate: an investor with a 2-period horizon and $b > 1$ makes a higher time-0 risky asset allocation than for a single-period horizon. The reason is that the changes in the risky asset's expected return covary negatively with the risky asset return and thus "hedge" some of the risk of time-2 wealth, so a 2-period investor with $b > 1$ allocates more to the risky asset at time-0 than a single-period investor.[5]

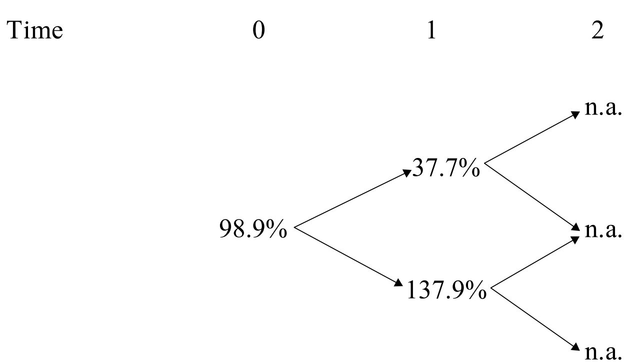

Figure 3 Risky asset allocations. Investor with $b = 3$ and 2-period horizon. Samuelson–Kritzman: Mean reversion and constant risk-free rate = 3.78%.

Note that the risky asset's variance is lower at time-1 than at time-0, but this drop does not drive the horizon effect. To see this, simply construct an example where the risky asset's *2nd-period* up-moves are 1.20 and down-moves are 0.96, with probabilities of 0.50. This example would have a lower risky asset variance in the second period, but no mean reversion. The time-0 risky asset allocation by an investor with $b = 3$ and a 2-period horizon is the same as for a single-period horizon, 87.0%, because there is no uncertainty in future investment opportunities.

4 Capital market model and the representative investor

Now let's re-cast the dynamic allocation decisions in a two-fund capital market model, interpreting the risky asset as the market portfolio. As in Sharpe (2007), a capital market setting often uses the perspective of a representative investor, who is not an actual investor, but rather a fictional aggregate of investors' wealths, horizons, and degrees of risk aversion. More risk averse investors hold long risk-free asset positions with a market portfolio position, lending to less risk averse investors who hold levered market portfolio positions. The representative investor allocates 100% to the market portfolio, and the aggregate risk-free asset position is zero. Given the market portfolio's return dynamics, the risk-free rate must be consistent with the representative investor's CRRA, horizon, and 100% allocation to the market portfolio. This consistency implies that the market-determined short-term risk-free rate is uncertain if the representative investor's CRRA is constant over time and the market portfolio does not follow a random walk.[6]

If the market portfolio's volatility is constant, a representative investor with a constant CRRA implies a fairly stable market risk premium in our illustrations. Although this stability is unrealistic, Munk *et al*. (2004) and Harris and Marston (2013) find that empirical *ex ante* market risk premium estimates are less volatile than the risk-free rate. So the stable market risk premium here seems like a more reasonable assumption than the Samuelson–Kritzman model's constant risk-free rate.

We assume that the representative investor has a 2-period horizon. However, the horizon for the final period is by necessity only a single period. The representative investor's degree of CRRA, denoted b^*, is equal to 3 for the illustrations to follow. With the assumed market portfolio dynamics, $b^* = 3$ implies market risk premium estimates in the 5–6% range, consistent with survey estimates by Welch (2009).[7]

First, we find the risk-free rates for the time-1 up- and down-state, given the representative investor's 100% allocation to the market portfolio. At the time-1 up-state, the representative investor's expected utility of time-2 wealth is $0.40\{1 - [1.24w_{1u} + (1 + r_{1u})(1 - w_{1u})]\}^{-2} + 0.60\{1 - [0.94w_{1u} + (1 + r_{1u})(1 - w_{1u})]\}^{-2}$. To find r_{1u} such that the optimal $w_{1u} = 1$, one can: (1) set the first derivative (with respect to w_{1u}) equal to zero; (2) substitute $w_{1u} = 1$; and (3) solve for r_{1u}. The solution is $r_{1u} = 0.0073$, or 0.73%. The market risk premium at the time-1 up-state is thus $0.06 - 0.0073 = 0.0527$, or 5.27%. A corresponding analysis for the time-1 down-state yields that $r_{1d} = 0.0583$, or 5.83%. The market risk premium at the time-1 down-state is thus $0.12 - 0.0583 = 0.0617$, or 6.17%.[8]

Given the 2-period horizon, we find the time-0 risk-free rate by working backward using Equations (1)–(4);[9] for the optimal w_0 to be 1, the time-0 risk-free rate must be 3.78%. The market risk premium at time 0 is thus $0.09 - 0.0378 = 0.0522$, or 5.22%. Figure 4 shows the evolution

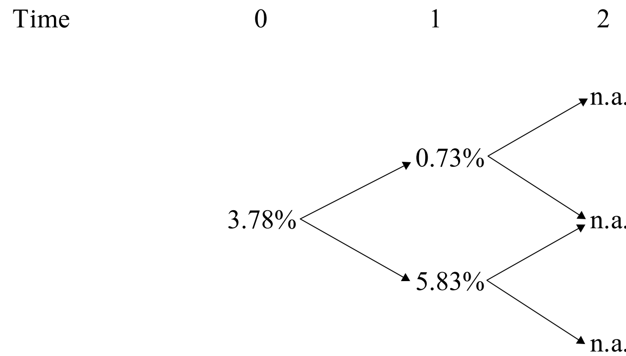

Figure 4 Single-period risk-free rates. Mean reversion; representative investor with 2-period horizon and $b^* = 3$.

pattern of the uncertain risk-free rates when the representative investor has $b^* = 3$ and a 2-period horizon.

Note that even with uncertain risk-free rates, the risky asset allocation is consistent with the direction of the Samuelson–Kritzman horizon effect, in that an investor with $b > 1$ makes a higher allocation to the mean-reverting market portfolio for a longer horizon. Here, an investor with $b = 3$ and a 2-period horizon makes a higher allocation (100%) to the market portfolio than an investor with $b = 3$ and a single-period horizon (87.0%, as shown earlier).[10] Note also that for the same time-0 risk-free rate of 3.78%, the investor's 100% risky asset allocation is higher than the investor would make in the Samuelson–Kritzman constant risk-free rate model, 98.9% as found earlier. We will elaborate on this point shortly.

The uncertainty in the future risk-free rates follows the pattern that when the market portfolio rises, the risk-free rate falls, and vice versa. This pattern is consistent with (small) positive empirical estimates of "bond betas" (e.g., Viceira, 2012). In the model here, the dynamic is due to the market portfolio's mean reversion and the relative stability of the market risk premium.

The time-0 short-term risk-free rate (3.78%) is higher, and the time-0 market risk premium

(5.22%) is lower, than if the representative investor has the same CRRA but only a single-period horizon (3.10%; and 9% − 3.10% = 5.90%). The reason is that a representative investor with a 2-period horizon (and $b^* > 1$) gets intertemporal "hedging" benefits from the uncertainty in both the future risk-free rate and the market portfolio's future expected return. So a representative investor with $b^* > 1$ and a 2-period horizon requires a lower time-0 market risk premium than with the same CRRA and a single-period horizon.[11] Therefore, for a given b^* and the assumed market portfolio dynamics, the time-0 risk-free rate is higher when the representative investor has a 2-period horizon rather than a single-period horizon.

5 Comparison of the models' horizon effects

To briefly summarize, assume the time-0 risk-free rate is 3.78%, the mean-reverting market portfolio dynamics are as described above, and you are an investor with $b = 3$ and a 2-period horizon. If you assume the risk-free rate will remain constant, as in the Samuelson–Kritzman model, your optimal initial market portfolio allocation is 98.9% of your wealth. If instead you are in the capital market setting where the representative investor has $b^* = 3$ and a 2-period horizon, and thus where the future risk-free rate is uncertain and the market risk premium reflects the intertemporal hedging benefit, you initially allocate 100% of wealth to the market portfolio. Both of these allocations display the horizon effect of being higher than if you had a single-period horizon, 87.0%.

We see that the difference between the horizon effects in the capital market and Samuelson–Kritzman models is relatively small when the investor has the same CRRA and horizon as the representative investor, but what about for investors with other levels of b? Table 1 summarizes a comparison between the horizon effects of the capital market and Samuelson–Kritzman models for various levels of CRRA. The Samuelson–Kritzman model assumes a constant risk-free rate of 3.78%. The capital market model assumes the risk-free rates in Figure 4.

Table 1 Comparison of horizon effects. Capital market model vs Samuelson–Kritzman model. Mean reversion; representative investor with 2-period horizon and $b^* = 3$.

b-Risk Aversion	1-Period Allocation	Initial Allocation Samuelson	Initial Allocation Capital Market	Samuelson % Above 1-Period	Capital Market % Above 1-Period	Capital Market % Above Samuelson
1	2.739	2.739	2.739	0	0	0
1.5	1.791	1.917	1.865	7.0%	4.1%	−2.7%
2	1.326	1.464	1.432	10.4%	8.0%	−2.2%
2.5	1.051	1.182	1.175	12.5%	11.8%	−0.6%
3	0.870	0.989	1	13.7%	14.9%	1.1%
3.5	0.743	0.851	0.883	14.7%	19.0%	3.8%
4	0.647	0.746	0.793	15.3%	22.6%	6.3%
4.5	0.573	0.664	0.723	15.9%	26.2%	8.9%
5	0.515	0.598	0.667	16.1%	29.5%	11.5%
5.5	0.467	0.544	0.622	16.5%	33.2%	14.3%
6	0.427	0.499	0.584	16.9%	36.8%	17.0%

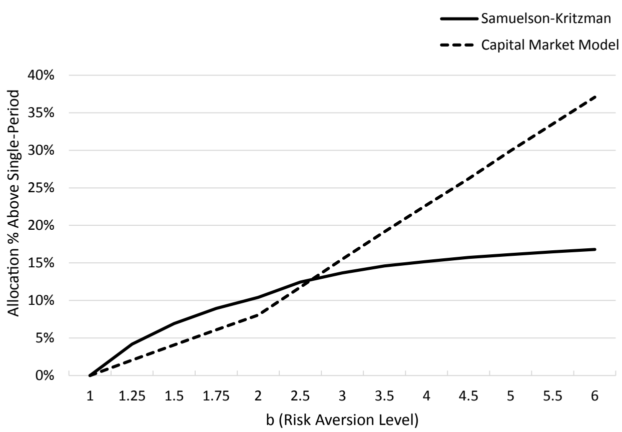

Figure 5 Comparison of horizon effects. Capital market model vs Samuelson–Kritzman model. Mean reversion; representative investor with 2-period horizon and $b^* = 3$.

Figure 5 shows a graphical comparison of the initial market portfolio allocations, expressed as percentage above the single-period allocation, given the time-0 risk-free rate of 3.78%.

In both the capital market and Samuelson–Kritzman models, we see that the horizon effect rises as risk aversion increases, and for CRRA levels lower than the representative investor's ($b < b^* = 3$), the horizon effect is relatively small. The horizon effect tends to be slightly higher in the Samuelson–Kritzman model up to a level of b that is near but below $b^* = 3$, around 2.75. For CRRA levels higher than the representative investor's, the horizon effect in the capital market model becomes substantially and increasingly stronger than in the Samuelson–Kritzman model. For an investor with $b = 6$, the initial market portfolio allocation in the capital market model, 0.584, is 17.0% higher than in the Samuelson–Kritzman model, 0.499.[12]

The reason for the horizon effect in the Samuelson–Kritzman model, where the risk-free rate is constant, is that changes in the market portfolio's expected return covary negatively with the market portfolio return, which partially hedges the risk of time-2 wealth; so a 2-period investor with $b > 1$ allocates more to the market portfolio at time-0 than a single-period investor. In the capital market model, the risk-free rate uncertainty further affects the risk of time-2 wealth. If $b < b^*$, neither of the intertemporal uncertainty effects makes much difference in expected utility of time-2 wealth, and thus in the initial allocation. If $b > b^*$, a 2-period investor makes a higher time-0 market portfolio allocation than if the risk-free rate is constant. This effect gets more and more substantial as b increases relative to b^*.

Note that for a given horizon, an investor who is more risk averse than the representative investor allocates like a traditional long-term investor who holds a positive fixed income position and a portfolio of stocks. For $b = 6$ for example, the allocation is 58.4% to stocks (the market portfolio) and 41.6% to fixed income (the risk-free asset). Investors who are less risk averse than the representative investor typify holders of levered stock portfolios.[13] These characterizations suggest that the difference between the models' horizon effect is important to traditional long-term investors, but not so much to levered investors.

Figure 5 compares the initial market portfolio allocations for a 2-period horizon in the capital market and Samuelson–Kritzman models for the illustrated mean-reverting market portfolio and representative investor with a 2-period horizon and constant CRRA $= 3$. The allocations are expressed as the percentage above the single-period allocation, given a time-0 risk-free rate of 3.78%. The Samuelson–Kritzman model assumes a constant risk-free rate of 3.78%. The capital market model assumes the risk-free rates in Figure 4.

6 Longer horizon illustration

Although two periods are sufficient to see the insights, one may extend the analysis to more periods. The representative investor has a single-period horizon for the final period, but could have, say, a 2-period horizon at all other times, because the representative investor is not a real investor but a fictional (weighted) average of all investors. In this scenario with more periods, the market risk premium would contain an intertemporal hedging effect for all periods except the last, if the representative investor's CRRA $\neq 1$.

As an example, assume a 3-period scenario where the probability of an up-state (down-state) is: 0.40 (0.60) at the time-2 up-state; 0.50 (0.50) at the time-2 middle-state; and is 0.60 (0.40) at the time-2 down-state. Figure 6 shows the risk-free rates if the representative investor continually has a 2-period horizon (until the final period) and constant $b^* = 3$.

Assume an investor with a 3-period horizon and $b = 4$. With the risk-free rates in Figure 6, you can work backward to find that the investor's expected utility of time-3 wealth is maximized with a time-0 market portfolio allocation of 0.853. If the risk-free rate is constant and equal to the time-0 risk-free rate, 3.78%, the expected utility of time-3 wealth is maximized with a time-0 market portfolio allocation of 0.797. Therefore, the capital market model's time-0 market portfolio allocation exceeds Samuelson–Kritzman model's allocation by 7.0%. Because the percentage difference between the two models' horizon effects is 6.3% when the investor with $b = 4$ has a 2-period horizon,[14] we see that the horizon effect difference between the two models is even larger for an investor who has a longer horizon, if risk aversion is higher than the representative investor's.

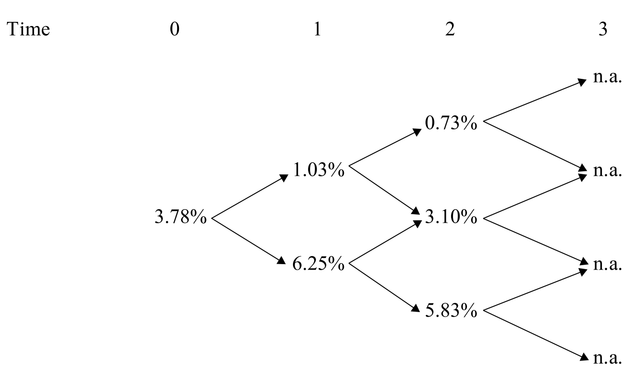

Figure 6 Single-period risk-free rates for 3-period example. Mean reversion; representative investor with 2-period horizon and $b^* = 3$.

7 Mean reversion and momentum

Because momentum, or positive serial correlation, is the opposite of mean reversion, it is not surprising that Samuelson (1991) showed that in a momentum market, an investor with CRRA > 1 will optimally make a lower initial risky asset allocation and expect to increase the allocation over time.[15] Illustrations of this horizon effect are straightforward extensions of the ones above.

Subsequent to the Samuelson (1991) and Kritzman (1994) publications, empirical support for momentum grew, e.g., Jagadeesh and Titman (1993), Carhart (1997), and Asness *et al.* (2014). Empirical support for mean reversion was challenged for a while, as reviewed by Bali *et al.* (2008), but was later revived, e.g., Gropp (2004), Bali *et al.* (2008), and Pástor and Stambaugh (2012).

Based on the empirical literature, including Fama and French (1988), Balvers and Wu (2006) (and others) have suggested that the reconciliation of the empirical support for both mean reversion and momentum is that mean-reversion strategies work for horizons of three to five years, whereas momentum strategies work for horizons of one month to one year. This reconciliation suggests that investors with $b > 1$ should: (1) for longer horizons, start with a higher initial stock allocation and expect to decrease the allocation over time; and (2) for relatively short horizons, start with a lower initial stock allocation and expect to increase the allocation over time.[16]

8 Conclusion

This note illustrates two-fund dynamic allocation in a capital market model with a representative investor who has constant relative risk aversion (CRRA) that is higher than 1 and stays constant over time. Given a mean-reverting market portfolio, the future risk-free rate is uncertain, and the market risk premium and the risk-free rate both reflect the intertemporal effects of uncertain future investment opportunities.

The illustrations show two main results. First, the capital market model yields a horizon effect that is consistent in direction with the horizon effect in Samuelson's (1991) constant risk-free rate model: an investor with CRRA > 1 makes a higher market portfolio allocation for a longer horizon if the market portfolio is mean-reverting. The second, and more interesting result, is that for investors with CRRA higher than the representative investor's, the horizon effect is substantially stronger in the capital market model than in Samuelson's model, due to the intertemporal hedging benefit of the future risk-free rate uncertainty.

Because an investor with CRRA above the representative investor's typifies a traditional long-term investor who holds a long fixed income position combined with a stock portfolio, and an investor with CRRA below the representative investor's typifies an investor who holds a levered stock portfolio, the findings here are insightful in a practical sense: A time diversification strategy, of allocating more to stocks for longer horizons, can be a good strategy for traditional long-term portfolio investors, but not for levered stock investors.

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Notes

1 A time-0 risk-free rate of 3.78% is reasonably consistent with the long-run average three-month Treasury bill rate, 3.50% (http://pages.stern.nyu.edu/~adamodar/New_Home_Page/datafile/histretSP.html). The time-0 market risk premium of $9\%-3.78\% = 5.22\%$ is slightly below the historical average (same site), but is in line with *ex ante* estimates, e.g. Welch (2009).

2 $U(W_2) = 0.25\{1 - [1.24w_0 + 1.0378(1 - w_0)] \cdot [1.24(0.87)+1.0378(1-0.87)]\}^{-2}+0.25\{1-[1.24w_0+1.0378(1-w_0)]\cdot[0.94(0.87)+1.0378(1-0.87)]\}^{-2}+0.25\{1 - [0.94w_0 + 1.0378(1 - w_0)] \cdot [1.24(0.87) + 1.0378(1-0.87)]\}^{-2}+0.25\{1-[0.94w_0+1.0378(1-w_0)] \cdot [0.94(0.87) + 1.0378(1 - 0.87)]\}^{-2}$.

3 McEnally's (1985) critique of the time diversification strategy is also insightful. The term "time diversification" is also used in various other ways. For example, Treynor (2003) uses the term to describe selling after market prices rise and buying after drops, so as to maintain constant dollar risk. Black and Scholes (1974) suggest that investing $1,000 into a risky asset in two successive periods is better than plunging $2,000 in the first period, because the first option has more "time diversification". *Across-time diversification* is allocating 100 percent to a risky asset for *X* percent of the

years in the horizon, which Samuelson (1990) showed is inferior to allocating X percent to the risky asset in all years, given the random walk case. Chung *et al.* (2009) further clarify the notion of time diversification in dynamic asset allocation.

4 $U(W_2) = 0.20\{1 - [1.24w_0 + 1.0378(1 - w_0)] \cdot [1.24(0.377) + 1.0378(1 - 0.377)]\}^{-2} + 0.30\{1 - [1.24w_0 + 1.0378(1 - w_0)] \cdot [0.94(0.377) + 1.0378(1 - 0.377)]\}^{-2} + 0.30\{1 - [0.94w_0 + 1.0378(1 - w_0)] \cdot [1.24(1.379) + 1.0378(1 - 1.379)]\}^{-2} + 0.20\{1 - [0.94w_0 + 1.0378(1 - w_0)] \cdot [0.94(1.379) + 1.0378(1 - 1.379)]\}^{-2}$.

5 Samuelson (1991) also showed: (1) a CRRA rebalancer with log utility, where $b = 1$, derives no benefit from mean reversion and thus allocates myopically to the risky asset; and (2) a CRRA rebalancer with $b < 1$ derives a negative benefit from mean reversion and thus allocates less to the risky asset for a longer horizon. Others who have addressed non-myopic allocation include Hakansson (1971) and Kim and Omberg (1996), with the latter providing the most thorough treatment of non-myopic allocation assuming a constant risk-free rate and investors who maximize expected utility of horizon wealth. Wachter (2002) models the dynamic allocations of investors who maximize utility of period-by-period consumption.

6 This assumes standard CRRA utility functions. The Campbell and Cochrane (1999) representative investor has a "habit formation" feature where risk aversion changes in such a way that the risk-free rate is constant. Illustrating that model is beyond the scope here.

7 Empirical estimates of b^* depend on the theoretical model used for the estimation. Single-factor models, like the CAPM, yield lower b^* estimates than multi-period models or models with additional factors like human capital. With monthly data, Campbell (1996) used a multiperiod model to estimate $b^* = 3.2$ if human capital is ignored, and $b^* = 7.8$ if human capital is 2/3 of the representative investor's total wealth. The first estimate is more appropriate for this analysis, which ignores human capital.

8 In a single-period mean–variance model, the market risk premium is equal to $b^* \cdot var(R_M)$. Because of the binomial framework, the scenario's time-1 market risk premia are more consistent with those found if the market's return were lognormally-distributed with the same continuously-compounded mean, μ, and variance, $var(r_M)$, where $r_M = ln(R_M)$. In this case, the market risk premium, $\mu + 0.5var(r_M) - r$, is equal to $b^* \cdot var(r_M)$, as in Rubinstein (1976) and Leland (1999).

9 $U(W_2) = 0.20\{1 - [1.24w_0 + (1 + r_0)(1 - w_0)] \cdot 1.24\}^{-2} + 0.30\{1 - [1.24w_0 + (1 + r_0)(1 - w_0)] \cdot 0.94\}^{-2} + 0.30\{1 - [0.94w_0 + (1 + r_0)(1 - w_0)] \cdot 1.24\}^{-2} + 0.20\{1 - [0.94w_0 + (1 + r_0)(1 - w_0)] \cdot 0.94\}^{-2}$.

10 Munk *et al.* (2004) got a similar result in a formal model with uncertain risk-free rates, but assume that the rates follow an exogenously-specified process, in contrast to the approach here where the risk-free rate must be consistent with the assumed market portfolio dynamics and representative investor's horizon and risk aversion.

11 Campbell (1996) and Guo and Whitelaw (2006) find that that the intertemporal risk of future investment opportunities is empirically negatively correlated with market returns and significant. Here, the covariance between the time-1 expected log of the time-2 market portfolio return, $E(r_{M1})$, and the log of the time-1 realized market portfolio return, r_{M1}, is -0.0038. Applying Campbell's (1996) Equation (9) after market aggregation and with consumption $= 0$, the time-0 market risk premium should be approximately equal to the market risk premium if the representative investor has a single-period horizon, 0.059, plus $(b^* - 1) \cdot covar(E(r_{M1}), r_{M1}) = 0.059 + (3 - 1)^*(-0.0038) = 0.0514$, which is reasonably consistent with the estimate in the illustration, 0.0522.

12 In an unreported robustness example where the market portfolio's unconditional expected rate of return is 10%, the initial standard deviation is 14%, and all else is the same, the risk-free rates are higher and the market risk premia are lower, but the results are almost identical to those in Table 1, especially for risk aversion levels above the representative investor's.

13 Investors who hold levered stock portfolios are exemplified by hedge funds, leveraged mutual funds, corporate and public pension funds that are (perhaps implicitly) funded by some amount of debt, and individuals who use margin or other forms of borrowing. Note that the market portfolio position of investors who are less risk averse than the representative investor is not necessarily equal to that of investors who are more risk averse. For example, assume a single-period mean–variance model where the allocation is b^*/b, one investor who has wealth of \$20 and $b = 1.50$, and another who has wealth of \$100 and $b = 3.75$. As we will see, the representative investor has $b^* = 3$. The first investor allocates $b^*/b = 2$, or 200% of wealth, to the market portfolio.

Thus, the first investor allocates \$40 to the market portfolio, borrowing \$20 from the second investor. The second investor allocates $b^*/b = 0.80$, or 80% of wealth, to the market portfolio, \$80, and lends \$20 to the first investor. The market portfolio's value is \$120, with one-third held by the first investor (\$40) and two-thirds held by the second investor (\$80). The market's representative investor thus has $b^* = 3$, because $(1/3)1.50 + (2/3)3.75 = 3$.

[14] You can see the 6.3% difference in Table 1 for 2-period investors in a 2-period scenario. In the 3-period scenario, the Samuelson–Kritzman model's initial allocation is exactly the same as in Table 1, because of the constant interest rate assumption, and the capital market model's initial allocation is only microscopically different than in Table 1, despite the difference in the time-1 risk-free rates in the 3-period and 2-period scenarios.

[15] This strategy is thus the *opposite* of a time diversification strategy, and (in principle) similar to a practitioner allocation strategy called *dollar-cost averaging* (even though Black and Scholes (1974) call a dollar-cost averaging example "time diversification"; see Footnote 3). A dynamic allocation scenario has otherwise not been found to justify dollar-cost averaging, which has been typically regarded by scholars as an inferior dynamic strategy (Constantinides, 1979; Knight and Mandell, 1993; Rozeff, 1994).

[16] The opposite strategies would be appropriate for investors with $b < 1$, if any exist.

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