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MEAN–VARIANCE OPTIMIZATION WITH PUBLIC AND PRIVATE ASSET CLASSES

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Liquidity has long been of great interest to investment professionals as well as academic researchers. The estimation of the illiquidity premium for infrequently traded asset classes, such as real estate and private equity, presents a challenge to the industry because of opaque information and sporadic trading activities. We propose using the autocorrelations of returns as a tool to estimate the transaction costs and illiquidity premium of private assets. This tool can also be used to adjust the risk of illiquid asset classes. At the end of this article, we show through an example that after making these adjustments to the estimates of expected return and risk, private and illiquid assets can be reasonably compared with public and liquid assets in the standard mean–variance optimization (MVO) process.

1 Introduction

The mean–variance optimization (MVO) framework is perhaps the most commonly used tool for making asset allocation decisions. The foundational work laid out by Markowitz over sixty years ago (Markowitz, 1952) provides an elegant framework to balance return and risk in the portfolio selection process. In the traditional MVO framework, impacts of liquidity are less emphasized, and investors are generally assumed to be able to readily trade all assets without market frictions. This assumption was reasonable sixty years ago as illiquid and private asset class investments such as private equity and private real estate were not considered mainstream investment vehicles. More recently, illiquid and private assets have garner increasing popularity with institutional investors seeking to enhance returns and to build a more diversified portfolio. However, investors face a tremendous challenge when mixing the liquid and the illiquid assets in constructing a diversified portfolio, which is analogous to mixing oil and water. This challenge, which is mostly due to different liquidity risk levels between public and private assets, has been

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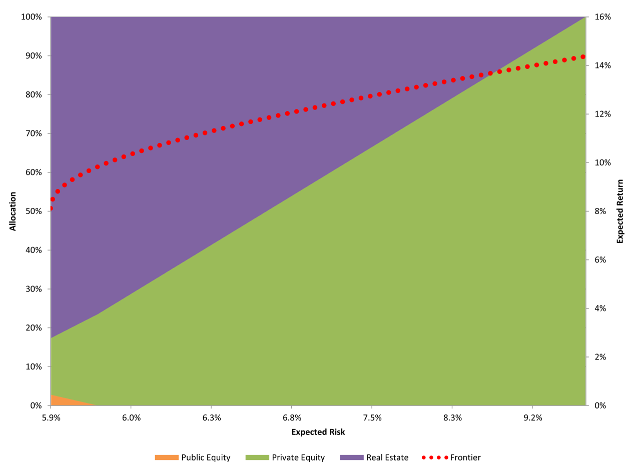
addressed by researchers developing approaches to enhance the MVO framework. Lo *et al.* (2003) proposed mean–variance-liquidity frontiers that can help investors significantly reduce liquidity risk while only modestly sacrificing expected return.

Liquidity risk is of particular concern for investors with liquidity requirements in the foreseeable future. The consequence of mis-managing this risk was prominently highlighted during the 2008 financial crisis, when institutional investors, led by Harvard University, were forced to offer their stake in private equity funds at discounts of at least 50% (Keehner and Kelly, 2008). However, investors that are able and willing to hold illiquid assets over a long time horizon might view this liquidity concern less as a risk and more as an opportunity to harvest a premium since rational investors require additional compensation for assuming additional risk. Therefore, holding all other variables constant, illiquid asset investors should be compensated with an additional risk premium (i.e., illiquidity premium) when compared with liquid asset investors. This concept has been extensively researched by both academia and practitioners. Most researchers use factor models to derive an illiquidity premium by constructing portfolios based on specific liquidity measures and comparing the returns of the most liquid portfolio with those of the least liquid portfolio. Amihud (2002) used bid–ask spreads to measure liquidity and suggested that an illiquidity premium is embedded in the excess return of stocks, especially for small stocks. Ibbotson *et al.* (2013) augmented the Fama–French model with an additional liquidity factor based on stock turnover. Khandani and Lo (2011) studied the relationship between illiquidity and positive autocorrelation in asset returns of a large sample of hedge funds and mutual funds. They documented the illiquidity premium ranges from 2.74% to 9.91% per year. Kinlaw *et al.* (2013) compared the risk-adjusted illiquid hedge fund returns with those of a liquid ETF tracking portfolio and estimated the illiquidity premium of these various strategies range from 1.63% to 4.86% per year. Franzoni *et al.* (2012) used a four-factor model that includes a liquidity factor to analyze private equity data from a proprietary database and reported a illiquidity premium of about 3% annually.

The illiquidity of private assets also has a significant and direct impact on transaction costs. Although Yoshimoto (1996) accounted for this relationship in the portfolio optimization problem by suggesting a nonlinear programming technique, estimating the actual transaction cost for private assets remains a challenge. The myriad of research in estimating transaction costs has been limited to public assets, such as equities and bonds. In the public financial markets, transaction costs are usually measured in the form of bid–ask spreads. In the equity space, Loeb (1991) estimated the commission-less transaction costs with a nonlinear model in which the two independent variables are the market capitalization and the percentage of outstanding shares traded. In the bond space, Ben Dor *et al.* (2011) proposed a new metric called Liquidity Cost Score (LCS) to quantify the trading costs of an individual bond. In both methods, the transaction costs are measured against a round-trip trade. These approaches for publicly traded assets cannot be extended to private assets because there is neither an exchange for private assets nor any readily observable bid–ask spreads like those in the public markets. In addition, the sheer size and timing of each private asset transaction increases the transaction costs dramatically as investors are required, at times, to offer tremendous discounts when selling private assets, particularly during market stresses—an impact that is also observed with public assets but to a lesser extent.

Illiquidity also subdues the risks of private assets, measured by volatilities and correlations, as returns are often smoothed. Two potential reasons for this smoothing effect: (1) the fundamentals and the capital structure of a business do not change quickly, and (2) the valuation of assets is often appraisal-based and performed quarterly or even annually. A consequence of this smoothing effect is an overly concentrated allocation to private assets as illustrated in Figure 1, which shows the optimal portfolio allocation of public and private assets under the MVO framework that estimates risks simply from historical data. As the figure shows, the optimal portfolios are comprised almost entirely of private equity and real estate, which usually makes these portfolios less implementable for large institutional investors because of market capacity limitations and liquidity constraints of investors. In addition to the impracticality of a concentrated private asset portfolio, it is perhaps also inappropriate to blindly apply the MVO framework. The smoothing effect of appraisal-based valuation results in an artificially low volatility and correlation estimate while the embedded illiquidity premium in private assets artificially overstates the expected return. This combination leads to a phantom phenomenon of a high risk-adjusted return forecast favored by the MVO framework. Despite this issue, we believe that the MVO framework is still an indispensable tool to construct portfolios with the caveat of being mindful of liquidity considerations when including private assets. The estimates of the two major inputs into



Estimates from historic quarterly data (09/1988–06/2014).

	Public Equity	Private Equity	Real Estate
Annualized return (%)	11.2	14.4	6.9
Annualized volatility (%)	15.8	9.9	6.1
Correlation			
Public Equity (S&P 500)	1.00		
Private Equity (Cambridge Associates U.S. Private Equity Index)	0.71	1.00	
Real Estate (NCREIF Fund Index - Open End Diversied Core Equity)	0.14	0.35	1.00

Figure 1 Efficient frontier and asset allocations with three assets.

the framework, expected risks and returns, warrant adjustments for illiquidity if historic data are involved in the estimation process. As Getmansky *et al.* (2004) pointed out, the illiquidity and smoothed returns are likely the explanation of autocorrelations observed in hedge fund returns; therefore, we believe that the autocorrelation is a helpful tool in making such adjustments.

Our paper contributes to the existing literatures in three areas: first, we show that autocorrelations observed in historical returns can be used to estimate illiquidity premium. Second, we discuss a way to infer the transactions costs of private assets using autocorrelations and the observed trading costs of public assets. Finally, we explicitly incorporate autocorrelations into a desmoothing model to obtain a less biased estimate of illiquid asset risks to uniformly and equitably compare the risks of liquid and illiquid assets. We illustrate this process by comparing the risk characteristics after applying our models to two major illiquid asset classes, private equity and real estate. The proxies used for these two asset classes are the Cambridge Associates U.S. Private Equity Index (PE Index) and the NCREIF Fund Index—Open End Diversified Core Equity (ODCE Index). We are aware of the inherent biases of these indices such as the selection bias attributed to the PE Index and the ODCE Index only representing a small portion of the investable real estate universe; however, we believe these biases will not have a material effect on our findings.

2 Illiquidity premium

There is abundant research on the existence of the illiquidity premium from both a theoretical and an empirical perspective. Amihud and Mendelson (1986) postulated that an illiquidity premium exists because investors are different with regards to investment time horizon and liquidity needs. Long-term investors are on one end of the spectrum of liquidity needs and they usually hold securities to maturity (e.g., bond investors) or for an extended time period (e.g., very long-term equity investors such as Warren Buffet). These investors do not have imminent liquidity needs and therefore are able to hold illiquid assets to harvest the illiquidity premium. On the other end of the spectrum are investors with liquidity needs in the foreseeable future and therefore prefer liquid securities. As long as there are different types of investors, or a clientele effect, the illiquidity premium will persist.

The empirical evidence also supports this postulation. Hibbert *et al.* (2009) conducted a literature survey that suggested a wide range of illiquidity premium. The majority of the analyses in their survey were focused on public assets, but they found that the illiquidity premium ranges from 0.1% to 6.5% depending on the credit quality of corporate bonds. Government bonds are more liquid, and thus, the illiquidity premium only ranges between 0.1% and 0.55%. The illiquidity premium of equities turned out to be quite high with a range of 3.5% to 7.7%.

There are different approaches to measure this illiquidity premium. For corporate bonds, Hibbert *et al.* (2009) summarized three major methods for estimating the illiquidity premium: microstructure, direct and structural model, and regression-based approach. For equity, researchers mainly use a regression-based approach. For private assets, there are two major methods in estimating the illiquidity premium in private assets. The first method is also a regression-based approach. Franzoni *et al.* (2012) performed a regression analysis of private equity returns between 1975 and 2006 from a proprietary database against four public factors, one of which is the liquidity risk factor. They estimated the beta of the liquidity risk factor to be about 0.64 and an illiquidity premium of about 3%. The second method is to compare private asset returns with those of a comparable public asset benchmark. Harris *et al.* (2014) used Burgiss cash flow data since 1984 and compared the performance of US Buyout funds with S&P 500 and found an outperformance of 3%.

We use a regression-based approach that is different than that employed by Franzoni *et al.* (2012) since our method does not rely on an external liquidity factor. Our method also has the added benefit of avoiding the benchmark requirement needed by the public asset benchmark comparison approach as it may be challenging to find appropriate benchmarks for private assets. For instance, the constituents in the PE Index and its public counterpart, such as S&P 500, are vastly different in terms of key risk characteristics such as sector and leverage. In addition, one can argue that part of the outperformance of private equity over a public equity index can be attributed to manager skill instead of the illiquidity premium, which is supported by this study as we will see in the later section.

Motivated by Lo (2001), our approach to illiquidity hinges on the following proposition.

Proposition I:

Asset returns are only forecastable to the extent that the forecastability cannot be arbitraged away due to illiquidity. As such, the observed forecastability of asset returns can be used as a proxy for asset class illiquidity.

How can one measure asset return forecastability? We revert to the autocorrelations observed in asset returns for guidance.

For investors in public and liquid assets, the past returns usually have no impacts on future returns; however, with private assets, we observe a strong relationship between the forecastability of the current reported returns and the past returns which we attribute to an illiquidity effect based on Proposition I. Because most investors in private assets buy and hold given the barriers preventing frequent transactions, these investors are being compensated for simply holding these illiquid assets. If investors are free to trade private assets without incurring significant costs, the impacts of past returns should be arbitraged away akin to what we observe in the public markets. It is also interesting to observe that the past returns of the ODCE Index can explain about 80% of its current returns while for the PE Index, only about 17% can be explained. This suggests the ODCE Index is affected by past returns more than the PE index, which suggests that the ODCE Index is less liquid.

These findings are based on multi-linear regressions of returns, r_t, of the index against its past returns.

$$r_t = a + \beta_1 r_{t-1} + \beta_2 r_{t-2} + \beta_3 r_{t-3} + \beta_4 r_{t-4} \tag{1}$$

The returns in past four quarters ($r_{t-1}, \ldots, r_{t-4}$) are chosen because of significant autocorrelations in past four quarters of the ODCE Index as indicated in Table 1. As shown in Table 2, the regression coefficients of past returns are significant for both the ODCE and the PE Index, especially for returns lagged by one quarter. The betas of lag-1 return for the ODCE Index and the PE Index are 1.15 and 0.33, respectively. This suggests that investors of the ODCE Index are expecting 115% of the previous quarter returns to be carried over to the next quarter while investors of the PE Index are expecting only 33% of the previous quarter returns to carry over. The 1.15 beta value of the lag-1 return for the ODCE Index may not be intuitive since it produces an exponentially increasing return series holding all else constant. We resolve this issue by calculating a total beta of past returns, and although not all betas of past returns are statistically significant in the regressions, we include the full set so the analyses of both indices are consistent. We conjecture that one year will be a reasonable time period for the impacts of the past to dissipate. To sum up

Table 1 Sample autocorrelations.

Index	Lag-1	Lag-2	Lag-3	Lag-4	Lag-5	Lag-6	Q-Stat (p-value)
ODCE Index	0.87*	0.67*	0.47*	0.26*	0.06	−0.07	163.08
PE Index	0.38*	0.29*	0.14	0.11	−0.03	0.01	28.20
Barclays US Corporate High Yield	0.29*	−0.14	−0.01	−0.07	−0.07	−0.05	12.58 (0.05)
Barclays US Corporate Investment Grade	0.08	0.11	−0.07	−0.20	−0.09	−0.12	9.42 (0.15)
Russell 2000	−0.08	−0.08	−0.05	−0.07	−0.02	0.00	2.43 (0.88)
S&P 500	0.07	0.04	0.02	0.01	0.01	0.11	2.10 (0.91)

*Autocorrelations that are significant at 5% level.
Quarterly data (09/1988–06/2014) from Barclays Capital, Bloomberg, NCREIF, Cambridge Associates.

Table 2 Regression against Past Returns (t-statistics in parenthesis).

		Beta					
	Intercept	Lag-1	Lag-2	Lag-3	Lag-4	Sum	R^2
ODCE Index	0.38	1.15	−0.33	0.13	−0.17	0.78	0.80
	(2.28)	(11.58)	(−2.14)	(0.83)	(−1.68)	(14.70)	
PE Index	1.84	0.33	0.17	−0.03	0.03	0.49	0.17
	(2.86)	(3.24)	(1.60)	(−0.32)	(0.30)	(3.71)	

Regression against Past Returns and Exogenous Factors (t-statistics in parenthesis).

		Beta								
	Intercept	Lag-1	Lag-2	Lag-3	Lag-4	Sum	S&P 500	10-yr Treas. Yld	CPI	R^2
ODCE Index	−0.13	1.21	−0.32	−0.01	−0.09	0.80	0.04	0.69	0.61	0.87
	(−0.73)	(14.40)	(−2.50)	(−0.06)	(−1.07)	(18.23)	(2.87)	(2.83)	(4.17)	
PE Index	1.09	0.26	0.16	−0.02	0.07	0.47	0.40	1.45	0.03	0.66
	(2.22)	(3.82)	(2.23)	(−0.29)	(1.13)	(5.44)	(10.19)	(2.37)	(0.09)	

Quarterly data (09/1988–06/2014) from NCREIF, Cambridge Associates, Bloomberg and DataStream.

the total effect of previous returns, we calculate the expectations on both sides of Equation (1) assuming $E(r_t) = E(r_{t-1}) = \cdots = E(r_{t-4})$.

$$E(r_t) = E(a) + \sum_{i=1}^{4} \beta_i E(r_{t-i})$$
$$= E(a) + \sum_{i=1}^{4} \beta_i E(r_t) \quad (2)$$

We believe the assumption is reasonable for a long return series. The second component on the right side of the above equation is related with auto-correlations and can be viewed as a proxy of the percentage contribution from the illiquidity premium to the total return because of Proposition I. As we see from the top panel of Table 2, the total beta of the previous four quarterly returns, or illiquidity beta, is 0.78 for the ODCE Index and 0.49 for the PE Index with both betas being statistically significant. Combining this illiquidity beta with the historic annual return of 6.91% (14.38%) of the ODCE Index (PE Index) gives an estimated annual illiquidity premium of 5.4% (7.05%) for the ODCE Index (PE Index). The illiquidity premium of the PE Index is greater than that of the ODCE Index because the total return of the PE Index in the sample period (between 1988 and 2014) is higher even though the percentage attributable to the illiquidity premium is lower. We further analyze the returns of the two indices by introducing three exogenous factors: the S&P 500, the 10-year Treasury Yield and the US Consumer Price Index NSA (CPI). Our regression model is as follows:

$$r_t = a + \sum_{i=1}^{4} \beta_i r_{t-i} + \beta_S r_t^{SPX}$$
$$+ \beta_T \Delta r_t^T + \beta_C r_t^C \quad (3)$$

where r_t^{SPX}, Δr_t^T and r_t^C are the contemporaneous returns of the S&P 500, the change in 10-year Treasury yields and the percentage changes of the CPI Index, respectively. The regression coefficients (β_S, β_T and β_C) can be interpreted as the public equity market beta, the interest rate beta (or negative duration), and the inflation beta, respectively.

The results of the regression are shown in the bottom panel of Table 2. Although both indices have a positive exposure to the S&P 500, the beta for the ODCE Index is smaller, which suggests that real estate is less affected by public equity markets.

Both indices also react positively to changes of interest rates or have negative durations, which is contrary to the predictions of traditional theories such as the well-known dividend discount model. However, both the PE Index and the ODCE Index have a significant amount of debt at the fund level. For the PE Index, the majority of constituents are buy-out funds, so it is reasonable to believe that the value of a private equity fund will increase with interest rates since the value of debt will decrease with rising interest rates. For the ODCE Index, the average historical leverage between March 2000 and June 2014 is 0.22 which means debt was about 22% of the total gross assets of the index, so a positive relationship between the ODCE Index and nominal interest rates is not surprising.

The ODCE Index has a positive exposure to CPI because its constituents are commercial real assets, and landlords usually have the power to increase rents when inflation is expected to rise. The CPI exposure of the PE Index is statistically insignificant, which also agrees with the results of other researchers such as Boudoukh and Richardson (1993).

The introduction of three exogenous factors improves the explanatory power of the two index returns. The majority of historic returns in the sample period can be explained by the illiquidity premium, the equity market, the interest rate

Table 3 Regression against past returns and exogenous factors (*t*-statistics in parenthesis).

		Beta					S&P	10-yr		
	Intercept	Lag-1	Lag-2	Lag-3	Lag-4	Sum	500	Treas. Yld	CPI	R^2
NCREIF TBI Index	1.15	−0.18	0.16	0.21	0.00	0.19	0.12	−1.47	1.13	0.16
	(1.26)	(−1.68)	(1.50)	(2.01)	(−0.00)	(0.87)	(1.49)	(−1.08)	(1.61)	
LPX Buyout Listed Private Equity Index	−0.34	0.20	−0.20	0.10	−0.07	0.04	1.09	2.35	2.56	0.66
	(−0.25)	(2.72)	(−2.57)	(1.36)	(−0.97)	(0.27)	(8.09)	(0.99)	(2.16)	

Quarterly data (03/1995–06/2014) from NCREIF and Bloomberg.

change and the inflation rate. For the ODCE Index, the intercept becomes statistically insignificant. For the PE Index, the intercept drops to 1% but is still statistically significant. This suggests that investors may still benefit from investing in private equity funds due to fund managers management skills after accounting for these four factors. However, Harris *et al.* (2014) pointed out that the data from Cambridge Associates might be biased towards GPs that have performed well and are engaged in raising new funds. It is also interesting to observe that the regression coefficients of past returns barely change after introducing these three exogenous factors; therefore, our estimates of the illiquidity premium still holds for both indices even after introducing these exogenous factors.

As an insightful exercise to identify the source of the illiquidity premium, we applied our model to liquid indices that share similar risk characteristics with the ODCE Index and the PE Index. Although finding liquid indices that have the same constituents as the two private indices is difficult, there are comparable indices that can provide some insights. For the PE Index, the LPX Buyout Listed Private Equity (LPX BO) Index contains all major publicly listed private equity companies that provide capital for buyout deals. For the ODCE Index, NCREIF has a transaction-based index (TBI Index) with quarterly data that is based on properties that were in the NCREIF Property Index and were sold to that respective quarter. Both the LPX BO and the TBI Index are deemed to be more comparable to liquid indices; therefore, we expect the illiquidity premium for these indices to be smaller than their respective illiquid indices. As the results in Table 3 indicate, the illiquidity betas of both indices are statistically insignificant. These results suggest that there is minimal illiquidity premium in liquid private asset returns, and the observed illiquidity beta in non-tradable private asset returns is due to the lack of actual market transactions and appraisal-based reported returns.

3 Transaction costs

Although investors may be compensated with a premium for investing in illiquid assets, the transaction costs to realize this premium cannot be presumed to be the same as in the public markets. The transaction costs of public assets have been researched extensively. Harris (2003) identified, through empirical evidence and the theory of market microstructure, three major sources of liquidity immediacy, depth, and resilience. Amihud *et al.* (2005), on the other hand, provided a survey on the sources of illiquidity that is summarized as stemming from five major sources: exogenous transaction costs, demand pressure, inventory risk, private information, and search friction. There are also plenty of literatures on liquidity proxies, which include bid–ask spreads, unique roundtrip costs, return-to-volume measures, number of zero-return days, turnover, volatility and liquidity cost score (Ben Dor *et al.*, 2011; Hibbert *et al.*, 2009).

In sharp contrast, estimating transaction costs of illiquid assets remains very challenging particularly due to an absence of bid–ask spreads from a central exchange. Although the private equity secondary markets have grown substantially in recent years and have become a source of trading information, compared with the size of public equity markets, the secondary market of private equity is miniscule.

Although we cannot directly observe the bid–ask spreads of private assets, and obtaining transaction costs directly for private asset transactions is very difficult, we aim to estimate the transaction costs using statistical tools and the observable bid–ask spreads from public markets. Bid–ask spread data for public equity markets are widely available; however, in the fixed income space, it is only recently that data, akin to bid–ask spreads, are provided by institutions such as Barclays Capital LLC (Ben Dor *et al.*, 2011).

Lo (2001) suggested using a statistical metric called Q-statistic, Q_m, to proxy illiquidity:

$$Q_m = T(T+2)\sum_{k=1}^{m}\frac{\rho^2(k)}{T-k} \tag{4}$$

where T is the sample size, $\rho(k)$ is the lag-k autocorrelation and m is the number of lags included in the calculation of the Q-statistic.

In the last column of Table 1, we show the Q-statistic of six indices, including two public equity and fixed income indices, by using six lagged autocorrelations. As we expected, the Q-statistics of both the ODCE and the PE Index are significantly large while those of the two public equity indices are not statistically significant. Judging by the Q-statistics, the two fixed income indices are less liquid than the public equity indices but more liquid than the two private asset indices.

We then analyzed the relationship between Q-statistics and transaction costs of public assets. In Figure 2, we plotted the Q-statistics and transaction costs of five public assets: S&P 500 Index (S&P), Russell 2000 Index (Russell), Barclays US Aggregate Non Corporate Index (NonCorp), Barclays US Aggregate—Corporate Index (Corp), and Barclays US High Yield Index (HY). The Q-statistics are based on the trailing 80 quarterly returns in which four lagged autocorrelations are used to be consistent with our analysis estimating the illiquidity premium.

The transaction costs of the public equity indices are the market capitalization-weighted average transaction cost of each stock in the index. For each constituent in the index, the transaction costs are computed by the difference in end-of-day bid and ask price divided by bid price, which is consistent with the LCS calculation by Ben Dor *et al.* (2011). The bid and ask prices of each constituent are downloaded from Bloomberg, and we exclude transaction costs that are greater than 50% for the S&P 500 and 100% for the Russell 2000 Index. The transaction costs of the fixed income indices are represented by the LCS from Barclays Capital LLC. Since investment grade and high yield bonds are quoted differently, their respective LCS are calculated differently. The calculation details can be found in Ben Dor *et al.* (2011). These transaction costs used in our analysis represent the total cost for an investor to buy and then immediately sell a security.

The top scatter plot in Figure 2 shows the relationship between the Q-statistic and the transaction cost on five different dates. As we see from the chart, this relationship is not static and changes substantially over time, but in general, the greater

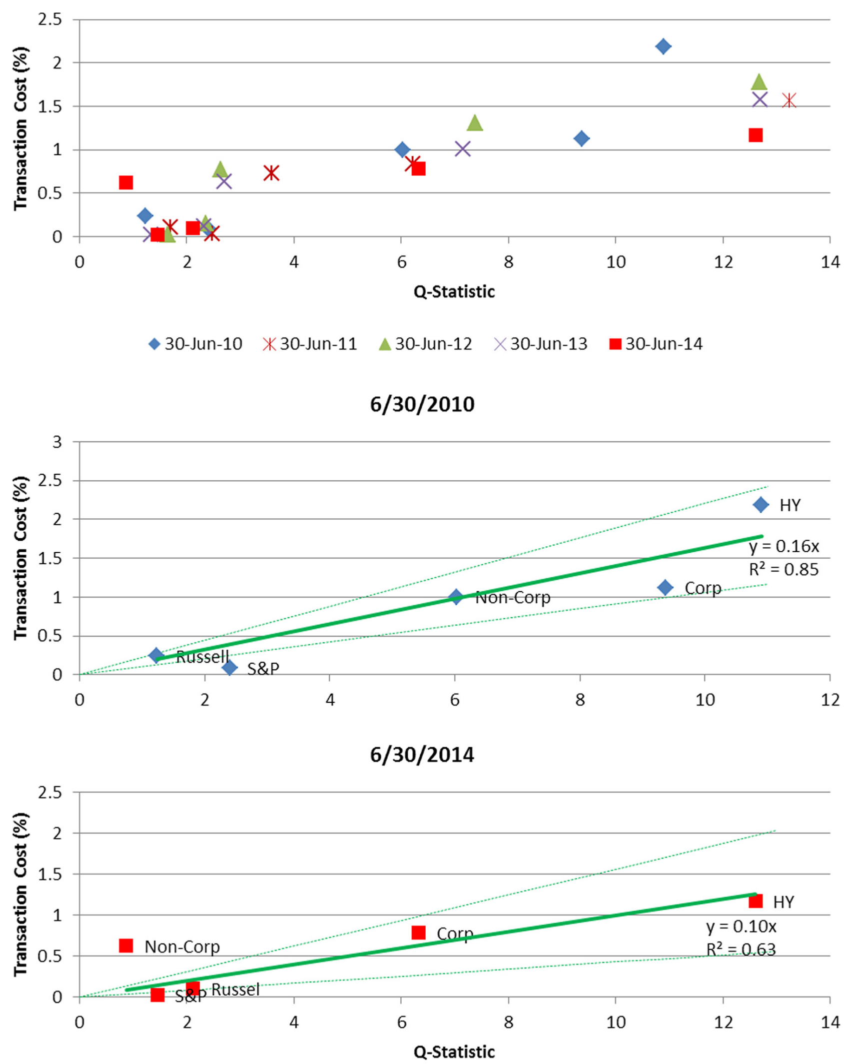


Figure 2 Relationship between Q-statistics and transaction costs.

the Q-statistic, the higher the transaction cost, which to a certain extent validates our Proposition I: Asset return forecastability can be used as a proxy for illiquidity. The middle and bottom charts show more details of the relationship at the end of June of 2010 and 2014, separately. We fit the data with a solid linear regression line and the dashed lines represent the 95% confidence interval of the regression. In the regression, we set the intercept to zero because the transaction costs should be close to zero if the assets are very liquid. Although the trend lines fit the majority of the data reasonably well, there are some exceptions. For example, in the bottom chart detailing 6/30/2014, the Q-statistic of Non-Corp is less than those of public equities but the implied transaction costs are higher. This is due to the fact that the Q-statistics are based on in-sample historical returns. The data sample used to calculate the Q-statistics on 06/30/2014 does not include the index returns in early 1994 when the Federal Reserve shocked the treasury market with a surprise interest rate hike. A larger sample size might address this issue better, but unfortunately, private assets generally do not have very long historical data. These results also suggest that Q-statistics can be used as a

Table 4 Estimated transaction costs (%).

	PE Index		ODCE Index	
Date	Q-Statistic	Estimated transaction cost (95% confidence interval)	Q-Statistics	Estimated transaction cost (95% confidence interval)
06/30/2010	24.87	4.06 (2.64–5.49)	134.79	22.03 (14.31–29.75)
06/30/2014	22.13	2.20 (0.95–3.46)	114.1	11.36 (4.89–17.82)

general guideline for liquidity, but transaction costs are greatly dependent on the market environment on specific trading days. It is also noteworthy that slopes are lower when liquidity improves. This implies that the relationship between the Q-statistics and transaction costs might be weaker when the overall market liquidity improves over time.

If we use the same regression models shown in Figure 2 to estimate the transaction costs of the PE Index and the ODCE Index on two different dates (Table 4), we see two very different values for the transaction cost depending on the date of the estimate. The results bring up an interesting question: how long are investors expected to hold their illiquid assets to realize the illiquidity premium after accounting for transaction costs? The answer, of course, depends on the market conditions. Using the ODCE Index as an example, if we use the estimated illiquidity premium of 5.4% from Section 2, the break-even holding period is expected to be at least four years ($\cong$22.03%/5.4%) in 2010. In 2014, the holding period is reduced to two years because of improved of market conditions.

Although the regression model that links Q-statistics with transaction costs could be tempting to use as a guideline for the overall transaction costs of a private asset index, investors should use this approach with caution as the estimates would be very crude for two reasons. First, because there are no actual transactional data points beyond high-yield bonds, extrapolating the regression line into the private and illiquid assets space is very challenging. Second, the regression model assumes that investors can buy and sell assets immediately. The immediate execution of transactions for most public assets take seconds or days at most; however, for private assets, this assumption may not be valid as transactions often take months or even longer depending on market conditions. In a buyer market, the acquiring party has significant negotiating leverage and may demand a longer due diligence period, which further lengthens the time to execute. Cheng *et al.* (2013) highlight this point in their research that the total risk of illiquid assets has two components: volatility of asset returns and volatility of time-on-market. Contrary to market conventions, the latter is too significant to be ignored. This time delay in completing a transaction, or time-on-market, adds another layer of uncertainty for private assets. Because of these two limitations, we believe that the estimated transaction costs with the regression model only serve as a lower limit and may require further adjustments.

4 Smoothing effect

Not only are transaction costs in private and public markets vastly different, the returns of private and public assets differ in many ways. First, compared with public assets, whose returns can be observed in seconds or even in milliseconds, the frequency of illiquid asset observations is usually low typically with returns that are reported on a quarterly or even annual basis. For example, the PE Index and the ODCE Index are reported on a quarterly basis. Second, in contrast to public assets, the returns of private assets are reported with lags, often on the order of several months, whereas public asset returns can be observed almost in real-time. Lastly, the most salient difference between public and private assets is how price is determined. Public equities are usually traded on central exchanges where prices are observations of actual trades. In contrast, private assets are usually traded between two parties in a private transaction. Although returns associated with these private assets are reported on a regular basis, actual transactions do not occur regularly. Therefore, in order to meet investor demands of regular reporting, most private assets are regularly valued by professional appraisers.

These aforementioned three distinct differences result in reported private asset returns that are less volatile, and more importantly, more persistent than reality. Statistically speaking, private assets returns are often strongly auto-correlated meaning investors can predict future returns with some degree of certainty and should be able to engage in arbitrage under certain conditions. Lo (2001) pointed out that if the market is efficient and frictionless then price changes should be completely random because investors will arbitrage away any predictability. However, commissions, borrowing costs, and short sale limitations make absolute arbitrage costly and infeasible. As a result, market participants still observe persistent autocorrelation in private asset returns, especially where liquidity barriers, such as high transaction costs, prevent arbitrage opportunities. Conversely, the autocorrelations of public asset returns are not significant because market frictions are much smaller and arbitrageurs can easily and quickly exploit opportunities. Given these conditions and characteristics, we propose using autocorrelations as the central tool to proxy illiquidity for this article.

As Table 1 indicate, both the PE Index and the ODCE Index show strong autocorrelations. The returns of the PE Index in the current quarter are significantly correlated with returns in the previous quarters. For the ODCE Index, the autocorrelations are even stronger. These autocorrelation observations are due to the fact that both indices are appraisal-based (NCREIF, 2014; Ang *et al.*, 2014) and professional appraisers are usually reluctant to change their valuations dramatically. As a result, smoothed returns often downward bias the volatility and correlation with liquid public assets. To address this bias, Geltner (1991) proposed applying an ARMA model to correct the smoothing effects observed in illiquid commercial real estate returns that cause the underestimation of risks. Fisher *et al.* (1994) applied a regression model to desmooth returns of a commercial property index so that the greater economic risks of the illiquid index can be revealed and compared with risks of liquid assets.

In this article, we introduce a desmoothing process that differs from the model proposed by Fisher *et al.* (1994) in that we explicitly include autocorrelations in our desmoothing models. Getmansky *et al.* (2004) showed that the smoothing effect will reduce the volatility of smoothed returns as well as the correlations between smoothed returns with other liquid asset returns. In order to rectify the smoothing effect, we propose using adjusted risk metrics (volatility and correlation), or "economic" risks, for a reasonable risk comparison between illiquid and liquid assets. To make a distinction between "economic" risks and "accounting" risks estimated directly from historical reported returns, we have the following proposition:

Proposition II:

The "accounting" risks (volatility and correlation) of illiquid assets, which are estimated directly from reported historic data, are usually biased downwards because of the existence of strong autocorrelations in observed returns. The economic risks, which counteract the bias of accounting risks, should be used for estimating risks of illiquid assets.

Motivated by the above proposition, we develop the following models to uncover the economic risks embedded in the smoothed returns (see Appendix A for more details):

Model 1:

$$r_t = \frac{r_t^s - \rho_1 r_{t-1}^s}{1-\rho_1} \quad \text{and} \tag{5}$$

Model 2:

$$r_t = \frac{1+\rho_1}{1-\rho_2} r_t^s - \frac{\rho_1}{1-\rho_1} r_{t-1}^s - \frac{\rho_2 - \rho_1^2}{(1-\rho_1)(1-\rho_2)} r_{t-2}^s \tag{6}$$

where ρ_1 and ρ_2 are the lag-1 and lag-2 autocorrelations of r_t^s respectively, i.e., $\rho_1 = \rho(r_t^s, r_{t-1}^s)$ and $\rho_2 = \rho(r_t^s, r_{t-2}^s)$. The difference between Model I and II is that in the former, we assume ρ_2 is equal to the square of ρ_1, and in the latter, we remove this assumption. From Table 1, we find that the autocorrelations decay at a faster rate than suggested by Model I (see Appendix A for more details). As indicated in Equations (5) and (6), both models adjust the original return series by autocorrelations—the higher the autocorrelation, the greater the required adjustment. When the first lag autocorrelation is close to one, the "true" returns are close to zero. In this article, we prefer Model II for two reasons. First, Model II includes both lag-1 and lag-2 autocorrelations. In Appendix A, we show that Model I actually implies that the lag-2 autocorrelation is the square of the lag-1 autocorrelation; however, this assumption might be too strong to be supported by empirical evidence. Second, most of private assets are appraised on a quarterly basis, and from our experience, the delay in reporting normally does not span over two quarters. In theory, the autocorrelations of further lags can be included in a desmoothing model; however, in doing so, the closed-form solution would be overly complex. Model II offers a reasonable balance between complexity and sufficient lagged autocorrelations.

By construction, the average returns, before and after desmoothing, are virtually the same; however, after desmoothing both the ODCE Index and the PE Index, we observe a significant increase in return volatility (Figure 3). The desmoothing effect on the ODCE Index is even more pronounced. For the ODCE Index (Table 5), the annual volatility calculated from a sample period between September 1988 and June 2014 increased from 6.14% to 16.14% (23.47%) if Model II (I) is used. The difference in the variances, before and after desmoothing with both models, is statistically significant at 5% level after performing both the Bartlett and Levene hypothesis tests. The dramatic increase of volatility is due to the high autocorrelations present in its returns. For return series with less autocorrelations, such as PE Index, the desmoothing impact is less pronounced: the annual volatility increased from 9.86% to 17.48% (14.78%) if Model II (I) is applied. The Bartlett and Levene tests also indicate the hypothesis of equal variance before and after desmoothing cannot be rejected at 5% level.

As we expected, the correlation between the ODCE Index and a public asset index, S&P 500, more than doubled after desmoothing. The differences in the correlation coefficients before and after desmoothing with both models are significant at 5% level based on the hypothesis

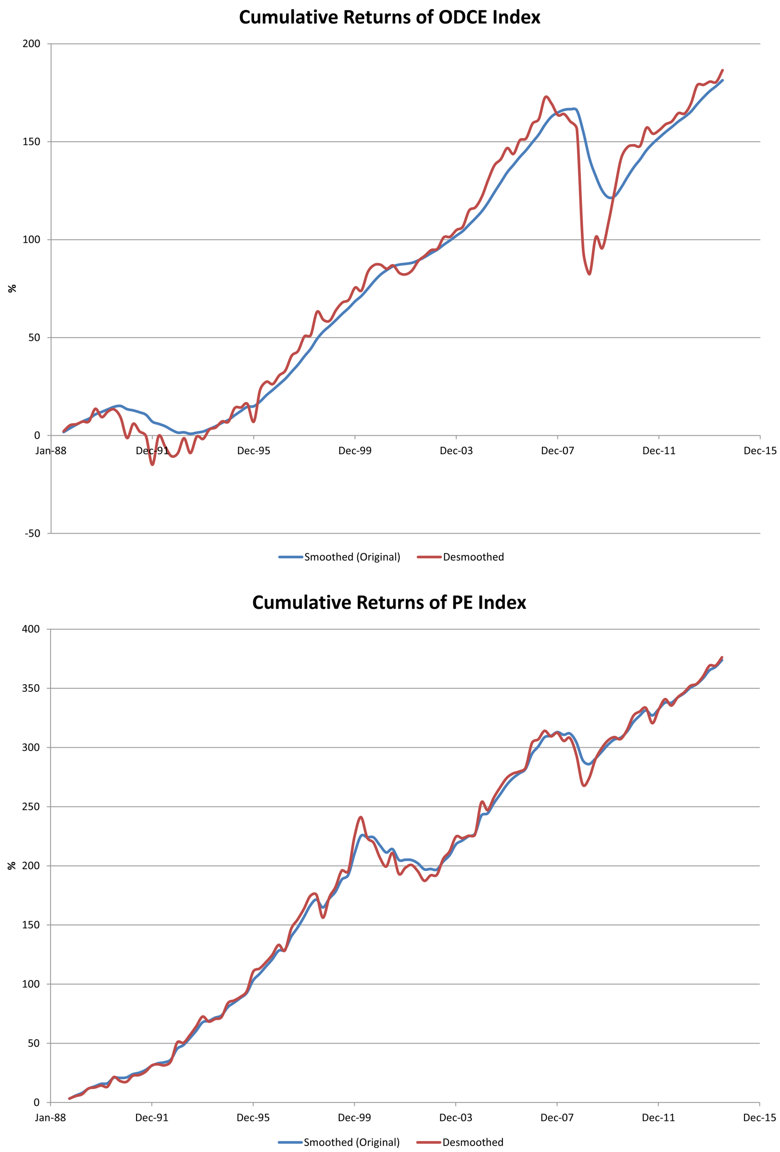


Figure 3 Cumulative return of private asset benchmarks (Model II used to desmooth original returns).

tests suggested by Meng *et al.* (1992). Interestingly, for the PE Index, the correlation with the S&P 500 does not change substantially after desmoothing (the differences are also not statistically significant at 5% level), which suggests that autocorrelation affects the correlations between public and private assets less than the individual volatilities. We found a similar pattern when we replace the S&P 500 with a public REIT index, i.e., FTSE EPRA/NAREIT Developed Total Return Index. Although the theoretical rationale why desmoothing has a different effect on the two return series is not immediately clear, we can shed some light through our simpler smoothing model, Model I. As suggested by Equation (B4) in Appendix B, the autocorrelations will lower the correlations between smoothed returns and liquid asset returns; however, the maximum autocorrelation after introducing the smoothing process is 0.5. One way to explore this effect is by examining

Table 5 Annualized return and volatility of private asset benchmarks.

	Smoothed	Desmoothed	
	Original	Model I	Model II
ODCE Index			
Annualized return (%)	6.91	7.22	7.05
Annualized volatility (%)	6.14	23.47	16.14
Correlation with S&P 500	0.14	0.35	0.33
Correlation with FTSE EPRA/NAREIT Developed Total Return Index*	0.13	0.35	0.43
PE Index			
Annualized return (%)	14.38	14.43	14.47
Annualized volatility (%)	9.86	14.78	17.48
Correlation with S&P 500	0.71	0.74	0.75
Correlation with FTSE EPRA/NAREIT Developed Total Return Index*	0.50	0.52	0.49

*Quarterly data (03/1995–06/2014) from NCREIF and Bloomberg.

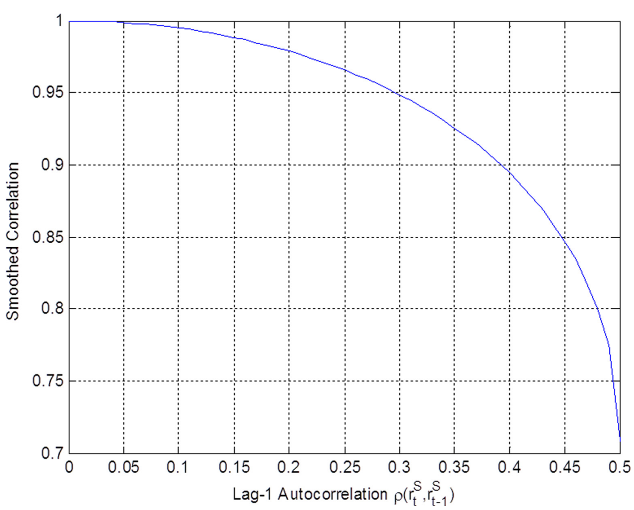
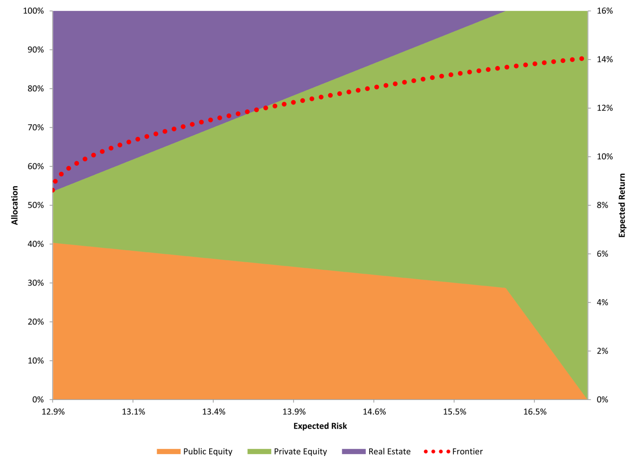


Figure 4 Impact of autocorrelations.

Equation (B4), in which the economic correlations are reduced by a fraction because of the smoothing effect. Once we know the autocorrelation, $\rho(r_t^s, r_{t-1}^s)$, we can solve for a through Equation (B6). In Figure 4, we plot the smoothed correlation against the lag-1 autocorrelation using the simple smoothing model (Equation (5)). In the figure, we assume the economic correlation, $\rho(r_t, r_t')$, is one. For unsmoothed returns, which has zero autocorrelation, we see the correlation is unchanged and remains one. After we introduce the smoothing effect, the autocorrelation becomes non-zero and the correlation between the smoothed returns, r_t^s, and the random returns, r_t', starts to decrease. Interestingly, we observe that the correlation between r_t^s and r_t' only decreases to about 0.9 even if the autocorrelation reaches 0.4. This chart suggests that autocorrelations have a marginal effect on the correlations between illiquid and liquid returns until a certain threshold. The empirical evidence shown in Table 5 also support this finding.

5 An illustration of adjusting private asset inputs in an MVO framework

In the previous sections, we discussed ways to estimate the illiquidity premium, transaction costs, and economic risks with desmoothing models. Armed with these tools, we are able to calculate the new efficient frontier under the MVO framework. We still believe that the



Estimates from desmoothed data (Q3 1988 to Q2 2014) after applying Model II and including estimated transaction costs			
	Public Equity	Private Equity	Real Estate
Annualized return (%)	11.2	14.1	4.9
Annualized volatility (%)	15.8	17.5	16.1
	Correlation		
Public Equity (S&P 500)	1.00		
Private Equity (PE Index)	0.75	1.00	
Real Estate (ODCE Index)	0.33	0.34	1.00

Figure 5 Efficient frontier and asset allocations with desmoothed data.

illiquidity premium, embedded in private asset returns, should be viewed as a component of the expected returns of the illiquid asset classes. However, investors need to consider the possibility of higher transaction costs associated with buying and selling illiquid assets. To illustrate this point, assume an investor with an expected 10-year investment horizon wanting to construct an optimal portfolio in Q2, 2010. Using the estimated transaction costs listed in Table 4, the investor could simply reduce the expected return with the estimated costs averaged over the entire investment time horizon. If the annual transaction cost is 0.4% (∼4%/10 years), the investor could simply subtract this cost from the expected return of the PE Index (14.4% in Figure 1) to arrive at the adjusted annual expected return of 14%.

Using economic risks and expected returns that incorporate expected transactions costs, we rerun the portfolio optimization process and the resulting efficient frontier and corresponding allocations are shown in Figure 5. Compared with the allocations shown in Figure 1, the allocations in Figure 5 are more evenly distributed among the asset classes, especially for the optimal portfolios with lower expected volatilities. This shows that the adjustments induced by autocorrelations enable investors to build a portfolio that is less concentrated in private assets and therefore rely less on ad-hoc investment constraints.

6 Conclusion

Investing in private assets presents many challenges to investors who have both public and private assets in their portfolios. Private assets and public assets are not directly comparable because private assets are usually illiquid, pricing is appraisal-based, and returns are smoothed which understates the economic risks. This challenge is analogous to mixing oil and water, which conventional wisdom tells us is difficult. Recently, a group of scientists found that oil and water can be mixed if a strong vibration or acceleration is applied to the oil base (Terwagne *et al.*, 2009). They call the resultant mixture a "Mayonnaise Droplet." Constructing a portfolio with liquid and illiquid assets requires a similar technology, i.e., "accelerating" the illiquid assets.

The methodology we present in this paper is our approach to "accelerate" the illiquid asset classes to address the issues commonly facing many institutional investors that invest in both public and private assets. First, we highlight the presence of an illiquidity premium through the lens of autocorrelation as past returns of illiquid assets are good indicators of their future returns. This relationship can be interpreted as compensation for buying-and-holding illiquid assets or an estimate of the illiquidity premium. Second, we analyzed and accounted for the difference in the transaction costs and time delays between the liquid and the illiquid assets. Unlike public assets, the transaction costs of private assets are unobservable, and each transaction may take a significant amount of time. As a result, despite the appeal of the illiquidity premium, the costs and the time-delay required to realize the benefit adds tremendous uncertainty. Therefore, we developed a model to estimate illiquid asset transaction costs to be used as a general guideline for private asset investors when making investment decisions. Third, we developed a desmoothing model based on the autocorrelations of the return in order to appropriately compare the economic risks of public and private assets. We apply the model to two private asset indices commonly used by institutional investors: the ODCE Index and the PE Index. After desmoothing the returns, the volatilities of both indices and their correlations with public equity indices, such as S&P 500, increased. These three components outline our methodology and provide a framework to adjust illiquid asset returns to allow an appropriate comparison with public assets.

Athough harvesting the illiquidity premium can be challenging given the unique risks associated with illiquid assets, our work demonstrates that the pursuit is a worthwhile endeavor and cannot be ignored. Our framework establishes a guideline to manage the risks associated with illiquid assets, but it is by no means a final solution for this challenging issue. Beyond the immediate implications of this work, the authors' hope is that this analysis will elicit more research to garner further knowledge into illiquidity to ultimately help investors obtain a more comprehensive perspective and a better approach to managing public–private investments.

Appendix A

We assume a smoothed return series, r_t^s, which is an exponential weighted average of an independent return series or desmoothed return series, r_t. This assumption is reasonable since people usually put more weight on their recent memories, which is also supported by the empirical evidence in Table 1. Therefore, we assume that r_t^s follows:

$$r_t^s = \beta \sum_{i=0}^{\infty} c^i r_{t-i} \tag{A1}$$

in which β and $c(0 < c < 1)$ are two coefficients to be determined.

In order to focus on uncovering the true risks, we further assume the long-term expected returns of smoothed and desmoothed return series are equivalent, i.e., $E(r_t^s) = E(r_t)$. Therefore, we have:

$$E(r_t^s) = E\left(\beta \sum_{t=0}^{\infty} c^i r_{t-i}\right) = \beta \sum_{i=0}^{\infty} c^i E(r_t) \tag{A2}$$

$$\beta \sum_{i=0}^{\infty} c^i = \beta \frac{1}{1-c} = 1 \tag{A3}$$

Given that $\beta = 1 - c$, Equation (A1) can be rewritten as:

$$\begin{aligned} r_t^s &= (1-c) \sum_{i=0}^{\infty} c^i r_{t-i} \\ &= (1-c)(r_t + c r_{t-1} + c^2 r_{t-2} + \cdots) \\ &= (1-c)\left(r_t + \frac{c}{1-c} r_{t-1}^s\right) \\ &= (1-c) r_t + c r_{t-1}^s \end{aligned} \tag{A4}$$

We then have the first method of desmoothing r_t^s,

$$r_t = \frac{r_t^s - c r_{t-1}^s}{1-c} \tag{A5}$$

Rather than obtaining c with a regression model, we can derive the value of c through the following process: calculating the covariance $cov(r_t^s, r_{t-1}^s)$ on both sides of Equation (A4) and considering r_t is independent of r_{t-1}, we have,

$$\begin{aligned} cov(r_t^s, r_{t-1}^s) &= cov((1-c) r_t + c r_{t-1}^s, r_{t-1}^s) \\ &= (1-c) cov(r_t, r_{t-1}^s) + var(r_{t-1}^s) \\ &= c \cdot var(r_t^s) \end{aligned} \tag{A6}$$

As a result,

$$c = \frac{cov(r_t^s, r_{t-1}^s)}{var(r_t^s)} = \rho(r_t^s, r_{t-1}^s) = \rho_1 \tag{A7}$$

in which $\rho(r_t^s, r_{t-1}^s)$ is the lag-1 autocorrelation of r_t^s. Now Equation (A5) can be rewritten as:

$$r_t = \frac{r_t^s - \rho_1 r_{t-1}^s}{1-\rho_1} \tag{A8}$$

From the above equation, we see that the autocorrelation is the major driver of smoothed returns, r_t^s. It also suggests that the higher the autocorrelation is, the higher weight it should be taken into account in order to uncover the true risks of illiquid assets.

In Equation (A8), we only use the lag-1 autocorrelation to desmooth r_t^s. It is interesting to observe that the assumption given by Equation (A1) implies lag-2 autocorrelation of r_t^s is the square of ρ_1, i.e., $\rho_2 = \rho_1^2$. From Table 1, it is reasonable to assume such for the ODCE Index but not so for the PE Index. Instead of making an explicit assumption, we can use both lag-1 and lag-2 autocorrelations of r_t^s, and generalize Equation (A5) as follows:

$$r_t = \frac{r_t^s - c_1 r_{t-1}^s - c_2 r_{t-2}^s}{1 - c_1 - c_2}, \tag{A9}$$

in which c_1, c_2 coefficients are to be determined and we assume $E(r_t^s) = E(r_t)$.

Again, we calculate the covariance $cov(r_t, r_{t-1}^s)$ and $cov(r_t, r_{t-2}^s)$ on both sides of Equation (A9),

$$\begin{aligned} 0 &= cov(r_t, r_{t-1}^s) \\ &= cov\left(\frac{r_t^s - c_1 r_{t-1}^s - c_2 r_{t-2}^s}{1-c_1-c_2}, r_{t-1}^s\right) \\ &= \frac{1}{1-c_1-c_2}(cov(r_t^s, r_{t-1}^s) - c_1 var(r_{t-1}^s) \\ &\quad - c_2 cov(r_{t-1}^s, r_{t-2}^s)) \\ &= \frac{1}{1-c_1-c_2}(cov(r_t^s, r_{t-1}^s) - c_1 var(r_t^s) \\ &\quad - c_2 cov(r_t^s, r_{t-1}^s)) \end{aligned} \tag{A10}$$

$$\begin{aligned} 0 &= cov(r_t, r_{t-2}^s) \\ &= \frac{1}{1-c_1-c_2}(cov(r_t^s, r_{t-2}^s) \\ &\quad - c_1 cov(r_t^s, r_{t-1}^s) - c_2 var(r_t^s)) \end{aligned} \tag{A11}$$

Solving Equations (A10) and (A11), we have,

$$c_1 = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}, \quad c_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \tag{A12}$$

In which ρ_1 and ρ_2 are lag-1 and lag-2 autocorrelations of r_t^s respectively, and Equation (A9) can be written as:

$$r_t = \frac{1+\rho_1}{1-\rho_2} r_t^s - \frac{\rho_1}{1-\rho_1} r_{t-1}^s - \frac{\rho_2-\rho_1^2}{(1-\rho_1)(1-\rho_2)} r_{t-2}^s \tag{A13}$$

The generalized process can be extended to include autocorrelations of more lags; however, the formula becomes very complex. In our opinion, Equation (A13) is a good balance of simplicity and effectiveness of desmoothing illiquid asset returns.

Appendix B

Assume we have an infinite random series, r_t, on which we apply a simple moving average model,

$$r_t^s = ar_t + (1-a)r_{t-1}, \tag{B1}$$

where a is a smoothing parameter ($0 < a < 1$). We also assume the volatility of r_t is σ and r_t is independent of r_{t-1}. Then we can calculate the variance on both sides of Equation (B1) and have the following:

$$\sigma_s^2 = var(r_t^s) = a^2\sigma^2 + (1-a)^2\sigma^2 \tag{B2}$$

Because of the condition $0 < a < 1$, we can easily prove that

$$\sigma_s < \sigma \tag{B3}$$

Suppose we have another infinite random, r'_t, whose volatility is σ and the correlation between r_t and r'_t is ρ but the lagged correlation between r_t (or r_{t-1}) and r'_{t-1} (or r'_t) is 0, then the correlation between smoothed r_t^s and r'_t is $\rho(r_t^s, r'_t)$. We can then show the following:

$$\begin{aligned}\rho(r_t^s, r'_t) &= \frac{cov(r_t^s, r'_t)}{\sigma_s\sigma'} \\ &= \frac{cov(ar_t + (1-a)r_{t-1}, r'_t)}{(\sqrt{a^2+(1-a)^2})\sigma\sigma'} \\ &= \frac{cov(ar_t, r'_t)}{(\sqrt{a^2+(1-a)^2})\sigma\sigma'} \\ &= \frac{a}{\sqrt{(a^2+(1-a)^2)}}\frac{cov(r_t, r'_t)}{\sigma\sigma'} \\ &= \frac{a}{\sqrt{(a^2+(1-a)^2)}}\rho < \rho \end{aligned} \tag{B4}$$

The first lag autocorrelation, $\rho(r_t^s, r_{t-1}^s)$, of smoothed series, r_t^s, can be calculated as follows:

$$\begin{aligned}\rho(r_t^s, r_{t-1}^s) &= \frac{cov(r_t^s, r_{t-1}^s)}{\sigma_s^2} \\ &= \frac{cov(ar_t + (1-a)r_{t-1}, ar_{t-1} + (1-a)r_{t-2})}{\sigma_s^2} \\ &= \frac{cov((1-a)r_{t-1}, ar_{t-1})}{\sigma_s^2} \\ &= \frac{a(1-a)\sigma^2}{a^2\sigma^2 + (1-a)^2\sigma^2} \\ &= \frac{a(1-a)}{a^2+(1-a)^2} > 0 \end{aligned} \tag{B5}$$

From Equation (B5), we can derive a once we know $\rho(r_t^s, r_{t-1}^s)$,

$$a = \frac{1}{2}\left(1 + \sqrt{\left(\frac{1-2\rho(r_t^s, r_{t-1}^s)}{1+2\rho(r_t^s, r_{t-1}^s)}\right)}\right) \tag{B6}$$

In deriving Equation (B6), we remove one of the two solutions since it does not satisfy the condition that autocorrelations will lower the correlation between smoothed and unsmoothed return series. Once we know the lag-1 autocorrelation, $\rho(r_t^s, r_{t-1}^s)$, we can analyze its impact on correlation between smoothed and unsmoothed return series through Equation (B5).

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