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EQUITY INDICES' RETURNS: CONTINGENT CLAIMS ON GDP STOCHASTIC MOVEMENTS

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This paper proposes an equity index contingent claim model. The model assumes that the equity broad-based market indices' stochastic movements are contingent to macroeconomic risk factors that are derived from Ho et al.'s (HPS, 2012, 2013) and Ho and Lee's (HL, 2015b, 2015c) theoretical models. The results show that these factors can explain the equity indices' returns reasonably well.

Our model accounts for the complex lagged effect of GDP growth rate modeled by HPS and estimated by HL, and determines the sensitivities of a market index to the stochastic GDP multiple factors. We show that the S&P Index seems to have anticipated the Great Recession and the higher growth rate of the current recovery. The results also show that the market premiums of Dow Jones and NYSE indices move mostly in tandem with those of S&P. However, such is not the case with NASDAQ and Russell. The model can be used for asset allocation and hedging in investment strategies, and we have provided multiple hedging strategies in this paper to illustrate some applications of our model.

1 Introduction

The relationship between equity market capitalization and GDP is a subject of voluminous research underscoring the importance of the subject. Market capitalization is often approximated by the value of a broad-based market index, such as the S&P Index, which is viewed as the present value of future outputs of an economy. For this reason, market capitalization and the real sector performance should be related empirically: a rise of the GDP growth rate may indicate a higher present value of future incomes of firms in an economy and hence a higher market capitalization. This relationship suggests that a market return model based on GDP growth rate can be established and such a model would be useful for economic policies and investment strategies. Indeed, a general idea of this relationship is

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already accepted in practice. For example, the market capitalization to GDP ratio is widely used as a long-term market valuation indicator.[1]

Yet, understanding the market stochastic performance as related to the real sector stochastic performance over an investors' portfolio time horizon ranging from three months to a year, not over a multiyear long-term horizon, remains relatively unexplored. Is the stock index cumulative return to date excessive relative to the economic recovery after the Great Recession? Can the market indices anticipate the growth of the real sector performance in the near future? How are the market indices valued relatively to each other at a certain date?

Current literature on real-sector-based market capitalization valuation models does not address these questions adequately. First, Dynamic Stochastic General Equilibrium models view GDP as output of the economy's aggregate asset, which is measured by the market capitalization as a proxy. These equilibrium structural models tend to be stylized omitting many economic factors important for this paper's objective, and therefore these top–down models have not provided empirical results relevant to the problems at hand (Kocherlakota, May 2010). Secondly, the bottom–up market capitalization valuation models are exemplified by dividend discount models, such as the Gordon Growth Model. This approach discounts the projected dividends of the corporations in the market indices by some assumed discount rates incorporating the market risk premiums and time value of money. Therefore, these models depend significantly on the model assumptions, which tend to be subjective. Thirdly, these are the empirically based models using a long historical sample period. For example, long historical averages of real earnings are used to identify "bubbles" (Campbell and Shiller, 1988); consumption growth process (Campbell and Cochrane, 1999), and consumption/savings profile in a permanent income model (Hansen *et al.*, 1999). These models typically assume unbiased expectation over a long historical time series and that the market structure remains constant. They then derive a general equilibrium model to determine the stock prices by utility maximization or valuation of the future dividends. Therefore, this empirical approach cannot capture the changing market expectations, investors' preferences, and economic policies during a long sample period, as discussed in Lucas critique (1976) in the context of discussing the relationship between macroeconomic policies and economic performance. The purpose of this paper is to fill this void and address the questions stated above.

This paper proposes a contingent claim equity market model that relates the dynamic stochastic behavior of market indices to that of the GDPs. This model enables investors to evaluate the risk and level of market indices, and it is not a general equilibrium model, a cash flow discount model nor an empirical model that assumes constant market structure. Instead, this model is based on relative valuation, analogous to the option-adjusted spread (OAS) fixed income model (Ho and Lee, 1986), a widely used fixed-income analytical model, with the following three salient features:

(1) Our contingent claim market model assumes that equity indices' stochastic movements are driven by a pre-specified set of GDP risk factors. By way of comparison, the OAS model assumes a set of common factors to bonds such as the three-factor principal movements or the key rate movements of the yield curve.
(2) Our model focuses on the dynamic stochastic movements of the indices contingent to those of the GDP. The non-stochastic component is implied only from the market, and that can be updated empirically using the market prices over a sample period. By way of

comparison, the OAS approach recognizes that the liquidity and credit risk premiums are unobservable and stochastic, and therefore the OAS is allowed to be adjusted over time.

(3) Our model is used to compare a market index performance relative to the GDP stochastic factors and also relative to other indices. By way of comparison, the OAS model is used to compare a bond value with the shape and level of the yield curve and with the values of other bonds.

Our contingent claim equity model is also consistent with much of equity research in practice. The stock market level is often compared with the "peak" or "trough" of previous years. A market index is also compared with other market indices, suggesting that the deviations come from the underlying real sector performance such as the technology sector compared with the utility sector. Our model deals with such relative valuation by time periods and by sector performances, using the GDP risk drivers as the common risk factors.

This paper shows that our model is empirically relevant. The GDP risk factors can explain some of the dynamic stochastic movements of the broad-based indices, using quarterly data from 2000Q1 to 2014Q2.[2] These results enable investors to identify the reasonableness of the stock market level and to hypothesize the stock market expected returns based on the real sector output forecasts. The main results are as follows. (1) The change in market capitalization depends importantly on the risk factors of GDP and not on the change in GDP per se; this may explain the lack of direct empirical relationship between GDP and market capitalization. (2) Real sector risk drivers can be used to model the market risks, a useful tool for risk modeling such as that for the Dodd–Frank Act stress test. (3) In comparing the Great Moderation period between 2002Q1 and 2007Q2 with the 2009 economic recovery, the model suggests that the equity market had anticipated the Great Recession and the real sector output growth starting from 2011; the result suggests that the current recovery is significantly different from the Great Moderation. (4) The broad-based market indices are affected by GDP risk factors differently resulting in different performance profiles, suggesting that investors can implement hedging strategies across these indices. For example, our result shows that the model can reduce 38.7% risk in hedging NASDAQ with S&P, based on our model hedge ratio as opposed to the one-on-one pair trade. This analysis is important since SPY (S&P), QQQ (NASDAQ), and IWM (Russell) are some of the most actively traded Exchange-Traded Funds. Examples of hedging strategies using these indices are suggested to illustrate some applications of our model.

This paper proceeds as follows. Section 2 describes the theoretical context of the model. Section 3 describes the data and the empirical results. Section 4 reports analyses and applications of the models. Section 5 contains the conclusions.

2 Descriptions of the models

The model first identifies the risk factors of GDP dynamic stochastic movements. Then the broad-based indices (S&P, NASDAQ, Russell 2000, NYSE, Dow Jones Index) are assumed to have the same common risk drivers of the GDP but depending on them differently.

2.1 GDP growth rate model

The theoretical macro-financial model that specifies the risk factors of GDP is based on Ho *et al.* (HPS, 2013). The Ho and Lee (HL, 2015a, 2015b, 2015c) model specifies the seven risk factors of

GDP dynamic stochastic movements. The importance of the HPS model for this paper is that they show theoretically that GDP growth rate has a lagged component induced by the capital market. And therefore, any estimation of independent risk drivers of GDP has to isolate this autocorrelation effect using the theoretical model. For clarity on exposition we first provide a summary of their results relevant to our paper here, explaining this macro-financial relationship with the real sector.

The HPS model considers a financial system that improves the allocation of real resources and enhances the performance of the real economy. But these benefits are offset in part by the risk of financial distress and the associated deadweight loss resulting from bankruptcy costs. The stochastic output is primarily generated from capital and labor. The capital is represented by the dynamics of the aggregate real productive assets K in the economy that depends on a number of factors: the production of these assets, the proportion of this production which is reinvested, and the production shocks that hit the economy. The aggregate real asset K evolves over time according to production, investment, and consumption in the economy. Using a multi-period discrete time model, they assume that the aggregate real asset K is a linear stochastic process at time t. The household leverage and the financial leverage induce a feedback effect to the real sector output. The stochastic dynamic total aggregate assets are used as collaterals to support the credit market. The size of the credit market is L. The household leverage, l_H, is defined as the total outstanding credit to the total aggregate asset L/K.

HL assumes that the financial system is a network with a ring topology that enables the flow of funds from the investors to the borrowers passing through the credit market. This financial network also allows for the flow of risk, which is the default risk, flowing through the financial system. The aggregate household asset must be equal to the aggregate household liability. But the household assets are separated into two classes: capital C and investments. The financial leverage (l_F) is defined as the total outstanding credit to capital L/C.

The default cost would pass from the aggregate household liability to the aggregate household assets via the financial sector. And the capital can be viewed as a junior tranche of the aggregate household asset that absorbs the default costs first. The financial sector intermediate transactions can lower the required rate of returns on investments and hence the lending rate to the borrowers from the household sector. This is the economic incentive to lower the capital ratio, C/L, enabling the financial institutions to perform a larger role in the economy. But on the other hand, the capital has to increase as a proportion to the total household liabilities in order to limit the expected bankruptcy cost, deadweight loss to the real sector. HL model shows that there is a cost to holding excess capital in a financial system and that there is an equilibrium financial sector size and the credit market induces a feedback effect $\pi'\sigma(\frac{K_{n-2}}{K_{n-1}})$ to the real sector growth.

Specifically, the model is:

$$
\begin{aligned}
\frac{\Delta Y_n}{Y_n} &= \pi + \sigma\varepsilon_{n-1} + \pi'\sigma\left(\frac{K_{n-2}}{K_{n-1}}\right)\varepsilon_{n-2} \\
&\quad + \beta^L \frac{\Delta L_n}{L_n} + \beta^G \frac{\Delta G_n}{G_n} + \beta^I \frac{\Delta I_n}{I_n} \\
&\quad + \beta^T \frac{\Delta T_n}{T_n} \qquad (1)
\end{aligned}
$$

where

Y_n the GDP
K_n be the aggregate real asset value at time n
$\pi = (h - c + (b - \alpha)l_H - \kappa l_F l_H)$ productivity

$\pi' = (\alpha + \kappa l_F + \gamma l_F^2) l_H$ coefficient of the feedback effect

$\beta^L, \beta^G, \beta^I, \beta^T$ are elasticities of labor, government expenditure, investment, trade imbalance to GDP, respectively.

G, I, and T are government expenditure, investment, and trade deficits, respectively, and the independent variables are their returns, de-trended with mean zero.

h the output per unit of the aggregate real asset

c the consumption and depreciation rate net of investments

b the positive effect of the household leverage

$\alpha = k^H \beta^H$ bankruptcy cost rate on household leverage

$\kappa = k^H \beta^H \beta^F$ combined financial and household bankruptcy cost

$\gamma = k^H (\beta^H)^2 \beta^F$ compounding dead weight loss of bankruptcy feedback effect

ε_n production risk, the uncertain real outputs generated by the capital (or total aggregate asset) of the production function

σ the standard deviation of ε_{n-1}.

These GDP risk drivers can be explained intuitively. The GDP growth rate should be affected by some underlying real sector output ε_n, and the growth of the economy results in the growth of the financial market, and that in turn leads to higher outputs, the lagged effect. The Cobb–Douglas model suggests that labor is another input to real output. The resulting unexplained variable can then be explained by trade, government expenditure, and investment according to the national income identity. Ho and Lee (2015b) have not found consumption to be significant as an explanatory variable as consumption effect is captured by the production function.

2.2 *Stock index contingent claim model*

The stock index contingent claim model assumes that the stock index quarterly in value depends on expectation of future real sector performance and that the observed changes in GDP risk drivers provide informational content to future performance.

Let $\frac{\Delta S_n}{S_n}$ be the return of the stock over a unit time period, ignoring dividends for clarity of exposition since the dividend payout is negligible for the purpose of this paper. The stock index value is measured in real terms, as all the GDP risk drivers are measured in real terms, not in nominal values. And let α, β_i (for $i = 1, \ldots, 7$) be constants, and then the stock index contingent claim on GDP risk factors is given as:

$$\begin{aligned}\frac{\Delta S_n}{S_n} &= \alpha + \beta_1 \varepsilon_{n-1} + \beta_2 \varepsilon_{n-2} + \beta_3 \frac{\Delta L_n}{L_n} \\ &\quad + \beta_4 \frac{\Delta G_n}{G_n} + \beta_5 \frac{\Delta I_n}{I_n} + \beta_6 \frac{\Delta T_n}{T_n} \\ &\quad + \beta_7 \delta^{Y_n} + \delta^S \end{aligned} \tag{2}$$

Given that Equation (2) is specified with the production ε_{n-1} and its lagged value ε_{n-2}, the equation seems to suggest that the stock index does not follow a martingale process. However, that is not the case. The production risk and the feedback effect in this equation should be interpreted differently from that of the GDP attribution equation. In the stock index equation, the lag factor is relevant because of the timing of the release of GDP data. They are not reported at the end of the cycle, but in the middle of the following cycle. Therefore, Equation (2) shows that the stock index return over a quarterly cycle depends on the reported macroeconomic data for the past cycle during the period and the market consensus expected reported value, assuming unbiased expectation, of the production risk of the current cycle. Therefore, the production ε_{n-1} is the expected production to be released for the current quarter and "feedback" ε_{n-2} is the reported production for the prior cycle. The equity market level is formed by both pieces of information.

This issue will be discussed further in the result section.

The intercept term α is the quarterly constant return of the index value, in real terms, over a sample period. This is the value of the non-stochastic component of risk premium and risk-free returns. This term is called "productivity" to be consistent with the Ho and Lee (2015a).

The discrepancy between expectation and the actual observed values is captured by the residuals δ^S, which we will call the "market factor". The market factor measures the returns of the stock index isolated from the real sector factors. This factor may capture the premium in the equity market when the real sector is expected to grow significantly in the future. This market factor may also capture the stochastic behavior of the equity market risk premium.

3 Empirical evidence: Data description and estimated models

3.1 Data description

For our analysis, all economic data are real terms based on 2009 price. The GDP deflator (implicit price deflator for GDP) is a measure of the level of prices of all new, domestically produced, final goods and services in an economy. Like the consumer price index (CPI), the GDP deflator is a measure of price inflation/deflation with respect to a specific base year; the GDP deflator of the base year 2009 itself is equal to 100.

The quarterly time series of the GDP are obtained from the Federal Reserve Board. We use the household net worth as proxy to the aggregate real asset. Household net worth is the sum of the market value of assets owned by every member of the household minus liabilities owed by household members. Wealth in US is commonly measured in terms of the net worth. Here we use only household net worth rather than the sum of the corporate and the household net worth to avoid the double counting.[3]

For the broad-based stock indices, we have chosen S&P 500, NYSE, Dow Jones, Russell 2000, and NASDAQ. We have collected the quarterly data covering 1999Q4–2014Q2 which is consistent with the GDP data. The S&P 500, or the Standard & Poor's 500, is based on the market capitalization of 500 large companies listed on the NYSE or NASDAQ. It is considered as one of the best representative indices in the US. The NYSE Composite covers all common stocks listed on the New York Stock Exchange including American depository receipts, real estate investment trusts, and foreign listings. The number of stocks in the NYSE is over 2000, of which over 1,600 are from the US corporations and over 360 are foreign listings. The foreign companies are among the 100 largest companies in the index, because more than half (55) are non-US issues. The Dow Jones Industrial Average is based on the 30 publicly owned companies in the US. The Russell 2000 Index is a small-cap stock market index of the smallest 2000 stocks in the Russell 3000 Index. It is the most widely quoted measure of the overall performance of the small-cap to mid-cap companies. The NASDAQ Composite is based on the common stocks and American depository receipts and not limited to the US companies like the NYSE Composite. It is often followed in the US as an indicator of the performance of stocks of technology companies and growth companies.

3.2 Empirical models

The empirical stock index models consist of the GDP dynamic stochastic process, the S&P contingent claim model, and other stock index models. We will describe them in this section.

The estimation is based on the sample period 2000Q1–2014Q2.

3.2.1 The GDP dynamic process

For completeness of our exposition, we first recap the GDP model (Ho and Lee, 2015b) here. HL assumes that the equilibrium leverages are determined by the structural parameters of the market and they are constant over the sample period. The production risk ε_n with constant standard deviation is the only factor explaining the stochastic variations in the real asset value K_n. The investment, trade, and government time series are de-trended.

The estimated results show that the GDP growth rate $(\Delta Y_n/Y_n)$ is a combination of six risk factors: production risk (ε_{n-1}), feedback (ε_{n-2}), labor input $(\Delta L_n/L_n)$, government actions $(\Delta G_n/G_n)$, investment rate $(\Delta I_n/I_n)$, and trade imbalances $(\Delta T_n/T_n)$. In addition to these risk factors, we have the intercept π the "productivity" and δ_n the residual term to complete the GDP dynamic process specification.

The results can be re-written as a model of the quarterly changes of GDP as reported below:

$$\begin{aligned}\frac{\Delta Y_n}{Y_n} &= 0.00328 + 0.06745\varepsilon_{n-1} \\ &\quad + 0.03501\left(\frac{K_{n-2}}{K_{n-1}}\right)\varepsilon_{n-2} \\ &\quad + 0.87240\frac{\Delta L_n}{L_n} + 0.21724\frac{\Delta G_n}{G_n} \\ &\quad + 0.14532\frac{\Delta I_n}{I_n} - 0.01285\frac{\Delta \mathrm{T}_n}{T_n} + \delta_n \end{aligned} \tag{3}$$

We call production risk (ε_{n-1}), feedback (ε_{n-2}), labor input $(\Delta L_n/L_n)$, government actions $(\Delta G_n/G_n$, investment rate $(\Delta I_n/I_n)$, and trade imbalances $(\Delta T_n/T_n)$ and residual δ_n the GDP risk factors.

3.2.2 The S&P contingent claim model

An equity index model is assumed to be a contingent claim on the GDP risk drivers. Equation (4) is estimated using the real value of the S&P Index. The estimation results are summarized below:

$$\begin{aligned}\frac{\Delta S_n}{S_n} &= \underset{(0.28)}{0.0014} + \underset{(15.23)^{**}}{3.1826\varepsilon_{n-1}} + \underset{(2.71)^{**}}{0.642\varepsilon_{n-2}} \\ &\quad + \underset{(1.89)^{*}}{2.193\Delta L_n/L_n} + \underset{(0.14)}{0.092\Delta G_n/G_n} \\ &\quad + \underset{(1.08)}{0.197\Delta I_n/I_n} - \underset{(-1.44)}{0.118\Delta T_n/T_n} \\ &\quad - \underset{(-1.94)^{*}}{4.020\delta^{Y_n}} + \delta^S \\ &\quad \text{Adjusted } R^2 = 83.67\%. \\ &\quad ** \text{ and } * \text{ indicate } 1\% \text{ and} \\ &\quad 5\% \text{ significance, respectively.}\end{aligned} \tag{4}$$

The results show that the model explains the S&P Index quarterly returns reasonably well with the adjusted R^2 of over 83.67%. Consistent with the attribution of the GDP, the reported production of the previous cycle and the expected production to be reported in the current cycle, investment and trade factors are significant. The GDP residual factor lacks explanatory power and that suggests that the other GDP risk drivers can provide most of the S&P dynamic stochastic movements.

3.2.3 Other stock index models

The contingent claim model for other stock indices (NYSE, NASDAQ, Russell 2000, Dow Jones) is estimated based on the same seven S&P risk factors with the additional market factor δ^S. The market factor captures the high correlations

Table 1 Indices model comparison.

		intercepts	Production	Feedback	Labor	Investment	Trade balance	Government	GDP residual	Market factor	Adjusted $\mathbf{R^2}$
	Factor	1	ε_{n-1}	ε_{n-2}	$L(n)$	$I(n)$	$T(n)$	$G(n)$	δ_n^Y	δ_n^S	
S&P	**Coefficients**	**0.0018**	**3.182**	**0.645**	**2.193**	**0.197**	**−0.118**	**0.092**	**−4.02**		0.837
	t-Statistic	0.28	15.23	2.71	1.89	1.08	−1.44	0.14	−1.94		
DJ	**Coefficients**	**0.005**	**2.779**	**0.503**	**1.409**	**0.298**	**0.006**	**0.523**	**−3.101**	**1.021**	0.924
	t-Statistic	1.66	20.96	3.34	1.92	2.57	−0.11	1.25	−2.36	11.14	
NYSE	**Coefficients**	**0.003**	**3.23**	**0.816**	**3.526**	**0.0724**	**−0.161**	**0.913**	**−2.123**	**0.979**	0.947
	t-Statistic	1.10	26.47	5.89	5.21	0.68	−3.38	2.37	−1.76	11.61	
Russell	**Coefficients**	**0.012**	**4.054**	**0.888**	**3.300**	**0.035**	**−0.159**	**1.989**	**−2.603**	**0.917**	0.898
	t-Statistic	2.42	19.56	3.77	2.87	0.19	−1.97	3.03	−1.27	6.41	
NASDAQ	**Coefficients**	**0.005**	**4.603**	**0.480**	**−0.078**	**0.111**	**−0.231**	**0.597**	**−3.23**	**1.111**	0.821
	t-Statistic	0.68	14.18	1.30	−0.043	0.39	−1.82	0.58	−1.01	4.95	

Note: The shaded cell indicates the significance at 5% or 1%.

across market indices due to market effect. The model is given below:

$$\frac{\Delta S_n}{S_n} = a + b\varepsilon_{n-1} + c\varepsilon_{n-2} + d\Delta L_n/L_n + e\Delta G_n/G_n + \mathrm{f}\,\frac{\Delta I_n}{I_n} + g\frac{\Delta T_n}{T_n} + h\delta^{Y_n} + i\delta^{S} + \delta \tag{5}$$

Summary results are presented in Table 1. The adjusted R^2 is reported in the last column.

Table 1 reports the coefficients and t-statistics of the model. The results show that the model has high explanatory power with Dow Jones, NYSE, and Russell model explaining 90% adjusted R^2 square. S&P and NASDAQ have similar explanatory power of approximately 82%. Production is significant for all the indices, while Russell and NASDAQ are most sensitive to the production factor with coefficients 4.054 and 4.603, respectively. The market factor of S&P is significant for all the other indices' coefficients approximately 1, suggesting that all the indices move almost in tandem in terms of broad-based market premium. However, each index may have its idiosyncratic movements, captured by the residuals, which will be discussed in Table 2.

Table 2 presents the percentage contribution to indices' returns by each risk driver. The production risk factor is significant in all the indices. The estimated and reported production combined can explain 80.69% of the variations in the S&P Index. The market effect (the residuals of the regression) can explain 14.25%. The two effects can explain over 95% of the variations.[4] The results show that the government expenditure and GDP residuals have minimal impact on indices' returns. Also, the market indices' movements are mostly based on the market estimated growth rate for the quarter and not on the reported GDP growth of the last cycle. And therefore consistent with research showing equity returns have minimal serial correlations.

The result seems to suggest that the large multinational firms are less affected by any trade imbalance between the US and the rest of the world. As expected, the market factor is important to other indices except NASDAQ. NASDAQ is less affected by the market factor and will be discussed partly in the next section.

4 Model analyses and applications

4.1 Stochastic dynamics of the S&P Index relative to the GDP risk drivers

For illustrative purpose, let us assume ε_{n-1} to be the same as ε_{n-2}. Then the sensitivity of the production to the returns of the S&P Index is 3.8246. The corresponding sensitivity to the GDP growth rate, according to Equation (4), is 0.10246. Therefore, the marginal return in S&P is 37.32 multiple of that of the GDP. This result suggests that the production value has significant impact on the

Table 2 Standard deviation decomposition (%) of the indices' returns.

	Production (%)	Feedback (%)	Labor (%)	Invest (%)	Trade balance (%)	Govern (%)	GDP residuals (%)	Market factor (%)	Residual (%)
S&P	78.22	2.47	1.15	2.27	0.50	−0.08	1.22	14.25	
DJ	69.57	1.80	0.60	3.87	0.00	−0.21	0.80	17.11	6.48
NYSE	75.37	3.19	2.43	0.71	0.74	−0.39	0.45	12.99	4.52
Russell	78.74	2.01	1.15	0.26	0.59	0.46	0.46	7.62	8.72
NASDAQ	73.68	0.42	0.01	0.72	1.57	−0.30	0.30	8.00	15.33

S&P valuation. If the production factor leads to a 0.1% marginal increase in GDP, then the S&P value would increase by 3.732% at the margin. Analogously, the multiple for labor, government, investment, and trades are 3.36, 0.42, 1.36 and 9.18, respectively. An increase in GDP based on a rise in government expenditure has much less impact on the market capitalization than that based on an increase in production.

4.2 *Cumulative returns in level of S&P Index from 2002Q1 to 2014Q1*

The quarterly proportional value change can be converted to cumulative returns over any time horizon. The derivation is provided in Appendix A. Table 1 shows that the model captures relevant information on the S&P dynamic stochastic movements. Figure 1 provides the attribution of S&P Index returns from 2002Q1 to 2014Q2. This period covers the Great Moderation period 2003–2007, Great Recession, and the Recovery 2008–2013.

Figure 1 shows that the S&P Index appreciated 33.38% adjusted for inflation over that period. Note that the nominal value appreciated 70.84% during the same period. The results depict the components of the cumulative returns of the S&P Index in real terms. Since the stochastic component of inflation rate is relatively low, the analysis of the stochastic real returns of the index remains appropriate for its nominal stochastic returns.

While the Great Recession officially lasted from December 2007 to June 2009, the estimated and reported GDP growth rates turned negative starting from 2007Q1. The Lehman bankruptcy occurred on September 13, 2008, resulting in the market experiencing the significant down turn in 2008Q4. The results show that the S&P Index market factor fell from -2.96% (2002Q2) to -30.23% (2006Q1), suggesting that the S&P Index had anticipated the Great Recession.

The recovery began 2009Q2 but the Euro-Crisis also started the same time. Bailout packages were approved from Greece and Ireland in May and

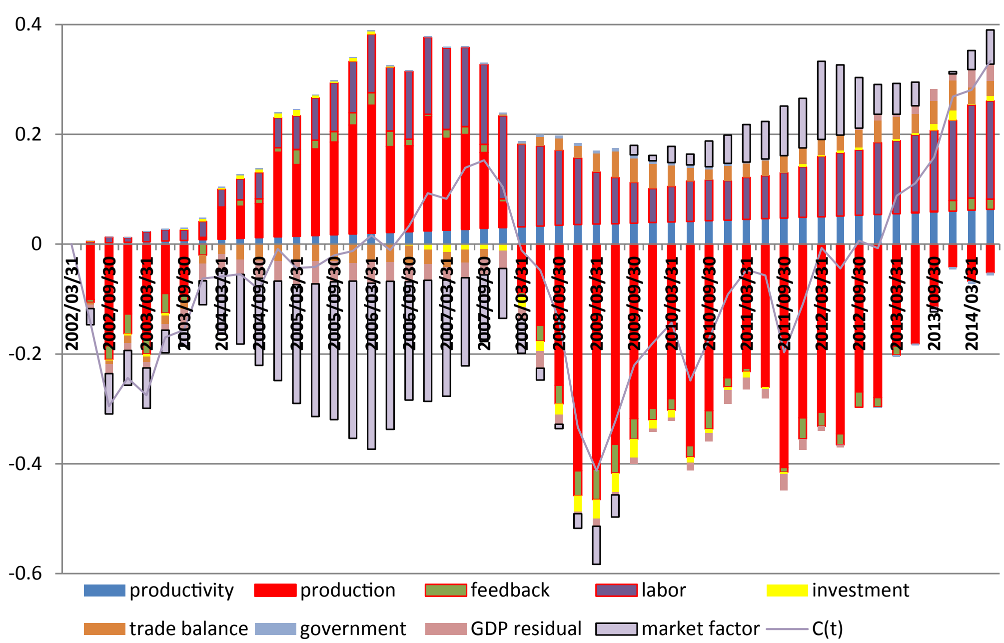

Figure 1 Attribution of cumulative proportional price change of the S&P Index.

November, respectively. 2011Q3 market fell as a result of concerns with Italian and Spanish economies. The results show that the uncertainties and the contraction of the European economies had lowered the stochastic production. In this period, the production risk accounted for much of the S&P Index cumulative value stochastic movements. Starting from 2011Q4 the recovery began. The market factor has also increased, suggesting that the market is anticipating a sustained recovery.

Note that the chart shows that inflation-adjusted value changes with the S&P Index. Therefore, the result shows that the model can be used to interpret the S&P returns in terms of the real sector performance. This relation has many applications. For example, the model can be used to simulate market risk generated by the real sector output risks. This methodology can be used in Dodd–Frank Act stress test, providing the relevant risk drivers for simulation as well as the magnitude of the risks.

4.3 *Comparing 2002 and 2009 economic recoveries*

Financial reporting often describes the market level by referring to a historic time point, such as, the "S&P has reached the record high since 2000" as discussed in Introduction. Therefore, market capitalization is often described relative to a historical level. Our contingent claim model in essence follows this approach. While the model is estimated over a relatively long sample period, the analysis can be based on a chosen starting point. Below, we discuss this relative valuation based on 2002Q1 as the start date. This date is chosen because at that time, the US economy just recovered from the Internet bubble and the September 11 tragedy. Using this period, we can compare the economic recoveries of two periods. The 2009 economic recovery is often referred to as the "Great Moderation v 2.0", suggesting that the recovery is similar to that of the period from 2002 to 2006, which is called the Great Moderation. Figure 1 shows that in fact there are some significant differences. For example, from 2009 to 2012, the market anticipated recovery with the market factor increasing from −6.90% (2009Q1) to 14.3% (2012Q1), while more recently the three quarterly cycles of 2013 showed minimal change in market factor (−0.6% and 0.4% in 2013Q3 and 2013Q4, respectively), suggesting that the S&P Index was consistent with the real sector positive recovery rate by 2014. By way of contrast, the market factor continued to deteriorate from −3.0% (2002Q2) to −30.2% (2006Q1), suggesting that the market may be anticipating the Great Recession despite the reported growth in GDP.

The results show that the increase in the S&P Index can be explained mostly by the cumulative increase in production since 2009. In the Great Moderation, the S&P, in real terms, has increased from 957 (2002Q3) to 1,483 (2006Q4)) from the lowest point to the peak. In the Great Moderation v 2.0, the S&P, in real terms, has also gone up from 796 (2009Q1) to 1,366 (2012Q3). By way of comparison, the production increases from −18.5% (2002Q3) to 23.3% (2006Q1) in the Great Moderation and from −40.1% (2009Q1) to −4.10% (2011Q1) in Moderation v 2.0. The results suggest that the equity increased more significantly in the current recovery even though the production has increased less than that in the Great Moderation period. This observation suggests that the market currently anticipates the real output to continue to grow.

In comparing these two recoveries, the results provide insights into the underlying economics of the current S&P level. In particular, the results suggest that researchers should focus on the underlying factors of the real sector growth in the current economic recovery. It seems that the market tends to be more optimistic with the current sector growth than that in the previous recovery. Maybe the current recovery depends more on structural change in the US real economy with higher energy production, greater technological innovations, and other factors. By way of contrast, the previous recovery may have relied much on housing production.

4.4 *Relative valuation of stock indices*

Using the methodology used for S&P Index, we extend the analysis for S&P to NYSE, Russell 2000, NASDAQ, and Dow Jones indices (Figures 2–5). A comparison of these indices using our relative valuation model enables us to identify the similarities and differences among these widely used stock indices. Even though the indices are considered as a well-diversified market portfolio, each index indicates its own characteristics as a result of its portfolio composition and its weighting scheme. The charts depict the differences in behavior among the indices.

The results show that (1) NASDAQ has significant premium ("residuals") relative to the other indices recently and therefore the results may

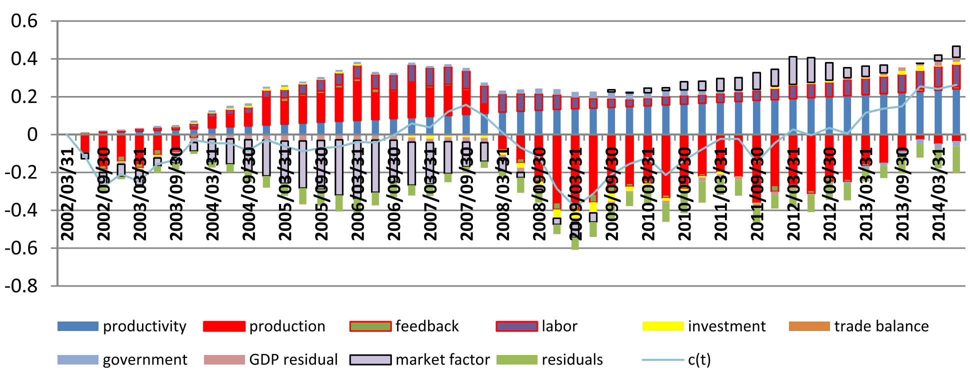

Figure 2 Dow Jones dynamic stochastic movements.

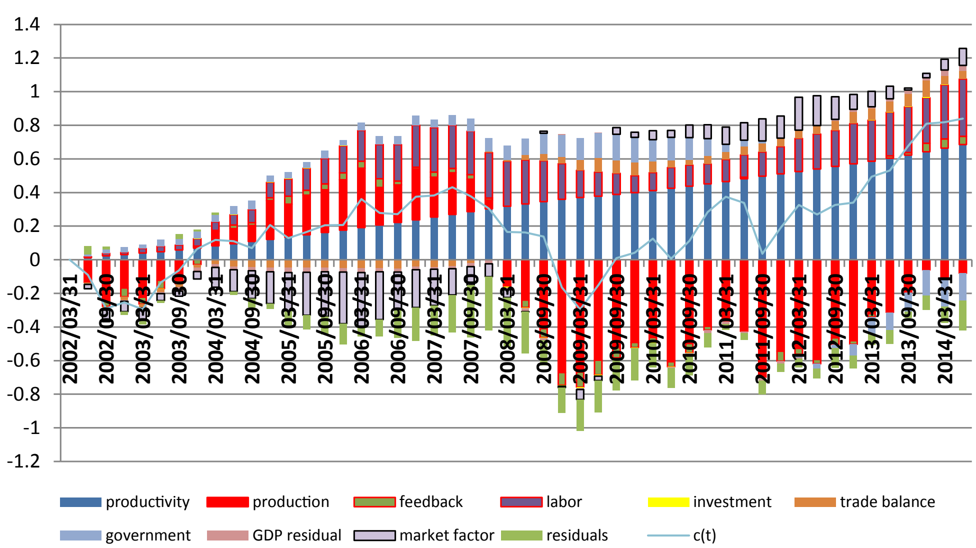

Figure 3 Russell 2000 dynamic stochastic movement.

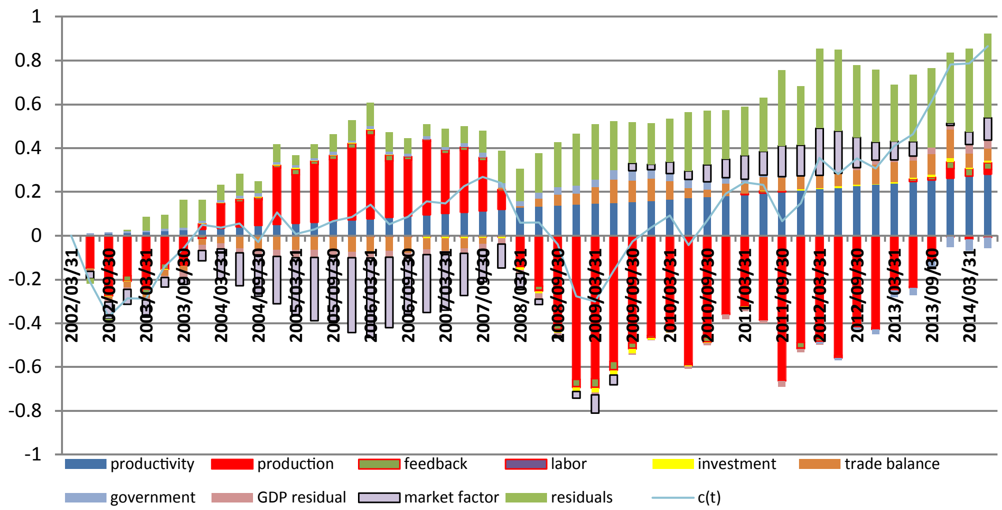

Figure 4 NASDAQ dynamic stochastic movements.

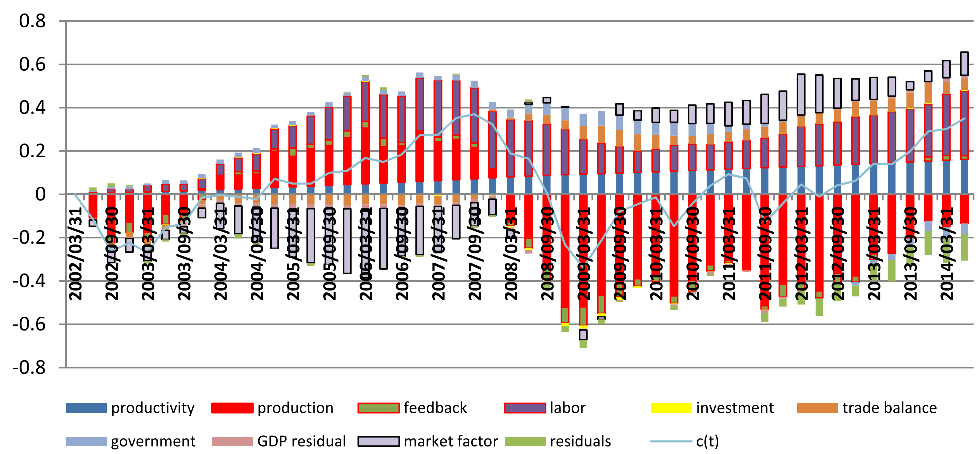

Figure 5 NYSE dynamic stochastic movements.

suggest that the NASDAQ market anticipates relatively high growth rates in technology stocks; from 2002Q2 to 2014Q2 the residuals of NASDAQ increased from −2.33% to 38.38%. By way of contrast, the residuals of Russell fell from 5.10% to −17.46% during the same period. (2) DJ rose less than that of the S&P in the current recovery and in the Great Moderation period, suggesting that the large stocks tend to anticipate slower growth rates in a recovery; the residuals of DJ fell from −1.36% (2002Q4) to −5.45% (2006Q4) in the Great Moderation period, and changed from −6.43% (2009Q1) to −8.90% (2012Q4)) in the current recovery. (3) NYSE movements are similar to that of S&P except allowing for less market premium in the current recovery, suggesting that the NYSE Index may be affected by their international corporation exposure as the US has been performing relatively better than the rest of the world; the residuals of NYSE dropped from −3.87% (2009Q1) to −5.00% (2012Q4) in the current recovery, while there was minimal change in the Great Moderation period from 1.74% (2002Q3) to 1.149% (2006Q2). (4) The Russell Index value appreciation depended on the production significantly in the current recovery; the Russell Index appreciation attributed to the residuals is higher in this current recovery than that in the Great Moderation, where the residual dropped from −1.99% (2002Q3) to −19.81% (2006Q4) and it rose from −19.04% (2009Q4) to −3.79% (2013Q3). Note also that the Russell residual then dropped to −17.46% by 2014Q2, 13.67% change in value only nine months later. When compared with NYSE and DJ, the result shows that Russell's residual risk is much more significant, consistent with the small stock behavior.

These results suggest that the model can also be used for index hedging and index allocation investment strategies. Since the stock indices react differently from the macroeconomic factors, we may be able to formulate a portfolio of the stock indices which is more sensitive or immunized to certain factors. For example, if an investment seeks to sell NASDAQ Index but would like to isolate from the macroeconomic risk factors, an investor may buy the S&P with an equal dollar amount of $100. However, according to Table 1, the production coefficients of S&P and NASDAQ are 3.182 and 4.603, respectively. Therefore, the hedge ratio should be 0.6913 (= 3.182/4.603), or selling $69.13 in NASDAQ and buying $100 in S&P. Using end of quarter nominal prices of the indices over our sample period, we calculate the risk of both hedging strategies. The result shows that the standard deviation of the quarterly dollar returns of this hedging strategy based on our hedge ratio is $4.08, while the hedge strategy using equal dollar amount is $6.56. That is, there is a 37.8% reduction in risk using the contingent claim model.

For clarity of exposition, we have confined our discussion of hedging strategies to pairing of only two indices. But of course, an optimized portfolio of ETFs can be formulated, including other sector ETFs using the analytical framework provided in this paper. Note that in using a contingent claim model, we assume that the structural relationships of stochastic movements of the indices in the coming investment horizon are the same as those estimated from the sample period.

Using Tables 1 and 2 in comparing the indices' characteristics, the results suggest that S&P Index is most exposed to macroeconomic factors (80.69% contribution to the S&P stochastic movements) without any residual risks, whereas other indices have the same market risk in addition to the residual risks. That may explain that the SPY ETF is the most active and widely held fund. By way of comparison, NYSE and Dow Jones being very similar to S&P, they have much less trading activity. But Dow Jones has very similar market risk to that of S&P, but lower exposure to the macroeconomic factors – may be because of its international exposure – and therefore DIA (Dow Jones ETF) can be used as a market risk hedge for S&P to gain a higher exposure to macroeconomic factors.

Our results also show that Russell and NASDAQ have significant residual risks which represent their exposures to small stocks and technology risks, respectively. Therefore, S&P can be used to isolate their exposures to the macroeconomic factors. This result may explain the high trading activity in IWM and QQQ ETFs because of the distinctiveness of these two indices.

So far, we have used quarterly data to estimate the S&P risk factors in Equation (5). This equation cannot be updated in the current month because their monthly data are not available. Our approach can be extended to incorporate monthly analysis by estimating the production and household net worth using monthly real sector output data, such as production index and housing start. These models can be used to update the model estimation for the months between any two end-of-cycle data. Preliminary research has shown the feasibility of this approach but this discussion is beyond the scope of this paper and we will leave it for future research.

5 Conclusions

This paper proposes an S&P Index contingent claim model based on Ho *et al.'s* (2013) and Ho and Lee's (2015b) GDP eight risk factor model. The results show that the estimated and reported production risk combined can explain 80.69% of the S&P Index's return variations. The market factor explains 14.25%. And S&P contingent claim model is also used to study other indices. Our results show that the indices' returns can be explained satisfactorily by these risk factors, and our results can identify the distinctiveness of each broad-based index.

The model results show that the relationship between GDP and a market index cannot be explained simply by their ratios because the GDP growth rate has a complex lagged effect and a market index is affected by a set of stochastic GDP factors differently. Using this analysis, we can show that the current economic recovery is related to the market indices significantly different from that of the Great Moderation period. The results also indicate that the market index level seemed to have anticipated the Great Recession and the higher growth rate of GDP in 2013 and 2014. However, by 2014Q2, market premium has declined, suggesting that the market does not anticipate higher GDP growth. In comparing the market indices, the results show that Dow Jones and NYSE movements, isolated from the real sector factors, are mostly in tandem with those of S&P. However, such is not the case with NASDAQ and Russell. We show that using the model hedge ratio, the risk is reduced by 37.8%.

Our model can provide a tool for relative valuation of market indices. The residuals of the model may provide a measure of idiosyncratic index movements not explained by the major factors. And therefore, our model may suggest a measure of mispricing of a particular index relative to other indices using residuals. This use of "residuals" is similar to the use of the OAS in fixed income securities analysis, an analytical framework that uses a contingent claim relative valuation concept.

The purpose of this paper is to propose a contingent claim model for stock indices. The paper is not intended to provide empirical structural models of market capitalization. The model provides instead an alternative approach to value equity indices, suggesting many avenues for future research, such as risk modeling of equity returns and formulating asset allocation strategies.

Appendix A

Derivation of the attribution of cumulative returns $c(t)$

Let the cumulative returns of an equity index is

$$S(t) = (1 + r_1)(1 + r_2)(1 + r_3)\cdots(1 + r_t)$$

And therefore $c(t) = S(t) - 1$

Because of the compounding effect, each component of the index cannot be compounded as the above equation.

From Equation (5), we have $r_t = a_1F(1, t) + a_2F(2, t) + a_3F(3, t)\cdots a_nF(n, t)$

where the $F(i, t)$ is the i th factor for the t th quarter

Therefore, $c(1) = S(1) - 1 = r_1 = a_1F(1, 1) + a_2F(2, 1) + a_3F(3, 1)\cdots a_nF(n, 1)$

The index returns at time 2 can be derived recursively.

$$\begin{aligned} c(2) &= (1 + r_1)(1 + r_2) - 1 = S(1)(1 + r_2) - 1 \\ &= c(1) + S(1)r_2 \\ &= a_1[F(1, 1) + S(1)F(1, 2)] \end{aligned}$$

$$
\begin{aligned}
&\quad + a_2[F(2,1) + S(1)F(2,2)] \\
&\quad + a_3[F(3,1) + S(1)F(3,2)] \\
&\quad \cdots a_n[F(n,1) + S(1)F(n,2)] \\
c(3) &= (1+r_1)(1+r_2)(1+r_3) - 1 \\
&= S(2)(1+r_3) - 1 \\
&= c(2) + S(2)r_3 \\
&= a_1[F(1,1) + S(1)F(1,2) + S(2)F(1,3)] \\
&\quad + a_2[F(2,1) + S(1)F(2,2) \\
&\quad + S(2)F(2,3)] + a_3[F(3,1) \\
&\quad + S(1)F(3,2) + S(2)F(3,3)] \\
&\quad \cdots a_n[F(n,1) + S(1)F(n,2) \\
&\quad + S(2)F(n,3)] \\
c(4) &= (1+r_1)(1+r_2)(1+r_3)(1+r_4) - 1 \\
&= S(3)(1+r_4) - 1 \\
&= c(3) + S(3)r_4 = a_1[F(1,1) + S(1)F(1,2) \\
&\quad + S(2)F(1,3) + S(3)F(1,4)] \\
&\quad + a_2[F(2,1) + S(1)F(2,2) \\
&\quad + S(2)F(2,3) + S(3)F(2,4)] \\
&\quad + a_3[F(3,1) + S(1)F(3,2) + S(2)F(3,3) \\
&\quad + S(3)F(3,4)] \\
&\quad \cdots a_n[F(n,1) + S(1)F(n,2) \\
&\quad + S(2)F(n,3) + S(3)F(n,4)]
\end{aligned}
$$

The attribution of cumulative returns $c(t)$ for any t can be derived recursively as the above.

Notes

1 In 2001 Warren Buffett remarked in a Fortune Magazine interview that "it is probably the best single measure of where valuations stand at any given moment."

2 We have also tested the model using sample period from 1992 to 2014 and the main results reported here remain consistent. Since the application of the model focuses on comparing stock returns across different indices over a particular time period, the results are relatively robust to the sample period chosen.

3 We have collected the household net-worth from Federal Reserve Economic Data at St. Louis Federal Bank.

4 Let σ_i be the standard deviation of the ith risk attribute estimated over the sample period. Also let σ be the standard deviation of the quarterly GDP growth rate and Ω be the correlation matrix of the attributes. Since the attributions' variations explain the variations in the GDP growth rates, the following equation must hold: $\sigma = (\Sigma\Sigma\Omega_{ij}\sigma_i\sigma_j)/\sigma$.

In re-arranging, we get $\sigma = (\Sigma\sigma_i\Sigma\Omega_{ij}\sigma_j)/\sigma$

Let $\beta_i = (\Sigma\Omega_{ij}\sigma_j)/\sigma$

Then we have $\sigma = \Sigma\sigma_i\beta_i$

Let $C_i = \sigma_i\beta_i/\sigma$

Then we have derived the following *risk attribution equation*

$\Sigma C_i = 1$ and C_i is interpreted as the component of risk. That is, we can decompose the standard deviation of the GDP growth rate into the eight components ($\sigma_i\beta_i$ for $i = 1, \ldots, 8$).

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