Journal Of Investment Management, Vol. 13, No. 2, (2015), pp. 59–93

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GROWTH OPTIMAL PORTFOLIO INSURANCE FOR LONG-TERM INVESTORS*

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We solve the growth-rate optimal multiplier *of a portfolio insurance strategy in the general case with a locally risky reserve asset and stochastic state variables. The level of the optimal time-varying multiplier turns out to be lower than the standard constant multiplier of Constant Proportion Portfolio Insurance (CPPI) for common parameter values. As a consequence the outperformance of the growth-optimal portfolio insurance (GOPI) strategy does not come with higher risk. In the presence of mean reverting stock returns the average allocation to stocks increases with horizon and the optimal multiplier introduces a counter-cyclical "tactical" component to the strategy. Furthermore, we unveil a positive relationship between the value of the strategy and the correlation between the underlying assets.*

1 Introduction

Portfolio insurance is an effective risk management tool that can be implemented with derivatives contracts written on the performance of the portfolio (Leland and Rubinstein, 1976), or using dynamic asset allocation strategies. This paper addresses the question of the optimal parametrization of a dynamic asset allocation-based portfolio insurance strategy similar to the popular Constant Proportion Portfolio Insurance (CPPI) but with a (possibly) time-varying multiplier process.

Asset allocation-based portfolio insurance strategies are structured mainly with two key ingredients: the investor's risk-management performance constraint or *risk budget* and a *multiplier* parameter that determines the allocation to risky assets per unit of risk budget. While the risk budget of the strategy can be expressed in very explicit and intuitive ways, determining the *optimal* multiplier parameter is a more involved question that depends on the optimality criterion chosen.

*A former version of this paper circulated as "Growth Optimal Portfolio Insurance and the Benefits of High Correlation".

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In order to determine the multiplier process of the portfolio insurance strategy, we choose to maximize its long-term average growth rate. Growth optimal portfolios (Kelly, 1956; Latane, 1959) have interesting theoretical properties such as outperforming any other portfolio in the long run in terms of wealth, and maximizing the expected geometric return mean and the median of wealth in the long run (see Hakansson and Ziemba, 1995; Platen, 2005; Christensen, 2005, and the references therein). This methodology is also related to the growth-versus-security tradeoff approach to portfolio choice introduced by MacLean *et al.* (1992), in which maximizing the growth rate of portfolio value has a "natural pairing" with the probability that the value remains above a predefined path as the measure of security.

The CPPI strategy, introduced by Perold (1986) (see also Black and Jones, 1987; Perold and Sharpe, 1988), dynamically allocates wealth between a *safe* or reserve asset that replicates the Floor process of the strategy (i.e., the risk-management constraint) and a risky performance-seeking asset. Although some properties of the CPPI with a stochastic *locally* risky reserve asset were already studied in Black and Perold (1992), most studies on the multiplier of the CPPI including Black and Perold (1992) themselves, Basak (2002), Bertrand and Prigent (2002), Cont and Tankov (2009), Hamidi *et al.* (2008, 2009a, 2009b, 2009c), and Ben Ameur and Prigent (2013) assume that the safe asset is a risk-free bond yielding a constant interest rate. In fact, in models with no interest rate risk, long-term zero-coupon bonds behave as a short-term savings account (i.e., cash). However, in practical applications, the safe asset (i.e., the Floor replicating asset) for long-term investors will typically replicate a default-free bond with long maturity, and thus behave very different from cash.

Hence, we revisit the theoretical properties of the popular portfolio insurance strategy using a *locally* risky reserve asset different from cash, in a model with stochastic state variables, and provide new insights into the behavior of the strategy and an explicit expression for its growth-rate optimal multiplier. In particular, we study the time variation in the allocation and the growth-optimal multiplier implied by interest rate risk and mean reverting excess returns as well as the impact of the correlation between the risky asset and the stochastic reserve asset on the growth rate of the strategy, all of which have remained widely uncovered in the CPPI literature.[1]

Former studies on the theoretical properties of CPPI have also suggested using a varying multiplier instead of a constant one, when the volatility of the risky asset is time-varying.[2] For instance, Cont and Tankov (2009), Hamidi *et al.* (2008, 2009a, 2009b, 2009c), and Ben Ameur and Prigent (2013) address the question of estimating the maximum multiplier or upper bound that allows the strategy to cope with its guarantee for a given confidence level under discrete-time trading (or discontinuous prices). They find that an increase in the volatility of the risky asset implies larger expected losses, and thus it induces conditional estimates of the upper bound of the multiplier to decrease. We find that the volatility of the risky asset has the same impact on the growth-optimal multiplier. Hence, in order to further characterize the strategy we focus our numerical simulations on models with constant volatility in which the optimal multiplier varies due to interest rate risk and time-varying expected returns.

In Monte Carlo simulations we find that for standard parameter values, the level of the growth-optimal multiplier is actually lower than its upper bound and that the potential benefits of using the optimal multiplier are significant in terms of long-term performance and risk in all configurations considered. Thus, the investor does not need to take as much risk exposure as implied by the multiplier upper bound, in order to maximize the strategy's long-term performance i.e. the growth rate.

An important practical advantage of the CPPI approach over the competing Option Based Portfolio Insurance (OBPI), is its flexibility. In fact the CPPI tends to be more adapted to protect the value of a portfolio containing many different risky assets.[3] Furthermore, the option replication strategy tends to have a higher degree of model risk than the CPPI approach.[4] Comparison between the CPPI and the OBPI is out of the scope of this paper, and we refer the reader to Black and Rouhani (1989), Bertrand and Prigent (2001), and Pézier and Scheller (2013).

2 CPPI, risk budget and horizon

Portfolio insurance strategies, such as the CPPI, guarantee that the portfolio respects a given performance constraint by following an asset allocation rule that prevents the value of the portfolio, denoted V, to fall below a Floor value F. The allocation rule splits wealth between a risky or performance-seeking asset S, and a reserve or safe asset, R, and consists of maintaining at every time t an allocation of wealth to asset S equal to

$$E_t = mC_t. \tag{1}$$

Hence, the risk exposure[5] E_t, is equal to a constant multiple $m > 0$ of the available *Cushion* at every time t, defined as $C_t = V_t - F_t$. Remaining wealth is allocated to asset R, which replicates the dynamics of F. Hence, whenever V approaches F, wealth is reallocated to the reserve asset to prevent the portfolio from breaching its Floor value.

The CPPI was initially conceived as a capital guarantee strategy providing access to the upside potential of a risky asset, typically modeled as an equity index.[6] The reserve asset is a zero-coupon bond with maturity matching the investor's horizon T and the Floor process is defined as

$$F_t = V_0 B(t, T), \tag{2}$$

where $B(t, T)$ is the price at time t of a zero-coupon bond paying \$1 at T. By continuously trading to keep the allocation to the stock equal to (1), the investor recovers at time T with probability 1 the initial capital invested V_0, plus some extra value coming from the realized performance of the stock index (thus *insuring* initial capital). In this case the initial Cushion (and thus the initial allocation) is determined by the prevailing bond yield and the investor's horizon, since $C_0 = V_0(1 - B(0, T))$. Hence an investor with a longer horizon has a higher allocation to stocks everything else being equal. A more general version of the Floor[7] used in former studies such as Deguest *et al.* (2012) is

$$F_t = kV_0 \frac{B(t, T)}{B(0, T)}. \tag{3}$$

This variation allows the investor to choose a proportion of *discounted* terminal wealth to ensure at horizon, different from $B(0, T)$.[8] The risk budget or Floor parameter k is a subjective value that determines the risk exposure and the minimum possible value of terminal wealth. Thus it can be related to investors' "risk-aversion" (understood in a general sense of the term). Notice that, for a given Floor parameter k, the horizon effect of the original Floor (Equation (2)) is lost: two investors with the same parameter k but different horizons would have the same initial allocation.

Hence, consider the alternative parametrization of the capital guarantee Floor that disentangles risk-aversion and investment horizon effects:

$$F_t = \theta V_0 B(t, T), \quad \text{for } 0 \le \theta \le \frac{1}{B(0, T)}, \tag{4}$$

where θ represents the proportion of initial capital V_0 that the investor recovers at horizon with probability one (for $\theta = 1$, one recovers the original capital guarantee Floor 2). An investor with a risk aversion parameter θ equal to its upper bound, $\frac{1}{B(0,T)}$, allocates 100% of her wealth to the safe asset (zero risk at horizon) and obtain with certainty the zero-coupon yield available at $t = 0$, i.e., $y^{ZC(0,T)} = \frac{1}{T}\log(\frac{1}{B(0,T)})$. Conversely, an investor with risk budget parameter $\theta = 0$ would allocate 0% of wealth to the safe asset.

Using this risk-sensitive capital guarantee Floor, the initial stock allocation of the strategy presents the same characteristics that popular asset allocation recommendations reported by Samuelson (1963, 1989, 1994) and Canner *et al.* (1997). Indeed, two investors with equal (different) horizon T, but different (equal) risk budget parameter θ, would have different asset allocations at $t = 0$ (for a given multiplier value). In particular, the initial allocation to stocks increases with horizon and decreases with risk aversion.[9]

Although the proportion of initial capital θ to insure at terminal date chosen by investors is likely to increase with horizon, due for instance to expected inflation, its relation with horizon might not necessarily be totally determined by the prevailing yield curve.[10] For instance, the relationship between initial allocation and horizon might be simpler, by setting θ in terms of *minimum acceptable rate of return* over the entire investment period, $\underline{r_{0,T}} \leq y^{ZC(0,T)}$, for any investment horizon as follows:

$$\theta(\underline{r_{0,T}}) = e^{(\underline{r_{0,T}} \times T)}. \quad (5)$$

Whenever the investor wishes to hedge away inflation risk, the risk-management objective can be set in terms of *real* acceptable return per annum, by using an inflation-linked bond instead of a nominal one as the reserve asset (and Floor). In practice, however, inflation-linked bonds are not always available or tend to be expensive, increasing the opportunity cost of the strategy. Thus the minimum required performance $\underline{r_{0,T}}$ might be set according to the expected rate of inflation (e.g., the central bank inflation target).

Notice as well that the upper bound of the risk budget parameter depends on horizon. This implies that an investor with a longer horizon can afford to take a higher initial allocation to the risky asset than another investor with a shorter horizon. In other words, given the yield curve, the initial exposure that an investor can take is limited, although not entirely determined, by her horizon.

The CPPI strategy can be adapted to an Asset-Liability Management context, where the investor is a pension fund that needs to match future retirement payments for its pensioners. This application is thoroughly discussed in Martellini and Milhau (2009) who adapted the Floor to the ALM context as follows:

$$F_t = \theta L_T B(t, T), \quad \text{for } 0 \leq \theta \leq \frac{1}{B(0, T)}, \quad (6)$$

where the value of the pension fund's liability is summarized by a future payment at horizon L_T. The difference between the capital guarantee Floor (Equation (4)) introduced above and the ALM Floor (Equation (6)) is simply the reference base. For the latter the reference is L_T while for the former is initial capital, V_0. The reason is that, in an ALM context, the quantity of interest is the Funding Ratio, defined as the quotient of the current value of assets to the present value of the liability $L_t = L_T B(t, T)$. Using this change of reference in the Floor process definition, the strategy ensures that the funding ratio of the fund, $FR_t = \frac{V_t}{L_t}$, is maintained at all times above a minimum required level θ. In this case the safe asset is a Liability-Hedging portfolio matching the duration of the liability.

3 Growth optimal portfolio insurance

3.1 *Model assumptions and portfolio growth rate*

Consider the following model for financial variables and traded assets. Trading uncertainly is expressed by Wiener processes but there may be *additional* non-traded uncertainty present in the market, modeling randomness, for example, in covariances, rates of return, interest rates, or other quantities. The finite time span is denoted with $[0, T]$, where T is the horizon of the investor. For simplicity we assume *continuous* security prices. A self-financing *portfolio* investing in n assets is given by a vector process π, $\pi(t) = (\pi_1(t), \ldots \pi_n(t))$, such that $\pi_1(t) + \cdots + \pi_n(t) = 1$, where the component π_i represents the proportion or weight of the corresponding asset in the portfolio. Assets with nonnegative weights are *held* in the portfolio, while a negative value of π_i indicates a short-sale in the ith asset.

Let $\sigma(t) = (\sigma_{1\leq i,j\leq n}(t))$ denote the *covariance* matrix process, which is positive-definite at all times and its entry $(\sigma)_{i,j}$ is the covariance between the ith and jth assets.

Let S_π be the value of an investment in portfolio π and its growth rate process g_π. By definition of the growth rate it follows that (see Fernholz, 2002, Proposition 1.3.1)

$$\lim_{T\to\infty} \frac{1}{T}\left(\log S_\pi(T) - \int_0^T g_\pi(t)dt\right) = 0, \quad \text{a.s.} \tag{7}$$

Thus, the average growth rate measures the long-term performance of the portfolio, as it has a one-to-one relationship with its value over long horizons. Hence, the growth rate can be interpreted as the continuously compounded rate of return or as the continuous time version of the geometric return average. Given this relationship, our main theoretical results are asymptotic, and an arbitrarily long investment horizon is assumed. Fernholz (2002) shows that the growth rate of a portfolio is equal to

$$g_\pi(t) = \sum_{i=1}^{n} \pi_i(t) g_i(t) + g_\pi^*(t), \tag{8}$$

where

$$g_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right) \tag{9}$$

is called the *excess growth rate*, $g_i(t) = \mu_i(t) - \frac{1}{2}\sigma_{ii}(t)$ is the growth rate of the ith asset and $\mu_i(t)$ is the instantaneous expected rate of return.[11] It can be shown that the excess growth rate is equal to half the weighted average of the *relative variance* of each asset with respect to the portfolio (see Fernholz, 2002, Lemma 1.3.6), i.e.,

$$g_\pi^*(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^\pi(t), \tag{10}$$

where τ_{ii}^π are the diagonal entries of the *relative covariance* matrix with respect to the portfolio π, defined as $\tau_{ij}^\pi(t) = \sigma_{ij}(t) - \sigma_{i\pi}(t) - \sigma_{j\pi}(t) + \sigma_{\pi\pi}(t)$, and $\sigma_{i\pi}(t) = \sum_{j=1}^{n} \pi_j(t)\sigma_{ij}$. Fernholz (2002) shows that $\tau_{ii}^\pi(t) \geq 0$ at all times. Thus, for *unleveraged* portfolios, i.e., $0 \leq \pi_i(t) \leq 1$, for $i \in \{1, 2, \ldots, n\}$, the excess growth rate is always positive. As Equation (9) illustrates, this quantity is higher for higher volatilities of the individual assets and for relatively lower or negative correlations. Thus for *unleveraged* portfolios, there is a diversification benefit from holding uncorrelated or anticorrelated assets in the portfolio, everything else equal.

Throughout this analysis we assume continuous trading and nil transaction costs (see Brandl *et al.*, 2008; Balder *et al.*, 2009, for a treatment of the CPPI strategy under discrete-time trading and transaction costs).

3.2 Portfolio insurance and the benefits of high correlation

In what follows, we define a set of portfolio insurance strategies similar to the CPPI described in Section 2, the only difference being a (possibly) time-varying multiplier process $m = (m_t, t \in [0, T])$. We provide explicit expressions for the values of the portfolio insurance strategy and its Cushion process and for the growth rate of the Cushion. Then, we show that the correlation between the two assets of the strategy has a positive impact on the value of the strategy (everything else equal).

The set of portfolio insurance strategies considered invest in a reserve asset R driven by a Wiener process W_R and a performance-seeking asset driven by a Wiener process W_S, with dynamics:

$$d \log S_t = g_S(t)dt + \sigma_S(t)dW_S(t), \quad (11)$$

$$d \log R_t = g_R(t)dt + \sigma_R(t)dW_R(t), \quad (12)$$

where W_S and W_R have correlation ρ. The reserve asset (or portfolio) replicates the Floor process, i.e., $dR_t = dF_t$ at all times. Notice that the performance-seeking "asset" S may be in fact any portfolio S_π.

Definition ($\boldsymbol{PI_m}$): A $\boldsymbol{PI}_m$, with value process V_m^{PI} is a portfolio insurance strategy investing in a given couple of assets $(S, R)_{t\in[0,T]}$. The portfolio is determined by a given Floor process F_t, with dynamics $dF_t = dR_t$, for all $t \in [0, T]$, and a bounded and adapted multiplier process $m = (0 < m_t < \infty)_{t\in[0,T]}$. The $\boldsymbol{PI}_m(t)$ holds at all times a proportion $e_t = m_t(1 - \frac{F_t}{V_m^{PI}(t)})$ of the performance-seeking asset S and a proportion $(1 - e_t)$ of the reserve asset R.

The Cushion process of a given $\boldsymbol{PI}_m$ is defined as $\boldsymbol{C}_m = C_m^{PI}(t) = V_m^{PI}(t) - F_t$, for all $t \in [0, T]$. Notice that from the definition above and the continuous prices and trading assumption it follows that $C_m^{PI}(t) > 0$, at all times. The following Corollary presents explicit expressions for the value and for the growth rate of the Cushion process, that hold for all the Floor definitions mentioned in Section 2. The result follows from the definition of the Cushion process and the decomposition of a portfolio's growth rate (Equation (8)).

Corollary 1. *The Cushion process of a* $\boldsymbol{PI}_m$ *follows the dynamics of a portfolio holding a proportion* m_t *in the risky asset*, S, *and* $(1 - m_t)$ *in the reserve asset*, R, *at all* $t \in [0, T]$. *Thus, the value of the Cushion process is*

$$C_m^{PI}(t) = C_0 e^{\int_0^t g_m^{\text{cushion}}(s)ds + \int_0^t [m_s\sigma_S(s)dW^S(s) + (1-m_s)\sigma_R(s)dW^R(s)t]} \quad (13)$$

where C_0 *is defined by the Floor process of the* $\boldsymbol{PI}_m$ *and the growth rate process of the Cushion is*

$$g_m^{\text{cushion}}(t) = g_m^\star(t) + m_t(g_S(t) - g_R(t)) + g_R(t) \quad (14)$$

where

$$g_m^*(t) = \frac{1}{2}m_t(1 - m_t)(\sigma_S^2(t) + \sigma_R^2(t) - 2\sigma_{S,R}(t)). \quad (15)$$

The proof is presented in Appendix A.

Remark 1. Unlike the portfolios studied in Fernholz (2002), the Cushion process of a $\boldsymbol{PI}_m$ is a *leveraged* portfolio whenever the corresponding multiplier $m_t > 1$. While for unleveraged portfolios the excess growth rate is always positive, when $m_t > 1$ the term g_m^* is in fact negative. Given the relationship (Equation (10)), we call g_m^*

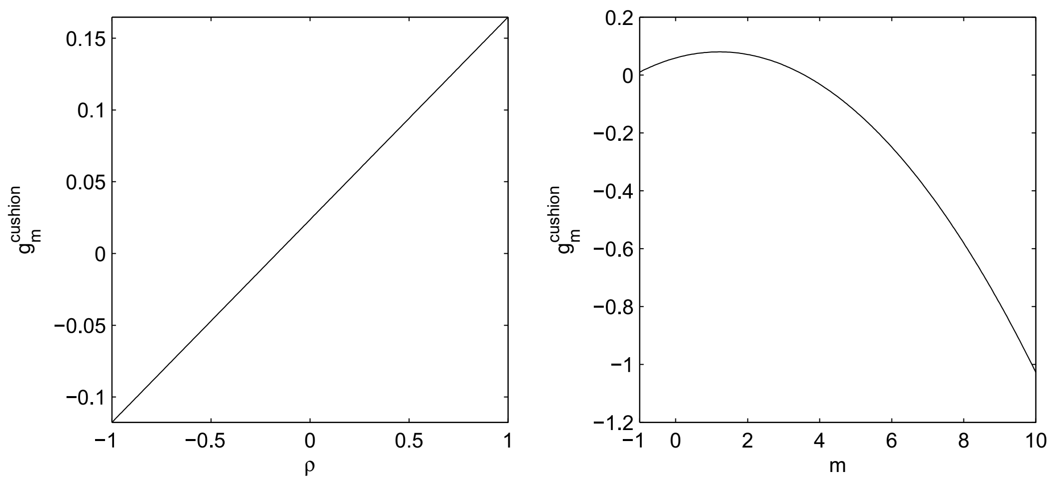

Figure 1 Cushion growth rate as function of correlation and multiplier. The left panel presents the Cushion growth rate (Equation (14)) as a function of the correlation between the reserve asset and the performance-seeking asset, everything else equal. The right panel shows the Cushion growth rate for several multiplier values. We use asset prices parameters of Table 1 for the risky asset and for the reserve asset and correlation we use the parameters in Table 2 with a horizon of $T = 5$ years.

the ***relative variance cost*** of the portfolio insurance strategy. From Equation (15) notice that the impact of a *higher* correlation between S and R, everything else being equal, is an *increase* in the growth rate of the portfolio, through a decrease in the relative variance cost.

The left panel of Figure 1 shows how the growth rate of the Cushion increases with the correlation between the performance-seeking asset and the reserve asset, everything else equal. This illustration uses a horizon of 5 years, a performance-seeking asset with parameters as in Table 1 and parameter values for the reserve asset as in Table 2 which correspond to a standard Vasicek model for bond prices. All the details of the models and parameters used are discussed in the Monte Carlo analysis of Section 4.

The right panel of Figure 1 shows how the Cushion growth rate varies for a reasonable range of multipliers at a given time t. The concavity of the curve observed implies that there is an optimal Cushion growth-optimal multiplier value (or process). The following Corollary provides an explicit expression for the value of the portfolio insurance strategy, which allows us to evaluate how it is impacted by the underlying assets correlation.

Corollary 2. *In the Black–Scholes (BS) model with constant parameters for asset prices, the value process of a* $\boldsymbol{PI}_m$ *with a constant multiplier process m is equal to*

$$V_m^{PI}(t) = F_0\left(\frac{R_t}{R_0}\right) + C_0\left(\frac{S_t}{S_0}\right)^m\left(\frac{R_t}{R_0}\right)^{1-m} e^{g_m^* t}. \tag{16}$$

The proof is presented in Appendix A.

Table 1 Geometric Brownian motion parameters: $\mu_S = \bar{r} + \bar{x}$ where $\bar{r}$, $\bar{x}$, and σ_S are borrowed from Munk *et al.* (2004).

Parameter	Interpretation	Value
Stock return process: $\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_S(t)$		
μ_S	Stock expected return	0.1017
σ_S	Stock volatility	0.1468

Table 2 Vasicek one-factor parameters from Munk *et al.* (2004).

Parameter	Interpretation	Value
Zero-coupon bond return: $\frac{dR_t}{R_t} = (r_t + \lambda_r \sigma_r D(r,t))dt + \sigma_r D(r,t) dW_R(t)$		
Nominal interest rate: $dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r dW_R(t)$		
κ	Degree of mean reversion of the interest rate	0.0395
$\bar{r}$	Long-run mean of the interest rate	0.0369
σ_r	Interest rate volatility	0.0195
λ_r	Premium on interest rate risk	0.2747
ρ	Correlation between W_S and W_R	0.0845

In order to further illustrate the impact of correlation and the multiplier on the value of the CPPI, consider the value of the portfolio in units of the reserve asset. In the Black–Scholes model, using the reserve asset as the numeraire, the value of the portfolio relative to the reserve asset is obtained by dividing Equation (16) by R_t (normalizing $S_0 = R_0$):

$$\frac{V_m^{PI}(t)}{R_t} = \frac{F_0}{R_0} + \frac{C_0}{R_0}\left(\frac{S_t}{R_t}\right)^m e^{g_m^* t}. \quad (17)$$

In an ALM context (see discussion in Section 2) this corresponds to the funding ratio (FR), obtained by replacing R_t by the present value of the liability $L_T B(t,T)$ in Equation (17) and normalizing $V_0 = 1$,

$$FR_t := \frac{V_m^{PI}(t)}{L_t} = \theta + (FR_0 - \theta)\left(\frac{S_t}{L_t}\right)^m e^{g_m^* t}. \quad (18)$$

Using Equation (18) (setting the minimum acceptable funding ratio to $\theta = 1$), in the left panel of Figure 2, we illustrate the important impact on the value of the strategy of the correlation between the performance-seeking asset and the reserve asset. To the best of our knowledge, the role of correlation is new to the literature of the CPPI. In effect, most papers on the properties

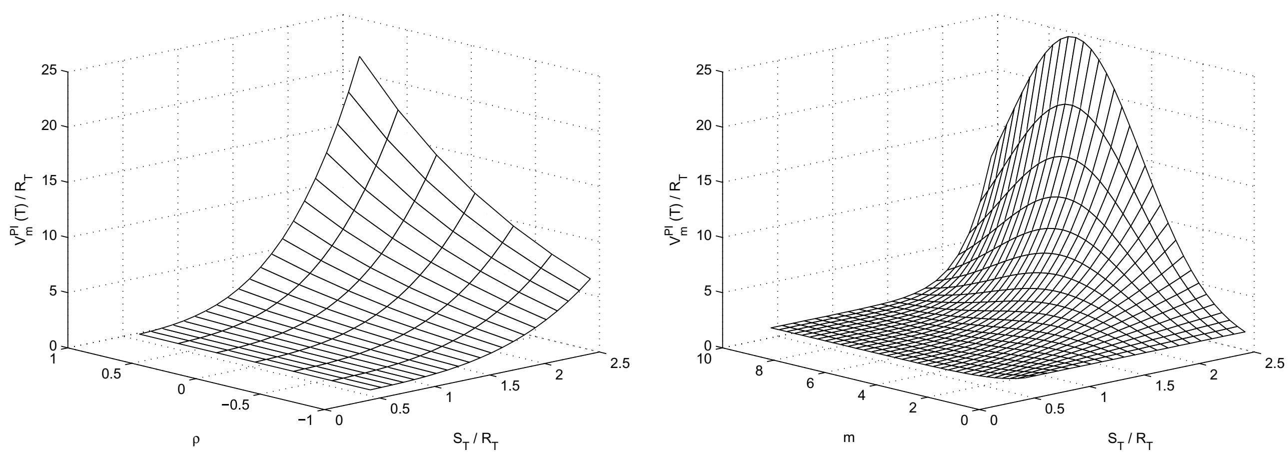

Figure 2 Relative value of CPPI under a Black–Scholes model as a function of correlation and the multiplier. The value of the CPPI is expressed in units of the reserve asset (i.e., funding ratio in an ALM context) as given by Equation (18) with $\theta = 1$ and $FR_0 = \frac{1}{B(0,T)}$. On the left panel the value of the strategy varies as a function of the correlation and the relative value of the risky asset, where the safe asset is the numeraire (volatilities are kept constant). On the right panel, the relative value of the portfolio varies as a function of the multiplier and the relative value of the underlying assets. In both panels, we use the parameters in Tables 1 and 2 and $T = 5$ years.

of the CPPI focused in the particular case of a safe asset with constant rate of return, in which case the reserve asset presents zero volatility and thus nil covariance with the risky asset, making the correlation irrelevant (not to be confused with the correlation between the Floor and the reserve asset).

The right panel of Figure 2 illustrates the impact of the multiplier on the value of the CPPI. This graph shows that for each set of parameter values there is a multiplier level that maximizes the value of the strategy. In what follows we derive an expression for the growth-optimal multiplier.

3.3 Growth-optimal multiplier

Following Fernholz (2002), we focus on the time-average values rather than the expected values of the processes under consideration because the former are actually observable. Thus, we define the Growth Optimal Portfolio Insurance strategy (GOPI) as the portfolio insurance strategy with the highest time-average growth rate over a long period of time among all strategies with the same couple of assets and Floor. The definition of the GOPI below is in the spirit of the definition of the (unconstrained) Growth Optimal Portfolio in Platen and Heath (2006, p. 373).

Definition (GOPI). A portfolio insurance strategy Class $\boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$ is defined as the set of $\boldsymbol{PI}_m$, defined by the couple of assets and Floor process $(S, R, F)_{t\in[0,\infty)}$ and all bounded and adapted multiplier processes $m = (0 < m_t < \infty)_{t\in[0,T]}$. A $\boldsymbol{PI}_m^\star \in \boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$ is called the Growth Optimal Portfolio Insurance strategy (GOPI) if, for all the portfolio insurance strategies $\boldsymbol{PI}_m \in \boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, the growth rates satisfy the inequality

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g^\star_{PI}(t)dt \geq \lim_{T\to\infty}\frac{1}{T}\int_0^T g_{PI}(t)dt, \quad \text{a.s.}$$

This definition is also consistent with the growth-versus-security tradeoff approach to portfolio choice of MacLean *et al.* (1992), in which the probability of maintaining a minimum wealth level over time constitutes a security criterion that, according to them, has a natural pairing with the growth rate maximization.

Remark 2. Notice that, from the definition of $\boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, the only difference between any two portfolio insurance strategies in a given class $\boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$ is the multiplier process. Thus for a given class $\boldsymbol{\mathcal{M}}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, the $GOPI$ is determined by the *growth-optimal multiplier* process, $m^\star = \{m^\star_t, t \in [0, T]\}$.

Corollary 3 in Appendix A shows that the multiplier that maximizes the log of terminal wealth over an arbitrarily long horizon also maximizes the growth rate of the portfolio insurance strategies considered (and that this is also the case for the Cushion process). Proposition 1 provides the growth-optimal multiplier of the Cushion and Lemma 1 shows that maximizing the growth rate of a portfolio insurance strategy V_m^{PI} is equivalent to maximizing the growth rate of its Cushion process C_m^{PI}. This implies that the growth-optimal multiplier for V_m^{PI} is also the growth-optimal multiplier of C_m^{PI} (see Identity 1 in Appendix A). This is intuitive in the light of Corollary 5 in the Appendix, which shows that the growth rate of this type of portfolio insurance strategy has a lower bound equal to the weighted average of the growth rate of the reserve asset and the corresponding Cushion's growth rate.[12] Corollary 4 in the Appendix shows that maximizing the growth rate of the "myopic" single-period problem is equivalent to maximizing the growth rate over any other investment horizon. These results lead to the growth-optimal multiplier of the portfolio insurance strategies considered.

Proposition 1. *For a given $\mathcal{M}_{(S,R,F)}$, the growth-optimal multiplier, which determines the corresponding GOPI strategy $V_m^\star = \{V_m^\star(t), t \in [0, T]\}$, is equal to*

$$m_t^\star = \frac{g_S(t) - g_R(t) + g^*(t)}{2g^*(t)} \quad (19)$$

for all $t \in [0, T]$, where $g^*(t) = \frac{1}{2}(\sigma_S^2(t) + \sigma_R^2(t) - 2\sigma_{S,R}(t))$.

Proof. Proposition 1 follows from Proposition 2, Identity 1 and Corollary 3 presented in Appendix A. □

Remark 3. Equation (19) implies that the growth-optimal multiplier $m^\star$ is independent of the Floor process F. Hence the growth-optimal multiplier process is the same for all classes $\mathcal{M}_{(S,R,F)}$ sharing the same pair of assets (S, R) and all possible risk budget levels and Floor processes satisfying $dF_t = dR_t$.

Remark 4. In the particular case with a locally riskless asset as the reserve asset with constant interest rate, then $\mu_R(t) = r$, $\sigma_R(t) = 0$, $\sigma_{S,R}(t) = 0$ for $t \in [0, T]$. In that less realistic case, the optimal multiplier Equation (19) simplifies to,

$$m_t^\star = \frac{\mu_S(t) - r}{\sigma_S^2(t)}. \quad (20)$$

Assuming constant interest rates, Black and Perold (1992) and Basak (2002) find a similar solution for an optimal multiplier in the case of a portfolio insurer investor with piece-wise CRRA preferences. Their optimal multiplier is equal to (20) times the inverse of the investor's relative risk aversion coefficient (which is equal to 1 in the log-utility case).

4 Monte Carlo simulation analysis

Using stochastic simulations of a discrete-time implementation of the portfolio insurance strategies in what follows we address three issues. First, we gauge the potential benefits of using an optimal multiplier instead of the common approach of using the maximal constant multiplier that allows the CPPI to respect its floor in discrete-time trading. Second, we evaluate the impact of interest rate risk and mean reversion on expected excess equity returns in the asset allocation of the strategy. Third, we analyze the relationship between horizon and the allocation of the GOPI strategy.

The models considered hereafter for the assets dynamics are special cases of the stochastic differential equations (SDEs) Equations (11) and (12) used to derive the optimal strategy in the previous section. Due to its relevance for practical applications (capital guarantee funds and ALM), we focus on a strategy using a zero-coupon bond with maturity equal to the investment horizon as the reserve asset, and an equity index as the performance-seeking asset. We simulate monthly data using parameter estimates from Munk *et al.* (2004) to model the (nominal) bond and stock index processes (further details are provided hereafter and in Tables 1–3), and consider horizons ranging from 5 to 20 years. In practice, leverage is often limited; following Pain and Rand (2008) we set a maximum leverage limit at 2.5 for both the GOPI and the CPPI, so the allocation to the stock index is equal to

$$e_t = \min\left(2.5, m_t\left(1 - \frac{V_t}{F_t}\right)\right),$$

$$\text{for all } t \in [0, T].$$

Thus, for allocations $e_t > 1$ the portfolio has also a short position on the reserve asset.

Unless indicated otherwise, we set θ as in Equation (5), with minimum required rate of return $r_{0,T} = 3\%$ for all T. We choose 3% because this

Table 3 Stock index with mean reversion in returns from Munk *et al.* (2004).

Parameter	Interpretation	Value
Stock return process: $\frac{dS_t}{S_t} = (r + x_t)dt + \sigma_S dW_S(t)$		
Time-varying expected excess return: $dx_t = \alpha(\bar{x} - x_t)dt - \sigma_x dW_S(t)$		
r	Constant interest rate	0.0369
σ_S	Stock volatility	0.1468
α	Degree of mean reversion in expected excess returns	0.0608
$\bar{x}$	Long-run equity risk premium	0.0648
σ_x	Excess return volatility	0.0069

value is lower than its upper bound, $y^{ZC(0,T)}$, in all model configurations considered (in the BS model, we use the constant interest rate equal to 3.69%, which is its long-term level as estimated by Munk *et al.*, 2004).

4.1 Constant maximal multiplier

A common way to determine the multiplier of the CPPI in practice is using the maximum value that would allow the Cushion to remain positive even in the "worst case scenario". Although in the former theoretical analysis we assumed continuous-time trading and prices, in practice trading can only happen in discrete time. A discretization of Equation (A1) in the Appendix shows that, for the Cushion to remain positive between any two trading moments t and $t+1$, the multiplier has to satisfy the following condition:

$$\frac{C_{t+1}}{C_t} - 1 = m\frac{S_{t+1}}{S_t} + (1-m)\frac{R_{t+1}}{R_t} - 1 > -1$$

$$\Leftrightarrow \frac{S_{t+1}}{S_t} > \frac{(m-1)}{m}\frac{R_{t+1}}{R_t},$$

or equivalently

$$m(r_S(t,t+1) - r_R(t,t+1)) \geq -(1 + r_R(t,t+1)). \qquad (21)$$

For $(r_S(t,t+1) - r_R(t,t+1)) < 0$ the inequality (Equation (21)) gets inverted. Thus, the maximum value for the multiplier that can guarantee in general the Cushion's positivity condition (Equation (21)) is:

$$m \leq \frac{-(1 + r_R(t,t+1))}{r_S(t,t+1) - r_R(t,t+1)}, \qquad (22)$$

for every t and $t+1$ for which the condition $(r_S(t,t+1) - r_R(t,t+1)) < 0$ is satisfied.

Most research *on* the properties of the CPPI assume a constant interest rate. By shutting down interest rate risk, the zero-coupon dynamics becomes the same as for a locally riskless asset (cash). In the particular case in which the reserve asset is cash, its returns have a minimum value of 0 (assuming positive interest rates) and are very small compared to the extreme returns of the risky asset. For this reason, the upper bound of the multiplier (Equation (22)) is reduced to:

$$m \leq \frac{-1}{r_S(t,t+1)}, \qquad (23)$$

for every t and $t+1$ such that $r_S(t,t+1) < 0$. Under the constant interest rate assumption, the left tail of the risky asset is the only matter of concern to guarantee that the strategy complies with its risk-management objective. Approaches to estimate the maximum multiplier as in Equation (23) include Bertrand and Prigent (2002), Cont and Tankov (2009), Hamidi *et al.* (2008, 2009a, 2009b, 2009c), and Ben Ameur and Prigent (2013).

Although former papers do not address the right-tail risk of the reserve asset, Equation (22) shows that, when the reserve asset is *locally* risky a sudden and significant increase in its value may also cause a Floor violation as well. Thus, the right tail of the distribution of the reserve asset might also be of critical importance for investors with long horizons, for which the reserve asset is a zero-coupon bond with long maturity.

In this paper we do not address parameter estimation issues and assume perfect foresight for both, the optimal multiplier of the GOPI strategy and for the constant maximum m of the CPPI. Thus, for all model configuration hereafter we compute the optimal varying multiplier $m^\star$ defined by Equation (19) at each time step and scenario and the constant maximal multiplier $m_{\max}$ as the upper bound defined in Equtaion (22) for every simulated scenario (determined a posteriori).

4.2 Optimal multiplier benefits

We now compare the performance (and hence their growth rates, as implied by Equation (7)) of the GOPI and CPPI strategies, under the same constraint on terminal wealth and with the same underlying assets. Thus, the two strategies have a similar risk profile, as measured by the worst possible outcome of terminal wealth. For completeness, we also compare their riskiness using an intermediate horizons metric (i.e., the maximum drawdown) and their allocation to the risky asset.

Table 4 presents the outperformance probability of the GOPI with respect to the CPPI across 10,000 scenarios with monthly time steps, for all models considered hereafter and each investment horizon. The probability of outperformance of the GOPI over the CPPI strategy obtained varies between 73% and 90% and is above 81% in 12 out of the 16 (4 × 4) model/horizon tests.

Table 4 GOPI outperformance probability under different model configurations and horizons.

Outperformance probability	5	10	15	20
Black–Scholes	0.881	0.817	0.770	0.729
Interest rate risk	0.844	0.810	0.828	0.850
Excess return mean reversion	0.896	0.839	0.791	0.751
Combined effect	0.865	0.841	0.859	0.886

The probability is calculated as the percentage of scenarios for which the GOPI strategy presented a higher cumulative return than the corresponding CPPI strategy. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

Table 5 presents a summary of the distribution of the outperformance for all models considered hereafter and over each horizon. The median outperformance of the GOPI across models and horizons is between 0.8% and 6% and the dispersion of the outperformance distribution decreases with horizon, which implies an economically

Table 5 GOPI outperformance distribution under different model configurations and horizons.

	Outperformance distribution (%)				
Horizon	5%	25%	50%	75%	95%
	Black–Scholes				
5	–20.971	0.377	0.839	1.561	3.641
10	–16.186	0.236	0.956	2.347	6.795
15	–11.158	0.089	1.037	3.082	9.259
20	–7.867	–1.801	1.052	3.756	11.100
	Interest rate risk				
5	–24.819	0.986	1.929	3.215	6.226
10	–18.341	1.232	3.053	5.366	10.059
15	–13.424	1.861	4.102	6.804	11.646
20	–9.800	2.472	4.981	7.726	12.254
	Excess return mean reversion				
5	–17.252	0.452	0.994	1.721	3.176
10	–14.266	0.373	1.371	2.835	5.695
15	–10.501	0.220	1.794	4.090	7.800
20	–7.792	0.010	2.147	5.265	9.543
	Combined effect				
5	–21.452	1.130	2.253	3.486	5.484
10	–15.294	1.691	3.819	5.941	9.154
15	–10.312	2.647	5.150	7.461	10.939
20	–7.244	3.537	6.121	8.282	11.594

The outperformance is calculated as the difference in average annualized return of the GOPI strategy and the CPPI strategy for each scenario over the corresponding horizon. The table presents the 5 quantiles of the outperformance distribution. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

meaningful advantage. Notice that there are some scenarios presenting an important underperformance with respect to the CPPI, as shown by the 5% quantile of return differences. This quantile is larger for the interest rate risk model with figures ranging from −25% for the 5 years horizon and −9.8% for the 20 years horizon. This large (rare) underperformance comes from important positive performance of the CPPI strategy as opposed to a large negative performance of the GOPI strategy, as shown in Table 6, where the 95% quantile of the distribution of returns of the CPPI strategy is between 20% and 34% (depending on horizon and model) compared to 7% and 21% for the GOPI. This is consistent with the higher values of the CPPI multiplier compared to the optimal multiplier observed in all model configurations and the convexity the CPPI strategy on the value of the risky asset.

Figure 3 displays the distributions of the optimal and maximal multipliers for all model configurations over the 15 years horizon. In fact, for all model configurations and horizons the optimal multiplier is lower than the maximal multiplier in more than 99.99% of scenarios (and in 100% of scenarios in most model/horizons combinations). When comparing with the minimum of

Table 6 GOPI and CPPI annualized return distribution under different model configurations and horizons.

GOPI return distribution (%)						CPPI return distribution (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	3.267	3.636	4.187	5.219	8.524	5	3.045	3.045	3.045	3.045	28.271
10	3.239	3.794	4.840	7.176	14.226	10	3.045	3.045	3.045	3.045	28.699
15	3.235	3.993	5.741	9.397	19.107	15	3.045	3.045	3.045	3.935	26.940
20	3.246	4.274	6.685	11.582	21.209	20	3.045	3.045	3.045	10.609	26.349
		Interest rate risk						Interest rate risk			
5	3.837	4.651	5.675	7.286	11.244	5	3.045	3.045	3.045	3.045	34.041
10	4.231	5.692	7.441	10.042	15.547	10	3.045	3.045	3.045	3.085	30.796
15	4.682	6.524	8.635	11.549	16.704	15	3.045	3.045	3.045	3.076	26.987
20	5.035	7.212	9.446	12.313	17.264	20	3.045	3.045	3.045	3.060	24.120
		Excess return mean reversion						Excess return mean reversion			
5	3.267	3.719	4.313	5.197	7.031	5	3.045	3.045	3.045	3.045	23.571
10	3.275	4.047	5.241	7.036	10.052	10	3.045	3.045	3.045	3.045	23.072
15	3.324	4.542	6.472	9.016	12.535	15	3.045	3.045	3.045	3.075	21.568
20	3.444	5.222	7.790	10.869	14.308	20	3.045	3.045	3.045	7.871	20.864
		Combined effect						Combined effect			
5	3.822	4.822	5.926	7.255	9.556	5	3.045	3.046	3.047	3.099	29.537
10	4.314	6.203	7.964	9.939	13.251	10	3.045	3.045	3.046	3.115	26.037
15	4.912	7.303	9.270	11.419	15.045	15	3.045	3.045	3.046	3.081	22.553
20	5.302	8.110	10.109	12.178	15.895	20	3.045	3.045	3.046	3.061	20.335

The table presents 5 quantiles of the annualized return distribution. The multiplier of the CPPI is the maximum feasible one for each scenario. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

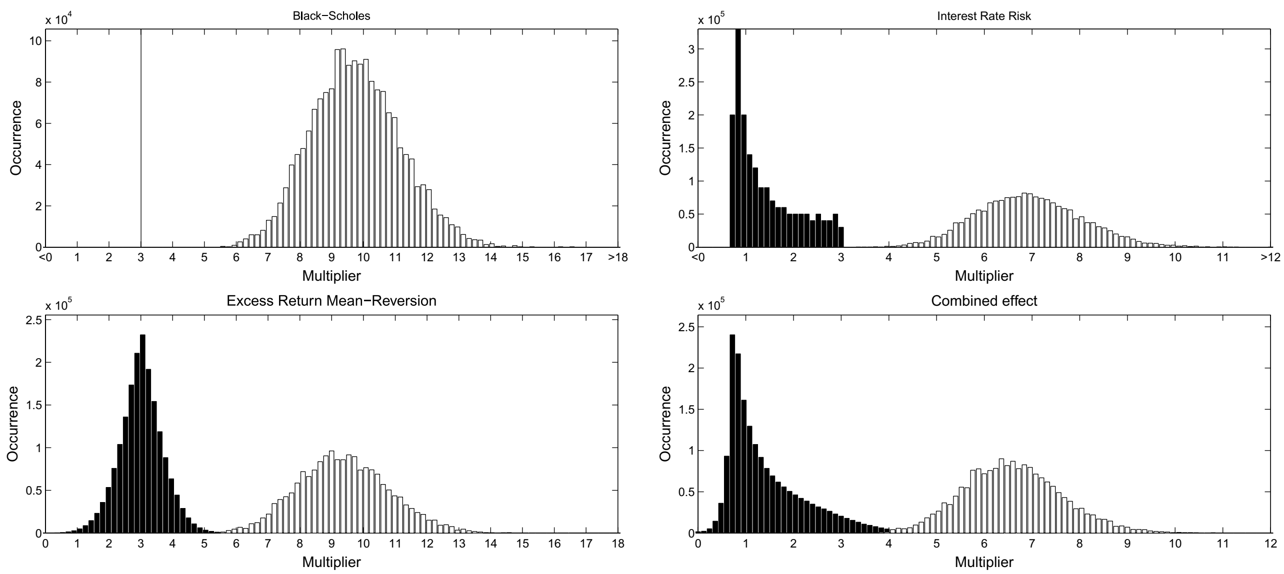

Figure 3 Distribution of the optimal and maximal multipliers across 10,000 scenarios over 15 years of monthly time steps, for the 4 different models. The black histograms correspond to the optimal multiplier and the white ones to the maximum multipliers. From left to right and then from top to bottom, the model configurations are: Black–Scholes, Interest Rate Risk, Excess Return Mean Reversion and Combined effects.

the maximal multipliers across scenarios (see Section 4.6) the optimal multiplier is lower in 95–100% of scenarios depending on the model and horizon pair considered.

Notice as well that the median performance of the GOPI range is $[4.2\%, 10.1\%]$ compared to $[3.045\%, 3.047\%]$ for the CPPI across models and horizons. In fact, except for the 95% quantiles, all other quantiles of the GOPI returns distribution are higher than the corresponding quantiles of the CPPI returns.

Table 7 presents a summary of the distribution of the maximum drawdown (MDD) of the two strategies across scenarios for all model configurations and horizons considered. The distribution of MDD shows an important reduction in the realized risk over intermediate horizons of the GOPI strategy with respect to the risk of the standard CPPI one. In fact 75 out of the 80 quantiles of MDD distributions we looked at (4 models $\times$ 4 horizons $\times$ 5 quantiles) are lower for the GOPI strategy than those for the CPPI strategy. Hence, the outperformance achieved by the optimal strategy does not come at the cost of a higher risk but it is achieved by exploiting the optimal trade-off between expected outperformance of the risky asset and the *relative variance cost* (see Remark 1) of this kind of allocation strategy.

As robustness checks we perform the simulations using different choices for the Floor (using Equation (3) with a constant parameter k across different horizons instead) and for the maximum CPPI multiplier (see Section 4.6 for details). Tables 11, 12, and 13 present the probabilities of outperformance, the outperformance distribution, and the performance distributions of the GOPI and CPPI strategies for all model configurations and horizons using Floor (3) and $k = 0.9$ for all horizons. Similar results for the alternative parametrization of the maximal multiplier are presented in Tables 18–20. The conclusions are qualitatively very similar. These results show an important benefit in terms of performance

Table 7 GOPI and CPPI MDD distribution under different model configurations and horizons.

GOPI MDD distribution						CPPI MDD distribution					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	0.004	0.009	0.016	0.031	0.073	5	0.017	0.030	0.062	0.174	0.418
10	0.019	0.044	0.087	0.159	0.314	10	0.043	0.069	0.146	0.367	0.588
15	0.042	0.104	0.202	0.332	0.520	15	0.070	0.110	0.228	0.475	0.681
20	0.074	0.193	0.333	0.468	0.639	20	0.097	0.153	0.321	0.561	0.746
		Interest rate risk						Interest rate risk			
5	0.038	0.051	0.064	0.082	0.116	5	0.057	0.094	0.150	0.269	0.460
10	0.103	0.134	0.163	0.204	0.281	10	0.153	0.235	0.331	0.460	0.631
15	0.163	0.207	0.250	0.304	0.402	15	0.261	0.370	0.477	0.588	0.738
20	0.210	0.265	0.314	0.375	0.489	20	0.372	0.497	0.599	0.698	0.821
		Excess return mean reversion						Excess return mean reversion			
5	0.006	0.010	0.015	0.023	0.039	5	0.018	0.031	0.063	0.170	0.413
10	0.026	0.053	0.080	0.112	0.171	10	0.043	0.072	0.148	0.367	0.591
15	0.062	0.128	0.185	0.246	0.350	15	0.071	0.114	0.232	0.482	0.687
20	0.120	0.231	0.304	0.385	0.509	20	0.098	0.161	0.326	0.573	0.745
		Combined effect						Combined effect			
5	0.035	0.047	0.059	0.074	0.103	5	0.054	0.091	0.142	0.258	0.457
10	0.094	0.125	0.151	0.183	0.239	10	0.151	0.230	0.324	0.456	0.629
15	0.150	0.197	0.238	0.288	0.377	15	0.260	0.367	0.474	0.588	0.739
20	0.197	0.254	0.307	0.370	0.485	20	0.375	0.496	0.598	0.699	0.820

The table presents 5 quantiles of the MDD distributions. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

(i.e., growth rate) and risk of using the growth-optimal multiplier instead of a maximum constant multiplier.

4.3 Black–Scholes model

In this base case configuration we consider a constant interest rate in which the dynamics of the reserve asset R is driven by the ODE, $dR_t = rR_t dt$ and its value given by

$$R_t = R_0 e^{rt}, \quad \text{for all } t \in [0, T] \tag{24}$$

where $r = 3.69\%$ is the constant interest rate, which is the long-run level of interest rate as estimated by Munk *et al.* (2004) for the Vasicek model (we use this level for comparison purposes with the other model configurations). Note that, in this case with no interest rate risk, the dynamics of the zero-coupon bond are the same as for a locally riskless asset (cash).

The performance-seeking asset follows a Geometric Brownian motion and satisfies:

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_S(t), \tag{25}$$

where $W_S(t) \sim N(0, t)$. The expected rate of return of the stock index is equal to the sum of the long-term levels for interest rate and excess return estimated in Munk *et al.* (2004), i.e., $\mu_S = \bar{r} + \bar{x}$. The volatility parameter is borrowed from Munk *et al.* (2004). The parameters values are given in Table 1.

Table 8 GOPI stock index allocation distribution under different model configurations and horizons.

GOPI stock index allocation (%)						CPPI stock index allocation (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	4.306	8.512	12.029	18.930	41.540	5	0.000	0.000	0.568	37.788	250.000
10	7.334	17.448	28.170	52.374	127.439	10	0.000	0.000	1.029	85.554	250.000
15	9.871	26.823	47.819	96.625	208.542	15	0.000	0.000	1.756	193.151	250.000
20	12.964	38.005	72.352	145.722	250.000	20	0.000	0.000	4.118	250.000	250.000
		Interest rate risk						Interest rate risk			
5	11.012	14.675	19.702	29.390	56.024	5	0.000	0.000	0.954	90.911	250.000
10	20.187	26.191	38.330	62.825	123.721	10	0.000	0.000	0.091	192.260	250.000
15	25.992	33.019	48.435	83.248	163.879	15	0.000	0.000	0.011	207.478	250.000
20	31.031	38.257	53.491	93.040	188.449	20	0.000	0.000	0.005	192.655	250.000
		Excess return mean reversion						Excess return mean reversion			
5	5.398	9.654	12.663	17.698	27.021	5	0.000	0.000	0.636	36.361	250.000
10	10.178	20.532	30.971	47.773	72.196	10	0.000	0.000	1.225	75.971	250.000
15	14.944	33.008	54.421	85.714	122.725	15	0.000	0.000	2.310	150.497	250.000
20	21.150	47.986	82.379	124.692	169.799	20	0.000	0.000	5.416	250.000	250.000
		Combined effect						Combined effect			
5	12.854	15.604	20.316	27.959	41.951	5	0.001	0.117	2.889	77.930	250.000
10	20.936	26.534	38.648	61.553	103.627	10	0.000	0.032	1.529	141.749	250.000
15	25.151	32.205	47.909	83.770	153.076	15	0.000	0.009	0.658	129.476	250.000
20	28.233	36.461	52.045	93.835	185.703	20	0.000	0.003	0.351	111.709	250.000

The table presents 5 quantiles of the stock index allocation distributions. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

In this model with constant parameters, the optimal multiplier is also constant over time and across horizons at 3.01. On the other hand, the maximum multiplier varies across scenarios between 5.5 and 16.5 (for the simulations over 15 years), as shown in the upper left panel of Figure 3.

The top panel of Table 8 presents a summary of the distribution of the time-average allocation of the GOPI strategy for several horizons. The median of the distribution presents an important increase with horizon, going from 12% to 72.4% for 5 and 20 years horizon. A similar behavior is observed in the other quantiles of the distribution, for which the 20 years horizon figures are roughly multiplied by 4 with respect to the ones corresponding to the 5 years horizon. This is consistent with the positive relationship between horizon and initial equity allocation discussed in Section 2.

4.4 *Impact of interest rate risk*

In order to evaluate the impact of interest rate risk on the strategy, we consider a zero-coupon bond as reserve asset driven by a Vasicek model (Vasicek, 1977). The nominal interest rate r_t is described by an Ornstein–Uhlenbeck process:

$$dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r dW_R(t), \qquad (26)$$

where $W_R(t) \sim N(0, t)$ with instantaneous correlation ρ with $W_S(t)$. The zero-coupon bond price follows:

$$\frac{dR_t}{R_t} = (r_t + \lambda_R(t))dt + \sigma_R(t)dW_R(t),$$

with $\lambda_R(t) = \lambda_r \sigma_R(t), \sigma_R(t) = \sigma_r D(r, t)$, and where $D(r, t) = \frac{1-e^{-\kappa(T-t)}}{\kappa}$ is the duration of the bond price. For a time to maturity τ, the price of the bond is given by:

$$R(r_t, \tau) = e^{-a(\tau)-b(\tau)r_t} \qquad (27)$$

where

$$a(\tau) = Y(\infty)(\tau - b(\tau)) + \frac{\sigma_r^2}{4\kappa}(b(\tau))^2$$

$$b(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})$$

and where $Y(\infty) = \bar{r} + \frac{\sigma_r \lambda_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2}$ describes the yield to maturity for a very long bond. The parameters values are borrowed from Munk *et al.* (2004) and presented in Table 2. All mean reverting variables are initialized at their long-term mean.

The performance-seeking asset is assumed to follow the SDE:

$$dS_t = (r_t + \bar{x})S_t dt + \sigma_S S_t dW_S(t), \qquad (28)$$

where $\bar{x} = 6.48\%$ is the long-run excess return of the stock index as estimated by Munk *et al.* (2004). In this model the expected rate of return of the safe asset $\mu_R(t) = r_t + \lambda_R(t)$ varies stochastically over time while its excess return increases deterministically with horizon; thus the expected outperformance of the stock index decreases (deterministically) accordingly,

$$\mu_S(t) - \mu_R(t) = \bar{x} - \lambda_r \sigma_r D(r, t). \qquad (29)$$

Thus, in the presence of interest rate risk (alone), from Equations (14) and (19) one can see that the impact of a decrease in expected outperformance, everything else equal, would be to reduce the optimal multiplier and the Cushion's growth rate values. In this model, the volatility of the reserve asset also increases with horizon deterministically. Figure 4 presents the total effect of horizon on the optimal multiplier (right panel) and on the initial allocation to the risky asset of the GOPI and CPPI strategies (left panel). Figure 4 illustrates

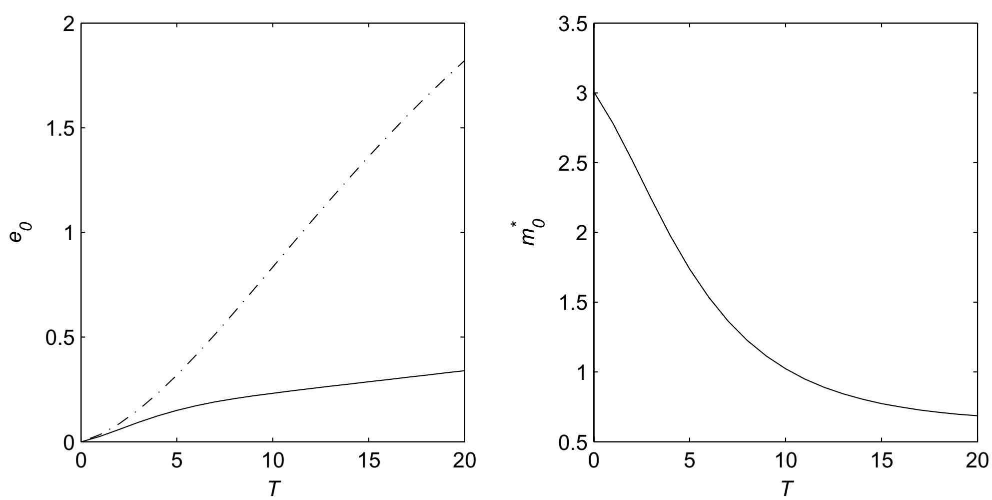

Figure 4 The right panel presents the optimal multiplier for different horizons. The left panel presents the initial allocation to the stock index of GOPI (continuous line) and CPPI (dotted line) strategies across horizons. The price of the zero-coupon bond is estimated using a Vasicek model with parameters from Munk *et al.* (2004), presented in Table 2.

that, parameterizing the Floor of the strategy as in Equation (4), the initial equity allocation of the GOPI strategy increases with horizon, even though the optimal multiplier value decreases with horizon. In contrast with the CPPI, under the considered model and parameters, the initial stock allocation of the GOPI strategy does not imply leverage for any of the horizons considered.

In spite of the negative relationship between the optimal multiplier and horizon induced by interest rate risk, the positive relationship between stock allocation and horizon is further confirmed by the time-average allocation of the GOPI across horizons, as shown in Table 8. The second panel (from top to bottom) of Table 8 presents a summary of the distribution of the time-average allocation of the GOPI strategy for several horizons in the presence of interest rate risk alone. In fact, the median of the distribution increases consistently with horizon, from 20% to 53% for 5–20 years horizon. The increase in the average stock allocation across scenarios in the presence of interest rate risk is less marked with respect to the Black–Scholes model with constant interest rate.

As we will see in Section 4.6, when the Floor is parameterized using a fixed level of initial Cushion regardless of horizon, as in Equation (3), in this model, the initial allocation of the GOPI strategy will in fact decrease with horizon due to the negative relationship of the optimal multiplier with horizon in Equation (29). However, as shown hereafter, the effect of mean reversion in expected returns can outweigh this effect and make the average stock allocation increase with horizon.

4.5 *Impact of mean reverting excess returns*

We now consider a set-up with constant interest rate in which the bond is driven by Equation (24), and the stock index value is driven by a mean reverting expected rate of return as follows:

$$\frac{dS_t}{S_t} = (\bar{r} + x_t)dt + \sigma_S dW_S(t),$$

where $\bar{r}$ is the constant long-run interest rate level and x_t a time-varying expected excess return driven by an Ornstein–Uhlenbeck process:

$$dx_t = \alpha(\bar{x} - x_t)dt - \sigma_x dW_S(t), \qquad (30)$$

as in Munk *et al.* (2004). The parameters values are given in Table 3. The (perfect) negative correlation between past realized returns and expected returns, which is imposed by using the same Brownian motion in both equations, has been shown to be an assumption that is close to empirical observations.[13]

We find that mean reversion in excess returns introduces a short-term "tactical" effect and a long-term "strategic" one on the allocation to the risky asset of the GOPI strategy. The former is a counter-cyclical movement of the optimal multiplier and the latter is an equity allocation increasing with the length of investment horizon.

In this model, the expected excess return of the risky asset increases following a negative return and vice versa, thus it presents a counter-cyclical dynamic. In the presence of mean reverting excess returns, Equations (19) and (30) imply that after a period of negative realized returns for the risky asset, the optimal multiplier value would increase, everything else equal. Due to mean reversion, after a series of negative returns, x_t is likely to be higher and push S_t up again. In this case, the optimal multiplier would be relatively higher than that at the beginning of the "bear market" period anticipating the positive returns.

In order to see this, Table 9 presents the relative change of the optimal multiplier, denoted as $\Delta m^{\star}$ during the maximum drawdown (MDD) period of the stock index for each 15-year simulated scenario. In fact, in this model, all quantiles of the

Table 9 Relative change in optimal multiplier during the maximum drawdown period.

$\Delta m^\star$ during MDD (%)	5%	25%	50%	75%	95%
Black–Scholes	0.000	0.000	0.000	0.000	0.000
Interest rate risk	2.634	6.639	12.828	25.273	65.038
Excess return mean reversion	19.964	29.220	39.084	54.481	92.229
Combined effect	28.380	47.746	72.104	116.550	267.948

The table presents 5 quantiles of the relative changes in the optimal multiplier distributions. We set $\theta = e^{0.03T}$ for $T = 15$ years.

distribution of $\Delta m^\star$ during each maximum drawdown period are positive, varying from a 20% increase for the 5% quantile up to 92% increase for the 95% quantile. This large increase contrasts with the constant multiplier (zero relative change) of the Black–Scholes model. Notice that the strictly positive $\Delta m^\star$ observed for the interest rate risk model does not depend on a tactical effect conditional on the MDD period, but it simply comes from a deterministic increase in the multiplier as approaching horizon, as previously discussed.

As it can be observed in Table 5, comparing the third and first panels (from top to bottom), the median of the outperformance distribution presents a very important increase in level with respect to the model with constant excess return for longer horizons, being 1.05% in the Black–Scholes model compared to 2.15% of the mean reversion model on the 20-year simulation. A similar increase is observed for the 75% quantile of the distribution. On the other hand, the 95% quantiles are lower than those for the BS model case, which shows that the dispersion of the outperformance distribution of the GOPI strategy decreases with horizon in the presence of mean reverting stock returns.

In order to further evaluate the potential benefits of the counter-cyclical movement of the optimal multiplier we look at the distribution of the outperformance of the GOPI strategy with respect to the CPPI for each scenario over the 15 years simulations between the start of the maximum drawdown period of the stock index until recovery of the index value at the beginning of the MDD period.[14] Table 10 presents a summary of the distribution of this conditional outperformance, for which the median, the 75% and the 95% quantiles are 1.4%, 9.2%, and 19.4% respectively, whereas the 5% and 25% quantiles are -10% and -2.3%, respectively. Thus the conditional outperformance presents a distribution with a very important right-skew and centered well above zero, which contrast with the distribution of conditional outperformance in the Black–Scholes model, for which the median value is 0.0%.

Table 10 GOPI outperformance from the beginning of the maximum drawdown period up to recovery.

Conditional outperformance (%)	5%	25%	50%	75%	95%
Black–Scholes	–9.817	–2.249	0.000	4.623	15.835
Interest rate risk	–18.744	–2.932	1.113	10.646	29.160
Excess return mean reversion	–9.938	–2.336	1.358	9.166	19.377
Combined effect	–16.417	–2.199	2.046	12.317	31.545

The table presents 5 quantiles of the GOPI outperformance distributions. We set $\theta = e^{0.03T}$ for $T = 15$ years.

Table 8 illustrates that the presence of mean reversion in expected returns induces a dramatic increase in the median of the distribution of the average allocation to the stock index of the GOPI strategy for longer horizons, compared to a model with constant expected returns (i.e., Black–Scholes setting). While at the 5 years horizon the median of the stock allocation distribution of the GOPI strategy is about 12% in both models, at the 20 years horizon the median of the distribution of average stock allocation is 82% in the presence of mean reversion compared to 72% in the Black–Scholes model. The average stock allocation distribution also presents a notable dispersion reduction around its median in the mean reversion model compared to the BS setting as horizon increases.

While the counter-cyclical effect on stock allocation observed follows directly from the mean reverting dynamics of equity expected returns and the optimal multiplier formula, the positive relationship between risk exposure and horizon might be surprising, given the "myopic" dynamics of the multiplier (see Corollary 4). However, this effect can be explained by the negative relationship between the volatility of S and the growth rate of the strategy, i.e., the relative variance cost (Remark 1) together with the volatility term structure of mean reverting stock returns (see for instance Campbell and Viceira, 2003).

This result contrasts with the allocation of the unconstrained growth-optimal portfolio but it is similar to the hedging-demand effect of an unconstrained CRRA expected utility maximization model. Indeed, Campbell and Viceira (2003) show that in the presence of mean reverting equity returns, the risk of investments in stocks decreases with horizon, thus implying a higher optimal allocation for long-term investors with CRRA$\neq 1$.

We also consider a *Combined effect* model in which the interest rate and reserve asset follow a Vasicek process as in Equations (26) and (27) and the stock index follows:

$$\frac{dS_t}{S_t} = (r_t + x_t)dt + \sigma_S dW_S(t),$$

and x_t follows the dynamics in Equation (30) as in Munk *et al.* (2004) (we use the same parameter values).

All former conclusions for the effect of mean reversion still hold and are actually strengthened in this configuration: the outperformance of the GOPI strategy is higher and increases with horizon (Table 5). The median of the average stock index allocation distribution is higher and increases with horizon with respect to the interest rate model (Table 8). The tactical component of the multiplier (Table 9) induces an important outperformance with respect to the CPPI during mean reverting return periods (Table 10).

4.6 *Robustness*

4.6.1 *Discounted terminal wealth floor*

In this subsection, we present the results obtained using the discounted terminal wealth Floor presented in Equation (3) with a parameter set at $k = 0.9$ across all horizons considered. The probability of outperformance of the GOPI over the CPPI strategy (Table 11) varies between 76% and 98% and is higher than with the previous Floor parametrization in the majority of model/horizon pairs (9 out of 16).

Table 12 summarizes the distribution of the GOPI outperformance, which confirms the important benefits of using the optimal multiplier instead of the maximal one. The median outperformance is around 1% across the different horizons for the Black–Scholes model. For the model with interest rate risk the median outperformance varies between 1.2% and 2.1%. The mean

Table 11 GOPI outperformance probability under different model configurations and horizons.

Outperformance probability	5	10	15	20
Black–Scholes	0.768	0.768	0.768	0.764
Interest rate risk	0.825	0.904	0.947	0.968
Excess return mean reversion	0.781	0.791	0.789	0.790
Combined effect	0.850	0.925	0.962	0.976

The probability is calculated as the percentage of scenarios for which the GOPI strategy presented a higher cumulative return than the corresponding CPPI strategy. The Floor parameter $k = 0.9$ for all horizons.

Table 12 GOPI outperformance distribution under different model configurations and horizons.

	Outperformance distribution (%)				
Horizon	5%	25%	50%	75%	95%
			Black–Scholes		
5	–26.229	0.418	1.813	3.563	7.866
10	–15.731	0.158	1.193	2.967	8.509
15	–11.107	0.086	1.043	3.100	9.314
20	–8.502	0.052	0.999	3.469	10.450
			Interest rate risk		
5	–25.896	1.047	2.141	3.588	6.986
10	–14.176	0.809	1.566	2.757	5.580
15	–2.711	0.699	1.310	2.331	4.745
20	0.177	0.623	1.165	2.103	4.432
			Excess return mean reversion		
5	–24.611	0.483	2.125	4.019	7.551
10	–14.815	0.293	1.719	3.727	7.568
15	–10.487	0.216	1.797	4.121	7.880
20	–8.180	0.214	1.997	4.734	8.532
			Combined effect		
5	–22.865	1.232	2.532	3.956	6.248
10	–9.053	1.048	1.908	2.915	4.674
15	0.190	0.927	1.580	2.396	4.055
20	0.225	0.836	1.397	2.126	3.844

The outperformance is calculated as the difference in average annualized return of the GOPI strategy and the CPPI strategy for each scenario over the corresponding horizon. The table presents 5 quantiles of the outperformance distribution. The Floor parameter $k = 0.9$ for all horizons.

reverting excess return model presents a median outperformance between 1.7% and 2% and higher 75% and 95% quantiles (around 4% and 8% respectively). The Combined effect model presents an outperformance distribution with less dispersion around its median value that for the mean reverting model and higher values than for the interest rate risk model. The performance

Table 13 GOPI and CPPI annualized return distribution under different model configurations and horizons.

GOPI return distribution (%)						CPPI return distribution (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	2.279	3.395	5.002	7.852	15.944	5	1.595	1.595	1.595	3.150	40.247
10	2.970	3.811	5.345	8.574	17.364	10	2.671	2.671	2.671	3.480	31.147
15	3.226	3.997	5.771	9.465	19.223	15	3.033	3.033	3.033	4.367	27.055
20	3.365	4.163	6.146	10.472	19.876	20	3.214	3.214	3.214	5.803	25.390
		Interest rate risk						Interest rate risk			
5	3.663	4.608	5.790	7.638	12.115	5	2.738	2.738	2.738	2.738	36.090
10	5.093	5.713	6.515	7.824	11.028	10	4.622	4.622	4.622	4.622	22.619
15	5.874	6.341	6.985	8.083	10.640	15	5.528	5.528	5.528	5.528	10.562
20	6.338	6.749	7.319	8.314	10.805	20	6.065	6.065	6.065	6.065	6.069
		Excess return mean reversion						Excess return mean reversion			
5	2.281	3.642	5.360	7.791	12.451	5	1.595	1.595	1.595	2.335	35.864
10	3.025	4.188	5.917	8.386	12.284	10	2.671	2.671	2.671	2.784	25.823
15	3.316	4.555	6.511	9.081	12.625	15	3.033	3.033	3.033	3.081	21.720
20	3.517	4.926	7.106	9.876	13.097	20	3.214	3.214	3.214	3.272	19.810
		Combined effect						Combined effect			
5	3.645	4.806	6.080	7.602	10.216	5	2.738	2.738	2.740	2.824	31.673
10	5.126	5.941	6.767	7.770	9.624	10	4.622	4.622	4.622	4.625	16.305
15	5.928	6.564	7.203	8.028	9.734	15	5.528	5.528	5.528	5.528	5.853
20	6.382	6.956	7.519	8.266	10.057	20	6.065	6.065	6.065	6.065	6.090

The table presents 5 quantiles of the annualized return distribution. The multiplier of the CPPI is the maximum feasible one for each scenario. The Floor parameter $k = 0.9$ for all horizons.

distributions presented in Table 13 confirm the important potential benefits of the GOPI relative to the CPPI using the discounted terminal wealth Floor with a fixed parameter k across horizons.

Table 14 presents a summary of the maximum drawdown (MDD) distribution of the two strategies. Similar to the test of former sections, in 14 out of 16 cases (4 models × 4 horizons) the median MDD is smaller for the GOPI strategy. More generally, 75 out of the 80 MDD quantiles (4 models × 4 horizons × 5 quantiles) presented are lower for the GOPI strategy than for the CPPI strategy. Notice that this is the case in particular for all the 75% and 95% quantiles.

The main difference with respect to the strategy using the former Floor setting is the effect of investment horizon on the strategy. Unlike the Floor providing a guarantee of a fixed return per annum regardless of horizon used before, the discounted terminal wealth floor with the same parameter k across horizons implies the same initial Cushion level regardless of horizon. In the model with interest rate risk, the initial risk

Table 14 GOPI and CPPI maximum drawdown distribution under different model configurations and horizons.

GOPI MDD distribution						CPPI MDD distribution					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	0.029	0.045	0.068	0.105	0.190	5	0.073	0.103	0.189	0.326	0.502
10	0.037	0.073	0.132	0.217	0.379	10	0.071	0.110	0.216	0.422	0.615
15	0.043	0.106	0.205	0.334	0.523	15	0.071	0.112	0.232	0.477	0.682
20	0.052	0.152	0.289	0.432	0.615	20	0.071	0.116	0.252	0.530	0.729
		Interest rate risk						Interest rate risk			
5	0.040	0.054	0.067	0.086	0.123	5	0.063	0.107	0.170	0.291	0.474
10	0.093	0.124	0.153	0.191	0.260	10	0.112	0.158	0.213	0.317	0.549
15	0.163	0.211	0.258	0.316	0.411	15	0.181	0.242	0.303	0.389	0.589
20	0.233	0.299	0.356	0.424	0.535	20	0.252	0.326	0.397	0.487	0.658
		Excess return mean reversion						Excess return mean reversion			
5	0.033	0.047	0.063	0.086	0.131	5	0.074	0.105	0.192	0.330	0.506
10	0.047	0.084	0.119	0.161	0.237	10	0.072	0.113	0.218	0.429	0.622
15	0.064	0.130	0.187	0.249	0.353	15	0.072	0.116	0.236	0.484	0.688
20	0.090	0.194	0.264	0.339	0.460	20	0.072	0.122	0.258	0.537	0.732
		Combined effect						Combined effect			
5	0.037	0.049	0.061	0.076	0.105	5	0.061	0.103	0.161	0.282	0.470
10	0.091	0.120	0.149	0.187	0.255	10	0.110	0.156	0.207	0.302	0.536
15	0.162	0.211	0.257	0.313	0.407	15	0.180	0.240	0.301	0.381	0.578
20	0.233	0.298	0.356	0.422	0.535	20	0.251	0.325	0.395	0.481	0.645

The table presents 5 quantiles of the maximum drawdown distributions. The Floor parameter $k = 0.9$ for all horizons.

exposure of the strategy decreases with horizon due to the negative relationship of the initial optimal multiplier and horizon (for the CPPI the initial allocation would be constant across horizons for a given k).

Table 12 shows that, with this horizon insensitive Floor parametrization, the median outperformance does not increase with horizon (see contrast with Table 5). This is also the case for the median stock allocation as we can see by comparing Table 15 with Table 8. However, notice that the dispersion around the median outperformance still decreases with horizon, as for the risk-sensitive capital guarantee Floor setting used before.

The tactical and strategic long-horizon effects of mean reverting stock returns on the allocation of the GOPI strategy still hold with the Floor parametrization with fixed k: the median allocation to stocks increases with horizon in the presence of mean reverting returns, everything else equal (see this by comparing the middle rows in the first and the third panel and in the second with the fourth panel of Table 15). Also note the outperformance of the GOPI with respect to the CPPI during the maximum drawdown plus

Table 15 GOPI stock index allocation distribution under different model configurations and horizons.

GOPI stock index allocation (%)						CPPI stock index allocation (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	13.220	25.388	35.042	52.730	101.203	5	0.000	0.000	14.832	200.646	250.000
10	11.255	26.292	41.649	74.276	160.449	10	0.000	0.000	4.527	197.390	250.000
15	10.051	27.285	48.578	97.860	209.738	15	0.000	0.000	1.886	201.744	250.000
20	9.840	29.464	57.791	124.435	247.032	20	0.000	0.000	1.238	236.612	250.000
		Interest rate risk						Interest rate risk			
5	12.808	17.000	22.766	33.789	63.484	5	0.000	0.000	2.139	116.659	250.000
10	8.632	11.478	17.191	29.588	67.031	10	0.000	0.000	0.000	16.836	250.000
15	6.620	8.923	13.868	26.050	67.033	15	0.000	0.000	0.000	0.260	250.000
20	5.722	7.878	12.347	24.252	69.719	20	0.000	0.000	0.000	0.000	147.877
		Excess return mean reversion						Excess return mean reversion			
5	16.571	28.703	36.736	49.165	67.669	5	0.000	0.000	15.964	184.703	250.000
10	15.605	30.852	45.532	67.320	94.842	10	0.000	0.000	5.263	164.961	250.000
15	15.216	33.568	55.252	86.757	123.807	15	0.000	0.000	2.470	156.628	250.000
20	16.104	37.448	67.034	107.496	153.208	20	0.000	0.000	1.659	164.317	250.000
		Combined effect						Combined effect			
5	14.924	18.061	23.457	32.143	47.780	5	0.001	0.148	3.876	98.037	250.000
10	9.170	11.776	17.498	28.890	53.459	10	0.000	0.006	0.205	12.806	250.000
15	6.770	8.934	13.929	26.117	57.708	15	0.000	0.000	0.019	1.126	217.993
20	5.776	7.765	12.157	24.294	61.956	20	0.000	0.000	0.003	0.300	104.776

The table presents 5 quantiles of the stock index allocation distributions. The Floor parameter $k = 0.9$ for all horizons.

Table 16 GOPI conditional outperformance, from the beginning of the maximum drawdown period up to recovery.

Conditional outperformance (%)	5%	25%	50%	75%	95%
Black–Scholes	–9.850	–2.253	0.000	4.677	15.803
Interest rate risk	–8.585	–1.450	–0.176	1.186	19.160
Excess return mean reversion	–9.977	–2.354	1.437	9.236	19.289
Combined effect	–6.503	–1.088	0.134	1.416	17.077

The table presents 5 quantiles of the GOPI outperformance distributions. The Floor parameter $k = 0.9$ for all horizons.

recovery period of the stock index (see last two rows of Table 16.

4.6.2 *Fixed maximum CPPI multiplier across scenarios*

So far we assumed perfect foresight over the maximum multiplier that allows the CPPI to stay above its Floor under discrete-time trading (Equation (22)), thus we used the simulated asset returns of each scenario to determine ex-post the maximum multiplier. In practice this parameter has to be estimated. Thus, we now present the results of a test similar to the ones previously discussed but using foresight only over the minimum of the maximum multipliers for the CPPI across all simulated scenarios (instead of using the upper bound of each scenario). This approach yields a multiplier that is numerically closer to the optimal multiplier in most scenarios and that is constant over time but also across scenarios. The qualitative conclusions from the former analysis still hold.

Table 17 presents the minimum maximum multiplier across all simulated scenarios for all model configurations and horizons. Due to the fact that we used a common seed on the random number generator across different model set-ups to generate dW_R and dW_S, the minimum maximum multipliers can be compared across different model configurations (notice that we used different seeds across investment horizons to increase heterogeneity of the data analyzed).

Table 17 Minimum of the maximum multipliers for the CPPI across all simulated scenarios in each model set-up and horizon, for $T = \{5, 10, 15, 20\}$.

Min. maximum multiplier	5	10	15	20
Black–Scholes	6.026	4.816	5.205	5.285
Interest rate risk	4.790	4.140	3.575	2.964
Excess return mean reversion	5.570	4.341	4.777	4.823
Combined effect	5.208	3.846	3.434	2.998

The difference between the maximum multipliers between models without interest rate risk and their counterparts with interest rate risk is important. For instance for the 20 years simulation, the maximum multiplier for the model with excess return mean reversion (only) is 4.8 compared to the Combined effect model which presents a maximum multiplier of 2.99. This result shows that disregarding the right-tail risk of the reserve asset in the presence of interest rate risk can significantly induce an overestimation of the maximum multiplier, and thus an important underestimation of the gap risk of the strategy.

A visible difference between the CPPI strategy using the perfect foresight multiplier and the one using the minimum maximal multiplier is that the latter presents a distribution of stock allocation across horizons and models much less concentrated on its bounds, i.e., [0, 2.5] as shown in the right panel of Table 22.

The probability of outperformance of the GOPI over the CPPI strategy (Table 18) varies between 60% and 86%. Table 19 presents the distribution of the GOPI outperformance with respect to the CPPI with the minimum maximal multiplier. Although the level of outperformance is lower than for the tests with the perfect foresight in each scenario of the maximal multiplier, the economic benefits of the optimal strategy remain clear. The outperformance probabilities and the level of outperformance are lowest for the Black–Scholes configuration and highest for the Combined effect model for most horizons. The median level of outperformance also increases with horizon, going from 0.86% p.a. for 5 years up to 4% p.a. for the 20 years horizon for the Combined effect model.

Table 18 GOPI outperformance probability under different model configurations and horizons.

Outperformance probability	5	10	15	20
Black–Scholes	0.696	0.656	0.635	0.603
Interest rate risk	0.644	0.705	0.762	0.799
Excess return mean reversion	0.715	0.731	0.714	0.677
Combined effect	0.722	0.753	0.815	0.859

The probability is calculated as the percentage of scenarios for which the GOPI strategy presented a higher cumulative return than the corresponding CPPI strategy. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

Table 19 GOPI outperformance distribution under different model configurations and horizons.

	Outperformance distribution (%)				
Horizon	5%	25%	50%	75%	95%
		Black–Scholes			
5	–11.269	–0.426	0.297	0.435	0.605
10	–9.116	–1.127	0.262	0.475	0.666
15	–7.406	–2.702	0.363	0.955	1.522
20	–5.503	–2.745	0.329	1.244	2.321
		Interest rate risk			
5	–12.959	–1.254	0.570	0.917	1.380
10	–13.907	–1.202	1.333	2.289	3.805
15	–11.491	0.344	2.348	3.698	5.951
20	–9.119	0.998	2.839	4.353	6.581
		Excess return mean reversion			
5	–6.999	–0.233	0.299	0.449	0.655
10	–7.036	–0.162	0.347	0.587	0.872
15	–7.420	–0.745	0.705	1.379	2.050
20	–5.878	–1.235	0.800	1.913	2.983
		Combined effect			
5	–12.579	–0.371	0.863	1.317	1.976
10	–11.089	0.038	1.613	2.732	4.424
15	–8.583	1.133	3.050	4.619	7.108
20	–6.711	1.928	3.986	5.643	8.027

The outperformance is calculated as the difference in average annualized return of the GOPI strategy and the CPPI strategy for each scenario over the corresponding horizon. The table presents the quantiles of the outperformance distribution. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

Table 20 GOPI and CPPI annualized return distribution under different model configurations and horizons.

GOPI return distribution (%)						CPPI return distribution (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	3.267	3.636	4.187	5.219	8.524	5	3.065	3.203	3.698	5.613	19.667
10	3.239	3.794	4.840	7.176	14.226	10	3.075	3.329	4.311	8.354	23.317
15	3.235	3.993	5.741	9.397	19.107	15	3.054	3.222	4.545	12.172	25.646
20	3.246	4.274	6.685	11.582	21.209	20	3.050	3.196	5.328	14.341	25.658
		Interest rate risk						Interest rate risk			
5	3.837	4.651	5.675	7.286	11.244	5	3.163	3.646	4.926	8.503	24.192
10	4.231	5.692	7.441	10.042	15.547	10	3.114	3.610	5.269	11.133	28.188
15	4.682	6.524	8.635	11.549	16.704	15	3.093	3.487	5.049	10.790	26.095
20	5.035	7.212	9.446	12.313	17.264	20	3.105	3.572	5.298	10.465	24.266
		Excess return mean reversion						Excess return mean reversion			
5	3.267	3.719	4.313	5.197	7.031	5	3.086	3.280	3.815	5.389	14.082
10	3.275	4.047	5.241	7.036	10.052	10	3.145	3.552	4.528	7.208	16.935
15	3.324	4.542	6.472	9.016	12.535	15	3.100	3.493	4.841	9.785	19.588
20	3.444	5.222	7.790	10.869	14.308	20	3.104	3.586	5.765	11.833	19.615
		Combined effect						Combined effect			
5	3.822	4.822	5.926	7.255	9.556	5	3.143	3.536	4.570	7.587	21.855
10	4.314	6.203	7.964	9.939	13.251	10	3.207	3.856	5.428	9.530	23.033
15	4.912	7.303	9.270	11.419	15.045	15	3.161	3.651	4.967	8.854	21.381
20	5.302	8.110	10.109	12.178	15.895	20	3.147	3.599	4.776	8.305	20.662

The table presents 5 quantiles of the annualized return distribution. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

Table 21 GOPI and CPPI MDD distribution under different model configurations and horizons.

GOPI MDD distribution						CPPI MDD distribution					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
		Black–Scholes						Black–Scholes			
5	0.004	0.009	0.016	0.031	0.073	5	0.011	0.023	0.050	0.124	0.304
10	0.019	0.044	0.087	0.159	0.314	10	0.030	0.070	0.167	0.314	0.512
15	0.042	0.104	0.202	0.332	0.520	15	0.058	0.148	0.323	0.485	0.662
20	0.074	0.193	0.333	0.468	0.639	20	0.089	0.236	0.437	0.576	0.738
		Interest rate risk						Interest rate risk			
5	0.038	0.051	0.064	0.082	0.116	5	0.049	0.075	0.108	0.172	0.326
10	0.103	0.134	0.163	0.204	0.281	10	0.123	0.187	0.270	0.394	0.564
15	0.163	0.207	0.250	0.304	0.402	15	0.194	0.283	0.385	0.510	0.680
20	0.210	0.265	0.314	0.375	0.489	20	0.256	0.359	0.464	0.588	0.754

Table 21 (*Continued*)

GOPI MDD distribution						CPPI MDD distribution					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
	Excess return mean reversion						Excess return mean reversion				
5	0.006	0.010	0.015	0.023	0.039	5	0.011	0.023	0.047	0.105	0.264
10	0.026	0.053	0.080	0.112	0.171	10	0.030	0.072	0.150	0.271	0.477
15	0.062	0.128	0.185	0.246	0.350	15	0.064	0.166	0.319	0.475	0.655
20	0.120	0.231	0.304	0.385	0.509	20	0.108	0.281	0.445	0.581	0.732
	Combined effect						Combined effect				
5	0.035	0.047	0.059	0.074	0.103	5	0.048	0.076	0.111	0.180	0.347
10	0.094	0.125	0.151	0.183	0.239	10	0.119	0.176	0.250	0.365	0.544
15	0.150	0.197	0.238	0.288	0.377	15	0.187	0.269	0.365	0.493	0.672
20	0.197	0.254	0.307	0.370	0.485	20	0.253	0.352	0.457	0.583	0.756

The table presents 5 quantiles of the MDD distributions. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

Table 22 GOPI stock index allocation distribution under different model configurations and horizons.

GOPI stock index allocation (%)						CPPI stock index allocation (%)					
Horizon	5%	25%	50%	75%	95%	Horizon	5%	25%	50%	75%	95%
	Black–Scholes						Black–Scholes				
5	4.306	8.512	12.029	18.930	41.540	5	1.542	8.662	20.436	43.652	167.319
10	7.334	17.448	28.170	52.374	127.439	10	3.295	17.418	39.123	94.324	250.000
15	9.871	26.823	47.819	96.625	208.542	15	2.087	20.532	62.023	187.043	250.000
20	12.964	38.005	72.352	145.722	250.000	20	1.733	25.363	90.186	250.000	250.000
	Interest rate risk						Interest rate risk				
5	11.012	14.675	19.702	29.390	56.024	5	5.329	20.310	40.690	72.391	187.756
10	20.187	26.191	38.330	62.825	123.721	10	5.731	30.662	76.857	151.095	250.000
15	25.992	33.019	48.435	83.248	163.879	15	4.663	30.674	87.478	179.199	250.000
20	31.031	38.257	53.491	93.040	188.449	20	6.138	36.714	99.562	183.822	250.000
	Excess return mean reversion						Excess return mean reversion				
5	5.398	9.654	12.663	17.698	27.021	5	2.279	9.605	19.445	39.286	126.747
10	10.178	20.532	30.971	47.773	72.196	10	5.878	19.979	38.018	79.372	214.698
15	14.944	33.008	54.421	85.714	122.725	15	5.568	27.611	63.804	153.750	250.000
20	21.150	47.986	82.379	124.692	169.799	20	6.828	38.175	94.824	241.430	250.000
	Combined effect						Combined effect				
5	12.854	15.604	20.316	27.959	41.951	5	4.558	18.659	40.440	73.824	194.296
10	20.936	26.534	38.648	61.553	103.627	10	8.647	33.513	73.015	131.644	250.000
15	25.151	32.205	47.909	83.770	153.076	15	7.411	32.959	81.770	159.003	250.000
20	28.233	36.461	52.045	93.835	185.703	20	7.247	33.800	87.307	168.166	250.000

The table presents 5 quantiles of the stock index allocation distributions. The Floor parameter $k = 0.9$ for all horizons.

Table 23 GOPI conditional outperformance, from the beginning of the maximum drawdown period up to recovery.

Conditional outperformance (%)	5%	25%	50%	75%	95%
Black–Scholes	–1.281	0.299	2.170	5.880	11.577
Interest rate risk	–6.765	1.576	4.838	10.270	23.082
Excess return mean reversion	–1.179	1.001	4.123	9.708	16.423
Combined effect	–6.779	2.422	6.080	12.495	26.491

The table presents 5 quantiles of the GOPI outperformance distributions. We set $\theta = e^{0.03T}$ for $T = \{5, 10, 15, 20\}$.

This comes with an increase in performance of the GOPI strategy with horizon, while the CPPI strategy presents a stable annualized performance across horizons (see Table 20). The MDD of the optimal strategy remains lower than for the CPPI strategy, for most quantiles of the distributions, as shown in Table 21 (in fact for all quantiles except one across models and horizons).

In this setting, the benefits of the counter-cyclical tactical component of the GOPI strategy are in fact more apparent when comparing the strategy with the CPPI with the minimal maximum multiplier, as shown in Table 23 (compared to Table 10). The median outperformance between the start of the MDD period up to recovery increases with respect to the models without mean reversion by more than 1% and is equal to 6% for the Combined effect model, compared to 4.83% for the interest rate risk model.

5 Conclusion

In this paper we study a class of CPPI-type strategies with a time-varying multiplier process. We derive an explicit formula for the multiplier that maximizes the time-average growth rate of the strategies and of the growth rate of the Cushion process, in the general case with a locally risky asset and stochastic state variables. Most former studies of the CPPI assumed a constant interest rate, neglecting the covariance between the underlying assets of the strategy. Our results illustrate that a higher correlation between the safe and the risky asset implies an increase in the value of the strategy, everything else being equal.

Furthermore, we find important potential benefits in terms of performance and risk in using the optimal strategy over the standard CPPI with constant multiplier. We find that interest rate risk induces a negative relationship between the optimal multiplier and horizon, due to the higher expected return of long maturity bonds. On the other hand, mean reversion in equity returns induces a "strategic" increase in the allocation to stocks for longer horizons and introduces a counter-cyclical short-term "tactical" component to the strategy that contrasts with the pure trend-following behavior of the CPPI allocation. These characteristics of the long-term optimal portfolio insurance allocation strategy introduced here are consistent with common features of popular asset allocation advice from professional advisors, such as the horizon puzzle and the risk-sensitive bond/stock allocation ratio puzzle reported by Samuelson (1963) and Canner *et al.* (1997).

Acknowledgments

The author would like to thank Maxime Bonelli for excellent research assistance, and Mireille

Bossy, Lionel Martellini, Gideon Ozik, and Vijay Vaidyanathan for their valuable comments. Much of this article was written when the author was at Koris International. All views expressed in this article are those of the author and do not necessarily represent the views of Koris International, Edhec-Risk Institute or Optimal Asset Management.

Appendices

A Proofs and technical results

Proof of Corollary 1

Proof. From the Cushion process definition of a given $\boldsymbol{PI}_m$, the dynamics of $\boldsymbol{C}_m$ follows

$$\begin{aligned} dC_m^{PI}(t) &= d(V_m^{PI}(t) - F_t) \\ &= dV_m^{PI}(t) - dF_t \\ &= V_m^{PI}(t)\left(\frac{m_t C_m^{PI}(t)}{V_m^{PI}(t)}\frac{dS_t}{S_t} \right. \\ &\quad \left. + \left(1 - \frac{m_t C_m^{PI}(t)}{V_m^{PI}(t)}\right)\frac{dR_t}{R_t}\right) - F_t\frac{dR_t}{R_t} \\ &= m_t C_m^{PI}(t)\frac{dS_t}{S_t} + (1-m_t)C_m^{PI}(t)\frac{dR_t}{R_t} \\ &= C_m^{PI}(t)\left(m_t\frac{dS_t}{S_t} + (1-m_t)\frac{dR_t}{R_t}\right). \end{aligned} \tag{A1}$$

Equation (A1) proves the conjecture that a given Cushion process follows the dynamics of a portfolio with weights $\pi(t) = (m_t, 1-m_t)'$. Fernholz (2002) shows that the portfolio log-value process satisfies

$$d\log S_\pi(t) = g_\pi(t)dt + \sum_{i,k=1}^{n} \pi_i(t) b_t^{i,k} dz_t^k, \tag{A2}$$

almost surely, where the predictable stochastic process $b^{i,k} = \{b_t^{i,k}\}$ is the *volatility* of the ith security with respect to the kth Wiener process z^k (notice that the covariance matrix elements are $\sigma_{ij}(t)dt = \sum_{v=1}^{n} b_t^{i,v} b_t^{j,v} dt$).

Result (Eq. (13)) in the Corollary follows from integrating Equation (A2) for the Cushion "portfolio" while result (Eq. (14)) follows from Equation (8). □

Proof of Corollary 2

Proof. In the Black–Scholes model with constant parameters, for a set of n risky assets, A_i for $1 \le i \le n$, Wise (1996) (Equation A3) shows that the value process of a portfolio with constant weights π is

$$V_\pi^{FM}(t) = V_0 e^{g_\pi^* t} \prod_{i=1}^{n} \left(\frac{A_i(t)}{A_i(0)}\right)^{\pi_i}. \tag{A3}$$

The result follows from Corollary 1, Equation (A3) and the assumption $dF_t = dR_t$ for $t \in [0, T]$. □

Proposition 2. *For a given $\mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, the Cushion growth-optimal multiplier $m^\star_{\text{cushion}}$ is defined as*

$$m^\star_{\text{cushion}} = \underset{m}{\operatorname{argmax}} \lim_{T\to\infty} \frac{1}{T} \times \int_0^T g_m^{\text{cushion}}(t)dt \quad a.s.$$

and is equal to

$$m^\star_{\text{cushion}}(t) = \frac{g_S(t) - g_R(t) + g^*(t)}{2g^*(t)}, \quad \textit{for all } t \in [0, T]. \tag{A4}$$

Proof. Deriving the Cushion growth rate at any time t (Equation (14)) with respect to m_t and equating to zero yields:

$$m^\star_{\text{cushion}}(t) = \frac{g_S(t) - g_R(t) + g^*(t)}{2g^*(t)}.$$

Notice that the quadratic coefficient of the second degree polynomial $g^{\text{cushion}}(t)$ of m_t is $-g^*(t) < 0$ for all t. Hence, the growth rate of the Cushion is a concave function of m_t for all t. Thus,

$m^\star_{\text{cushion}}(t)$ maximizes the Cushion growth rate, $g_m^{\text{cushion}}(t)$ for all $t \in [0, T]$, and the process $m^\star_{\text{cushion}} = \{m^\star_{\text{cushion}}(t), t \in [0, T]\}$ maximizes $\int_0^T g_m^{\text{cushion}}(t)dt$ for all $T \in [0, \infty)$. □

Corollary 3. *For all $\boldsymbol{PI} \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, with value process V_m^{PI}*

$$m^\star = \underset{m}{\text{argmax}} \lim_{T\to\infty} \frac{1}{T} \log V_m^{PI}(T) = \underset{m}{\text{argmax}} \lim_{T\to\infty} \frac{1}{T} \int_0^T g_{PI}(t)dt \quad a.s.$$

Furthermore, for the corresponding processes $\boldsymbol{C}_m^{\boldsymbol{PI}}$ of all $\boldsymbol{PI} \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$,

$$m^\star_{\text{cushion}} = \underset{m}{\text{argmax}} \lim_{T\to\infty} \frac{1}{T} \log C_m^{PI}(T) = \underset{m}{\text{argmax}} \lim_{T\to\infty} \frac{1}{T} \int_0^T g_m^{\text{cushion}}(t)dt. \quad a.s.$$

Proof. Proposition 1.3.1 of Fernholz (2002) states that, for any nonnegative portfolio π,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log S_\pi(T) - \int_0^T g_\pi(t)dt\right) = 0, \quad \text{a.s.}$$

from the additivity property of the limit it follows that

$$\lim_{T\to\infty} \frac{1}{T} \log S_\pi(T) = \lim_{T\to\infty} \frac{1}{T} \int_0^T g_\pi(t)dt, \quad \text{a.s.} \tag{A5}$$

which also holds for any $\boldsymbol{PI} \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$ and the corresponding $\boldsymbol{C}_m^{\boldsymbol{PI}}$ processes (which is also a portfolio process from Corollary 1). The Corollary follows. □

Lemma 1. For all $\boldsymbol{PI}_m \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, with value process $V_m^{PI}(t)$ and for any $t \in [0, T]$

$$m_t^{\star\star} := \underset{m_t}{\text{argmax}} \frac{1}{t} \log V_m^{PI}(t) = \underset{m_t}{\text{argmax}} \frac{1}{t} \log C_m^{PI}(t) \quad a.s. \tag{A6}$$

Proof. The first-order condition of the optimization problem (Equation (A6)) at any time t is:

$$\frac{\partial}{\partial m_t}\left\{\frac{1}{t}\log V_m^{PI}(t)\right\}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\frac{\partial}{\partial m_t}\left\{\frac{1}{t}\log\left(F(t) + C_m^{PI}(t)\right)\right\}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\Leftrightarrow \frac{\frac{\partial}{\partial m_t}\{F(t) + C_m^{PI}(t)\}}{F(t) + C_m^{PI}(t)}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\Leftrightarrow \frac{\frac{\partial C_m^{PI}(t)}{\partial m_t}}{F(t) + C_m^{PI}(t)}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\Leftrightarrow \frac{\partial C_m^{PI}(t)}{\partial m_t}\bigg|_{m_t=m_t^{\star\star}} = 0.$$

Similarly we have:

$$\frac{\partial}{\partial m_t}\left\{\frac{1}{t}\log C_m^{PI}(t)\right\}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\Leftrightarrow \frac{\frac{\partial}{\partial m_t}\{C_m^{PI}(t)\}}{C_m^{PI}(t)}\bigg|_{m_t=m_t^{\star\star}} = 0$$

$$\Leftrightarrow \frac{\partial C_m^{PI}(t)}{\partial m_t}\bigg|_{m_t=m_t^{\star\star}} = 0.$$

Since this holds for all $t \in [0, T]$ the result follows. □

Corollary 4. *The multiplier process $m = \{m_t, t \in [0, T]\}$ that maximizes $\frac{1}{T}\log C_m^{PI}(T)$ is the same as the series of multipliers m_t that result from maximizing the "myopic" single-period problem of maximizing $\frac{1}{t}\log C_m^{PI}(t)$ for every $t \in [0, T]$.*

Proof. Let $\{\tilde{m}_t\}_{t\in[0,T]}$ be the multiplier process maximizing $\frac{1}{T}\log C_m^{PI}(T)$, thus:

$$\frac{\partial}{\partial m}\frac{1}{T}\log C_m^{PI}(T)\bigg|_{m=\tilde{m}} = 0.$$

Replacing the value of C_m^{PI} as in Equation (13) yields for all $t \in [0, T]$

$$\frac{\partial}{\partial m_t}\left\{\frac{1}{T}\int_0^T g_m^{\text{cushion}}(s)ds + \frac{1}{T}\int_0^T [m_s\sigma_S(s)dW^S(s) + (1-m_s)\sigma_R(s)dW^R(s)]\right\}\Bigg|_{m_t=\tilde{m}_t} = 0. \quad \text{(A7)}$$

The only term for which the derivatives of these two integrals are not zero is when $s = t$ for every $t \in [0, T]$, thus the result follows. □

Identity 1. *For all* $\boldsymbol{PI} \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, *with value process* $V_m^{PI}(t)$ *and for all* $t \in [0, T]$

$$m^\star = \underset{m}{\text{argmax}} \frac{1}{T}\log V_m^{PI}(T) = \underset{m}{\text{argmax}} \frac{1}{T}\log C_m^{PI}(T)$$

for $T \in [0, \infty)$ *almost surely.*

Proof. Notice that Lemma 1 holds for any $t \in [0, T]$, including $t = T \to \infty$. This fact together with Corollary 4 and Corollary 3 yields the identity. □

Corollary 5. *For any* $\boldsymbol{PI} \in \mathcal{M}_{(\boldsymbol{S},\boldsymbol{R},\boldsymbol{F})}$, *its time-average growth rate process* g_{PI}, *has a lower bound given by*

$$\frac{1}{T}w\int_0^T g_R(s)ds + \frac{1}{T}(1-w)\int_0^T g_m^{\text{cushion}}(s)ds \leq \frac{1}{T}\int_0^T g_{PI}(s)ds \quad a.s.$$

when $T \to \infty$, *where* $0 \leq w := \frac{F_0}{V_0} \leq 1$.

Proof. By Definition, $V_T = C_T + F_T$, using the fact that $dF_t = dR_t$ for all $t \in [0, T]$, and Equations (13), (12),

$$V_m^{PI}(T) = F_0 e^{\int_0^T g_R(t)dt + \int_0^T \sigma_R(t)dW^R(t)} + C_0 e^{\int_0^T g_m^{\text{cushion}}(t)dt + \int_0^T m_t\sigma_S(t)dW^S(t) + (1-m_t)\sigma_R(t)dW^R(t)}, \quad \text{(A8)}$$

for all $t \in [0, T]$. Dividing by V_0 on both sides of Equation (A8) we get,

$$\frac{V_m^{PI}(T)}{V_0} = we^{\int_0^T g_R(t)dt + \int_0^T \sigma_R(t)dW^R(t)} + (1-w)e^{\int_0^T g_m^{\text{cushion}}(t)dt + \int_0^T m_t\sigma_S(t)dW^S(t) + (1-m_t)\sigma_R(t)dW^R(t)}, \quad \text{(A9)}$$

where $0 \leq w := \frac{F_0}{V_0} \leq 1$. On the other hand, from Equation (A5)

$$\lim_{T\to\infty}\frac{1}{T}\log V_m^{PI}(T) = \lim_{T\to\infty}\frac{1}{T}\int_0^T g_{PI}(t)dt, \quad \text{a.s.} \quad \text{(A10)}$$

Thus, normalizing $V_0 = 1$, applying the log, multiplying by $\frac{1}{T}$ and taking the limit for $T \to \infty$ on both sides of Equation (A9), from Jensen's inequality and result (Equation (A10)), the Corollary follows. □

Notes

[1] Martellini and Milhau (2009) study an asset allocation strategy in a model set-up including interest rate and inflation risk, where the objective is maximizing expected utility from the terminal *funding ratio* of a

pension fund (ratio of assets to liabilities) in the presence of a minimum funding ratio constraint (embedded in a piece-wise utility function), similar in spirit to the constraint considered here. The resulting strategy is not exactly structured as the CPPI strategy (due for instance to the extra cash position).

[2] The CPPI strategy was initially conceived with a constant multiplier parameter. Indeed, in a Black–Scholes model where asset dynamics have constant parameters, the optimal and maximum multipliers are constant over time as well.

[3] As Brandl *et al.* (2008) point out, in the option-based approach the investment manager needs to either buy or replicate put options to insure the portfolio. While options written on the single assets in the portfolio might be available on the market, usually an appropriate option on the whole portfolio will not be. Also, insuring the portfolio with options on all single assets is likely to be too expensive.

[4] Utility-maximizing OBPI strategies include Grossman and Vila (1989), Teplá (2001), El Karoui *et al.* (2005), and Deguest *et al.* (2012).

[5] The exposure can be expressed in terms of proportion of wealth as $e_t = m \times (1 - \frac{F(t)}{V(t)})$.

[6] In general, the risky asset may be any portfolio of tradable securities.

[7] In general, the Floor (see Black and Perold, 1992) is given by $F_t = kR_t$ for $0 \leq k \leq 1$, which limits the underperformance of the portfolio with respect to any given tradable stochastic Benchmark R, measured since initial date $t = 0$, i.e., $r_{0,t}^V - r_{0,t}^R \geq log(k)$, for all t (using log returns). Equation (3) is the special case in which the reserve asset is a zero-coupon bond.

[8] By setting $k = B(0, T)$ in Equation (3) one recovers the initial capital guarantee Floor (Equation (2)).

[9] Sørensen (1999), Brennan and Xia (2002), Wachter (2002), Munk *et al.* (2004) show that solving for the optimal *unconstrained* asset allocation model of Merton (1969, 1971, 1972), using a power utility function, yields a strategy that can explain the Samuelson and Canner *et al.* (1997) asset allocation puzzles, conditional to parameters satisfy some weak conditions.

[10] Inflation and bond yields might be related, since when inflation is rising, central banks often raise interest rates to fight inflation. However, the relationship is not necessarily one-to-one.

[11] The weight of the Cushion's growth rate is proportional to the initial risk budget.

[12] Wachter (2002) points out that "Empirical studies have found this correlation to be close to -1. Moreover, even perfectly correlated continuous-time processes are imperfectly correlated when measured in discrete time".

[13] In scenarios in which the stock index did not attain again its value at the beginning of the MDD period, we calculated the outperformance achieved between the beginning of the period and the latest simulated value of the corresponding scenario.

[14] Bertrand and Prigent (2002), Cont and Tankov (2009), and Hamidi *et al.* (2008) estimate the maximum multiplier in a constant interest rate model.

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Keywords: Portfolio insurance; asset allocation; risk management