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OIS DISCOUNTING, INTEREST RATE DERIVATIVES, AND THE MODELING OF STOCHASTIC INTEREST RATE SPREADS

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Before 2007, derivatives practitioners used a zero curve that was bootstrapped from LIBOR swap rates to provide "risk-free" rates when pricing derivatives. In the last few years, when pricing fully collateralized transactions, practitioners have switched to using a zero curve bootstrapped from overnight indexed swap (OIS) rates for discounting. This paper explains the calculations underlying the use of OIS rates and investigates the impact of the switch on the pricing of plain vanilla caps and swap options. It also explores how more complex derivatives providing payoffs dependent on LIBOR, or any other reference rate, can be valued. It presents new results showing that they can be handled by constructing a single tree for the evolution of the OIS rate.

1 Introduction

Before 2007, derivatives dealers used LIBOR, the short-term borrowing rate of AA-rated financial institutions, as a proxy for the risk-free rate. The zero-coupon yield curve was boot-strapped from LIBOR swap rates. One of the attractions of this was that many interest rate derivatives use the LIBOR rate as the reference interest rate, so these instruments could be valued using a single zero curve.

The use of LIBOR as the risk-free rate was called into question by the credit crisis that started in mid-2007. Banks became increasingly reluctant to lend to each other because of credit concerns. As a result, LIBOR quotes started to rise relative to other rates that involved very little credit risk. The TED spread, which is the spread between three-month U.S. dollar LIBOR and the three-month U.S. Treasury rate, is less than 50 basis points in normal market conditions. Between October 2007 and May 2009, it was rarely lower than 100 basis points and peaked at over 450 basis points in October 2008.

These developments led the market to look for an alternative proxy for the risk-free rate.[1] The

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standard practice in the market now is to determine discount rates from overnight indexed swap (OIS) rates when valuing all fully collateralized derivatives transactions.[2] Both LIBOR and OIS rates are based on interbank borrowing. However, the LIBOR zero curve is based on borrowing rates for periods of one or more months, whereas the OIS zero curve is based on overnight borrowing rates. As discussed by Hull and White (2013a), the credit spread for overnight interbank borrowing is less than that for longer term interbank borrowing. As a result the credit spread that gets impounded in OIS swap rates is smaller than that in LIBOR swap rates.

LIBOR incorporates a credit spread that reflects the possibility that the borrowing bank may default, but this does not create credit risk in a LIBOR swap. The reason for this is that the swap participants are not lending to the banks that are providing the quotes from which LIBOR is determined. Therefore, they are not exposed to a loss from default by one of these banks. LIBOR is merely an index that determines the size of the payments in the swap. The credit risk, if any, in the swap is related to the possibility that the counterparty may fail to make swap payments when due. The LIBOR swaps traded in the interdealer market are now cleared through central counterparties, which require both initial margin and variation margin. The counterparty credit risk in the swaps that are traded today can therefore reasonably be assumed to be zero.

Changing the risk-free discount curve changes the values of all derivatives. In the case of derivatives other than interest rate derivatives, implementing the change is usually straightforward. This paper focuses on how the switch from LIBOR to OIS discounting affects the pricing of interest rate derivatives. Other papers such as Smith (2013) have examined the nature of the calculations underlying the use of OIS discounting and the pricing of interest rate swaps. We go one step further by quantifying the impact of OIS discounting on several different interest rate derivatives in different situations

It might be thought that the switch from LIBOR to OIS discounting simply results in a change to the discount rate while expected payoffs from an interest rate derivative remain unchanged. This is not the case. Forward LIBOR rates and forward swap rates also change. One of the contributions of this paper is to examine the relative importance of discount rate changes and forward-rate changes to the valuation of interest rate derivatives in different circumstances. We first discuss how the LIBOR zero curve is bootstrapped when LIBOR discounting is used and when OIS discounting is used. This allows us to show how the transition from LIBOR discounting to OIS discounting affects the forward rates and the valuation of LIBOR swaps. We then move on to consider how the standard market models used to price interest rate caps and swap options are affected.[3]

This paper then considers how nonstandard instruments which provide payoffs dependent on LIBOR can be valued. It presents new results showing that it is possible to accommodate OIS discounting with a single interest rate tree describing the OIS zero curve and its evolution. It examines the impact of correlation between the OIS rate and the LIBOR–OIS spread on the pricing of Bermudan swap options.

2 Background

In this section, we review the procedures for bootstrapping a riskless zero curve from LIBOR swap rates. We start by examining how bonds and swaps are priced. This will help to introduce our notation and provide the basis for our later discussion of the impact of OIS discounting.

2.1 Interest rates and discount bond prices

Let $z(T)$ be the continuously compounded risk-free zero-coupon interest rate observed today for maturity T. The price of a risk-free discount bond that pays \$1 at time T is $P(T) = \exp[-z(T)T]$. A common industry practice is for a money market yield to be used for discount bonds with a maturity of one year or less. This means that[4]

$$P(T) = \frac{1}{1 + R(T)T} \qquad (1)$$

where $R(T)$ is the money market yield for maturity T so that

$$R(T) = \frac{1 - P(T)}{P(T)T} \qquad (2)$$

Consider a forward contract in which we agree to buy or sell at time T_i a discount bond maturing at time T_{i+1}. Simple arbitrage arguments show that the forward price for this contract, the contract delivery price at which the forward contract has zero value, is $P(T_{i+1})/P(T_i)$.

If $T_{i+1} - T_i$ is less than or equal to one year we can also define a money market yield to maturity on the forward bond. This is the forward interest rate. It is the rate of interest that must apply between times T_i and T_{i+1} in order for the price at T_i of a discount bond maturing at T_{i+1} to be equal to the forward price. The forward rate is

$$F(T_i, T_{i+1}) = \frac{P(T_i) - P(T_{i+1})}{(T_{i+1} - T_i)P(T_{i+1})} \qquad (3)$$

2.2 Interest-rate swaps

Consider one leg of an interest rate swap in which a floating rate of interest is exchanged for a specified fixed rate of interest, K. The start date is T_i and the end date is T_{i+1}. On the start date we observe the rate that applies between T_i and T_{i+1}. For a swap where LIBOR is received and the fixed rate is paid, there is a payment on the end date equal to $(R_i - K)L(T_{i+1} - T_i)$ where R_i is the LIBOR interest rate for the period between T_i and T_{i+1}.

The key to valuing one leg of an interest rate swap is the result that, when the numeraire asset is the risk-free discount bond maturing at time T_{i+1}, the expected future value of any interest rate (not necessarily a risk-free interest rate) between T_i and T_{i+1} equals the current forward interest rate (i.e. the interest rate that would apply in a forward rate agreement). This means that in a world where interest rates are stochastic, we can use $P(T_{i+1})$ as the discount factor, providing we also assume that the expected value of R_i equals the forward interest rate, $F(T_i, T_{i+1})$. The value, $S_i(K)$, of the leg of the swap that we are considering is therefore

$$S_i(K) = L[F(T_i, T_{i+1}) - K] \times (T_{i+1} - T_i)P(T_{i+1})$$

where L is the notional principal.

A standard interest-rate swap is constructed of many of these legs in which the end date for one leg is the start date for the next leg.[5] We define T_0 as the start date of the swap and T_i as the ith payment date $(1 \le i \le M)$. The total swap value is the sum of the values for all the individual legs. If the start date for the first leg of the swap is time zero $(T_0 = 0)$, the swap is a spot start swap. If the start date for the first leg of the swap is in the future $(T_0 > 0)$, the swap is a forward start swap. The total value of the M legs of the swap is

$$S(K, M) = L\sum_{i=0}^{M-1}[F(T_i, T_{i+1}) - K] \times (T_{i+1} - T_i)P(T_{i+1}) \qquad (4)$$

Using Equation (3) to replace the forward rates[6] this is

$$S(K, M) = L\sum_{i=0}^{M-1}[P(T_i) - P(T_{i+1})] - LKA \qquad (5)$$

where

$$A = \sum_{i=0}^{M-1} P(T_{i+1})(T_{i+1} - T_i)$$

is the annuity factor used to determine the present value of the fixed payments on the swap.

The breakeven swap rate for a particular swap is the value of K such that the value of the swap is zero. Using Equations (4) and (5) this is

$$\frac{\sum_{i=0}^{M-1} [P(T_i) - P(T_{i+1})]}{A} \quad \text{or} \quad \frac{\sum_{i=0}^{M-1} F(T_i, T_{i+1})(T_{i+1} - T_i)P(T_{i+1})}{A} \tag{6}$$

For a forward start swap, the breakeven swap rate is called the forward swap rate. For a spot start swap, the breakeven swap rate is simply known as the swap rate. We denote the swap rate for a swap whose final payment date is T_M as K_M.

2.3 *Notation*

In what follows we will use the subscript LD to denote quantities calculated using LIBOR discounting and the subscript OD to denote quantities calculated using OIS discounting. For example, $P_{\mathrm{LD}}(T)$ and $P_{\mathrm{OD}}(T)$ are the prices of risk-free discount bonds providing a payoff of \$1 at time T when LIBOR discounting and OIS discounting are used, respectively. Also, $F_{\mathrm{LD}}(T_i, T_{i+1})$ and $F_{\mathrm{OD}}(T_i, T_{i+1})$ are forward LIBOR rates between times T_i and T_{i+1} when LIBOR discounting and OIS discounting are used, respectively.

2.4 *Bootstrapping LIBOR with LIBOR discounting*

When LIBOR is assumed to define riskless rates, swap rates can be used to bootstrap the LIBOR zero curve. From Equation (5), when LIBOR discounting is used, the value of a swap that starts at T_0 and ends at T_M is

$$S_{\mathrm{LD}}(K, M) = L \sum_{i=0}^{M-1} [P_{\mathrm{LD}}(T_i) - P_{\mathrm{LD}}(T_{i+1})] - LKA_{\mathrm{LD}} \tag{7}$$

where

$$A_{\mathrm{LD}} = \sum_{i=0}^{M-1} P_{\mathrm{LD}}(T_{i+1})(T_{i+1} - T_i) \tag{8}$$

The swap value is zero when $K = K_M$. If $P_{\mathrm{LD}}(T_i)$ is known for all i from 0 to $M - 1$ it follows that when the swaps considered start at time zero

$$P_{\mathrm{LD}}(T_M) = \frac{P_{\mathrm{LD}}(T_{M-1}) + S_{\mathrm{LD}}(K_M, M-1)/L}{1 + K_M(T_M - T_{M-1})} \tag{9}$$

When $T_0 = 0$ Equation (9) allows the discount bond prices $P_{\mathrm{LD}}(T_1), P_{\mathrm{LD}}(T_2), \ldots$ to be determined inductively. The discount bond prices can then be turned into zero rates. The zero rate for any date that is not a swap payment date is determined by interpolating between adjacent known zero rates. Forward rates can be determined from Equation (3).

2.5 *Bootstrapping LIBOR with OIS discounting*

If OIS swap rates are assumed to be riskless, the riskless zero curve is bootstrapped from OIS swap rates. The procedure is similar to that just given for bootstrapping LIBOR zero rates where LIBOR rates are assumed to be riskless. One point to note is that OIS swaps of up to one-year's maturity have only a single leg resulting in a single payment at maturity.

If the zero curve is required for maturities longer than the maturity of the longest OIS swap, a natural approach is to assume that the spread between the OIS swap rates and the LIBOR swap rates is the same for all maturities after the longest OIS maturity for which there is reliable data. An alternative approach for extending the OIS zero curve is to use basis swaps where three-month LIBOR is exchanged for the average federal funds rate plus a spread. These swaps have maturities as long as 30 years in the U.S.[7]

Under OIS discounting, in order to determine the value of swaps and other derivatives whose payoffs are based on LIBOR it is necessary to determine the expected future LIBOR rate for the period between T_i and T_{i+1} when the numeraire asset is a zero-coupon OIS bond maturing at time T_{i+1}. These are the forward LIBOR rates, $F_{\mathrm{OD}}(T_i, T_{i+1})$ (i.e., the mid market rates that would apply in forward rate agreements when the market uses OIS discounting).

The bootstrapping process to determine the F_{OD}'s is straightforward. The value of a swap in which floating is received and a fixed rate of K is paid, with start date T_0 and payment dates $T_1, T_2, \ldots, T_M$ is

$$S_{\mathrm{OD}}(K, M) = L \sum_{i=0}^{M-1} [F_{\mathrm{OD}}(T_i, T_{i+1}) - K] \times (T_{i+1} - T_i) P_{\mathrm{OD}}(T_{i+1}) \quad (10)$$

This is zero when $T_0 = 0$ and $K = K_M$. It follows that when the swaps considered start at time zero

$$F_{\mathrm{OD}}(T_{M-1}, T_M) = K_M - \frac{S_{\mathrm{OD}}(K_M, M-1)/L}{(T_M - T_{M-1}) P_{\mathrm{OD}}(T_M)} \quad (11)$$

Since $S_{\mathrm{OD}}(K_1, 0) = 0$ the forward rates are determined sequentially starting with $M = 1$. Once all forward LIBOR rates are determined, the LIBOR discount bond prices for maturity T_j can be calculated as

$$\prod_{i=0}^{j-1} \{[1 + F_{\mathrm{OD}}(T_i, T_{i+1})](T_{i+1} - T_i)\}^{-1}$$

The zero rates can be determined from the discount bond prices. The zero rates so calculated are just a convenient tool for calculating the expected values of future LIBOR when OIS discounting is done.

Equation (6) shows that the forward swap rate for a swap with start date T_0 and payment dates $T_1, T_2, \ldots, T_M$ is

$$\frac{\sum_{i=0}^{M-1} F_{\mathrm{OD}}(T_i, T_{i+1})(T_{i+1} - T_i) P_{\mathrm{OD}}(t, T_{i+1})}{A_{\mathrm{OD}}} \quad (12)$$

where

$$A_{\mathrm{OD}} = \sum_{i=0}^{M-1} P_{\mathrm{OD}}(T_{i+1})(T_{i+1} - T_i) \quad (13)$$

is the annuity factor used to determine the present value of the fixed payments on the swap.

3 OIS discounting and LIBOR rates

To explore how LIBOR zero rates change when OIS discounting is used, we consider the case in which the zero curve is bootstrapped from swap market data for semi-annual pay swaps with maturities from six months to 30 years using the procedures outlined in the previous section. Three different sets of LIBOR swap rates are considered:

(1) 4 to 6: The swap rate is 4% + 2% × Swap Life/30.
(2) 5 Flat: The swap rate is 5% for all maturities.
(3) 6 to 4: The swap rate is 6% − 2% × Swap Life/30.

We assume that the OIS swap rates equal the LIBOR swap rates less 100 basis points. (We tried other spreads between the OIS and LIBOR swap rates and found that results are roughly proportional to the spread.)

Table 1 shows how much the LIBOR zero curve is shifted when we switch from LIBOR discounting to OIS discounting for the three term structures

Table 1 The basis point increase in LIBOR zeros caused by switching from LIBOR discounting to OIS discounting.

LIBOR rate maturity (Yrs)	Term structure 4 to 6	Term structure flat	Term structure 6 to 4
0.5	0.0	0.0	0.0
1	0.0	0.0	0.0
3	–0.1	0.0	0.1
5	–0.3	0.0	0.3
7	–0.6	0.0	0.6
10	–1.3	0.0	1.3
15	–3.5	0.0	3.2
20	–7.6	0.0	6.1
25	–15.7	0.0	9.7
30	–33.5	0.0	13.8

Table 2 The basis point increase in three-month LIBOR forward rates caused by switching from LIBOR discounting to OIS discounting.

Forward rate maturity (Yrs)	Term structure 4 to 6	Term structure 5 flat	Term structure 6 to 4
0.5	0.0	0.0	0.0
1	0.0	0.0	0.0
3	–0.4	0.0	0.4
5	–1.0	0.0	1.0
7	–2.0	0.0	2.1
10	–4.6	0.0	4.4
15	–13.0	0.0	10.7
20	–31.7	0.0	19.5
25	–76.8	0.0	29.7
30	–217.3	0.0	39.9

of swap rates. In an upward sloping term structure, changing to OIS discounting lowers the value of the calculated LIBOR zero rates. In a downward sloping term structure, the reverse is true. For a flat term structure the discount rate used has no effect on the zero rates calculated from swap rates. This last point can be seen by inspecting Equation (4) which determines the swap value. In a flat term structure, all the forward rates are the same so that we obtain a zero swap value if K equals the common forward rate. This is true regardless of the level of interest rates.

For short maturities the determination of the LIBOR zero curve is not very sensitive to the discount rate used. However, for maturities beyond 10 years, the impact of the switch from LIBOR discounting to OIS discounting can be quite large. The impact on forward rates is even larger. Table 2 shows three-month LIBOR forward rates calculated using OIS discounting minus the same three-month forward rate calculated using OIS discounting.

4 OIS discounting and swap pricing

Changing from LIBOR discounting to OIS discounting changes the values of LIBOR swaps. In the case of spot start swaps the value change can be expressed directly in terms of changes in the discount factors. First, consider the swap value under LIBOR discounting. Since the value of the floating side of an at-the-money spot start swap equals the value of the fixed side, the value of a spot start swap in which fixed is paid can be written as

$$S_{\mathrm{LD}}(K, M) = LA_{\mathrm{LD}}(K_M - K)$$

Similarly, when OIS discounting, is used the value becomes

$$S_{\mathrm{OD}}(K, M) = LA_{\mathrm{OD}}(K_M - K)$$

Note that, because the calibration process ensures that at-the-money spot start swaps are correctly priced, K_M is the same in both equations. As a result the price difference is

$$\begin{aligned} &S_{\mathrm{OD}}(K, M) - S_{\mathrm{LD}}(K, M) \\ &\quad = L(K_M - K)(A_{\mathrm{OD}} - A_{\mathrm{LD}}) \end{aligned}$$

Because OIS discount rates are lower than LIBOR discount rates the last term in this expression is positive so that, if the fixed rate being paid is below the market rate, the price change when OIS discounting is used is positive. If the fixed rate being paid is above the market rate, the price change is negative. For swaps where fixed is received, the reverse is true.

In general, this sort of simple decomposition of the value change is not possible. Two factors contribute to the value change: changes in the forward rates and changes in the discount factors used in the swap valuation. For all the interest rate derivatives we will consider in the rest of this paper, we define

V_{LD}: Value of derivative assuming LIBOR discounting

V_{OD}: Value of derivative assuming OIS discounting

V_{LO}: Value of derivative when forward rates are based on LIBOR discounting and the expected cash flows calculated from the forward rates are then discounted at OIS zero rates[8]

Calculating these three values allows us to separate the impact of switching from LIBOR discounting to OIS discounting into

(1) a pure discounting effect, $V_{\mathrm{LO}} - V_{\mathrm{LD}}$; and
(2) a forward-rate effect, $V_{\mathrm{OD}} - V_{\mathrm{LO}}$

As pointed out by Bianchetti (2010), a common mistake made by practitioners is that they calculate V_{LO} when they should be calculating V_{OD}. The forward-rate effect measures the error that this mistake gives rise to.

There are two special cases in which the discounting effect and the forward-rate effect can be easily determined. First, for at-the-money LIBOR spot start swaps (swap rate equals market swap rate), the calibration process used to generate the zero curves ensures that $V_{\mathrm{LD}} = V_{\mathrm{OD}}$, so that the two effects are equal in magnitude and opposite in sign. Second, if the term structure is flat (swaps of all maturities have the same market rate) then the forward rates based on both the LIBOR discounting and the OIS discounting are the same. In this case, $V_{\mathrm{OD}} - V_{\mathrm{LO}} = 0$ and all of the value change is a pure discounting effect. In all other cases both shifts in forward rates and changes in discount factors play a role in the value change that occurs when we switch from LIBOR to OIS discounting.

Table 3 shows the value differences for pay-fixed spot start swaps with a fixed rate that is 1% higher than the market rate for the same maturity. The discounting effect, $V_{\mathrm{LO}} - V_{\mathrm{LD}}$, makes the magnitude of the present value of every payment larger (farther from zero). When the term structure is not flat some of the expected payments are positive and some negative so the overall effect of changing the discount rates may be positive or negative. For all three term structures the overall effect of a switch to OIS discounting is over 35 basis points for 10-year swaps, over 100 basis points for 20 year swaps, and about 200 basis points for 30 year swaps. The table shows that for non-flat term structures, the forward-rate effect can be quite large and calculating forward rates using the pre-crisis approach can therefore lead to large errors. Indeed, for the upward sloping term structure we consider, the forward rate effect is bigger than the pure discounting effect when the swap lasts for 20 or 30 years.

Table 4 shows the value differences for forward start swaps where the swap life is 10 years. The fixed rate is paid and LIBOR is received. The fixed rate is 1% higher than the corresponding forward swap rate based on LIBOR discounting. Again, for upward and downward sloping term structures and long maturities, the impact of using

Table 3 The increase in spot-start swap value caused by switching from LIBOR discounting to OIS discounting.

	Term structure 4 to 6			Term structure 5 flat			Term structure 6 to 4		
Swap life (Yrs)	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$
1	−0.7	0.0	−0.7	−0.7	0.0	−0.7	−0.7	0.0	−0.7
3	−4.5	−0.3	−4.8	−4.7	0.0	−4.7	−4.8	0.3	−4.6
5	−10.6	−1.3	−11.9	−11.5	0.0	−11.5	−12.4	1.3	−11.2
7	−17.9	−3.6	−21.5	−20.8	0.0	−20.8	−23.5	3.4	−20.1
10	−29.3	−10.3	−39.6	−38.1	0.0	−38.1	−46.3	9.7	−36.7
20	−32.9	−81.7	−114.5	−112.6	0.0	−112.6	−180.4	69.6	−110.8
30	117.7	−299.7	−182.0	−192.6	0.0	−192.6	−411.1	207.1	−204.0

The swaps pay a fixed rate equal to the market swap rate plus 1%. Price differences are measured in basis points.

Table 4 The increase in forward-start swap value, where the swap life is 10 years, caused by switching from LIBOR discounting to OIS discounting.

	Term structure 4 to 6			Term structure 5 flat			Term structure 6 to 4		
Start time T_0 (Yrs)	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$
1	−35.4	−13.7	−49.1	−43.9	0.0	−43.9	−51.7	12.7	−39.0
3	−45.7	−22.2	−67.9	−53.9	0.0	−53.9	−60.9	20.4	−40.5
5	−53.6	−33.2	−86.7	−61.8	0.0	−61.8	−68.3	29.8	−38.5
7	−59.2	−46.5	−105.7	−68.0	0.0	−68.0	−74.4	40.8	−33.6
10	−63.5	−71.3	−134.8	−74.5	0.0	−74.5	−81.6	59.9	−21.7
20	−44.0	−218.1	−262.1	−80.0	0.0	−80.0	−98.0	137.5	39.5

The swaps pay a fixed rate equal to the forward swap rate based on LIBOR discounting plus 1%. Price differences are measured in basis points.

the wrong forward rates in conjunction with OIS discounting can lead to large errors.

5 OIS discounting and swap option pricing

An interest rate swap option is an option that gives the option holder the right to enter into a swap with a specified fixed rate, K, at some future date T_0 ($T_0 > 0$). The swap lasts from T_0 to T_M. In the case of LIBOR discounting, for a swap where LIBOR will be received and fixed paid, the standard market model gives[9]

$$V_{\mathrm{LD}} = A_{\mathrm{LD}} L[K_{\mathrm{LD}} N(d_{1,\mathrm{LD}}) - KN(d_{2,\mathrm{LD}})]$$

where A_{LD} is given by Equation (8) and K_{LD} is the current forward swap rate for the swap underlying the option (given by Equation (6)) when LIBOR discounting is used,

$$d_{1,\mathrm{LD}} = \frac{\ln[K_{\mathrm{LD}}/K]}{\sigma_{\mathrm{LD}}\sqrt{T_0}} + 0.5\sigma_{\mathrm{LD}}\sqrt{T_0}$$

$$d_{2,\mathrm{LD}} = \frac{\ln[K_{\mathrm{LD}}/K]}{\sigma_{\mathrm{LD}}\sqrt{T_0}} - 0.5\sigma_{\mathrm{LD}}\sqrt{T_0}$$

where σ_{LD} is the volatility of the forward swap rate and N is the cumulative normal distribution function. In practice, this model is used to imply volatilities and calculate volatility surfaces from option prices.

When LIBOR will be paid and fixed received the valuation becomes

$$V_{\mathrm{LD}} = A_{\mathrm{LD}} L[KN(-d_{2,\mathrm{LD}}) - K_{\mathrm{LD}} N(-d_{1,\mathrm{LD}})]$$

In the case of OIS discounting the standard market model becomes

$$V_{\mathrm{OD}} = A_{\mathrm{OD}} L[K_{\mathrm{OD}} N(d_{1,\mathrm{OD}}) - KN(d_{2,\mathrm{OD}})]$$

where A_{OD} is given by Equation (13) and K_{OD} is the current forward swap rate for the swap underlying the option given by Equation (6) when OIS discounting is used,

$$d_{1,\mathrm{OD}} = \frac{\ln[K_{\mathrm{OD}}/K]}{\sigma_{\mathrm{OD}}\sqrt{T_0}} + 0.5\sigma_{\mathrm{OD}}\sqrt{T_0}$$

$$d_{2,\mathrm{OD}} = \frac{\ln[K_{\mathrm{OD}}/K]}{\sigma_{\mathrm{OD}}\sqrt{T_0}} - 0.5\sigma_{\mathrm{OD}}\sqrt{T_0}$$

where σ_{OD} is the volatility of the forward swap rate.[10] When LIBOR will be paid and fixed received, the OIS-discounting valuation becomes

$$V_{\mathrm{OD}} = A_{\mathrm{OD}} L[KN(-d_{2,\mathrm{OD}}) - K_{\mathrm{OD}} N(-d_{1,\mathrm{OD}})]$$

This version of the standard market model, like the original version given above, can be used to imply volatilities and calculate volatility surfaces from option prices.

To examine the effect of the switch to OIS discounting on the prices of swap options we consider options with lives between 1 and 20 years on a 5-year swap when the volatility for all maturities is set equal to 20% for both LIBOR and OIS discounting. In every case the option strike price, K, is set equal to K_{LD}. For the LIBOR discounting case, the options are exactly at-the-money while in the OIS discounting case the options are approximately at-the-money.

Table 5 shows the value differences for options to enter into swaps in which floating is paid for the three-term structures of swap rates. Table 6 shows similar results for options to enter into swaps in which fixed is paid. The effect of changing the discount factors is about the same for both pay fixed and pay floating swaps. However, changing forward rates has a much more pronounced effect on the value of options on swaps in which the option holder pays the fixed rate. This is because pay-fixed (pay-floating) swap options provide payoffs when swap rates are high (low). The forward swap rate is assumed to follow geometric Brownian motion and so a change in the starting forward swap rate has a greater effect on high swap rates than on low swap rates.

Table 5 The increase in the price of an option to enter into a 5-year pay-floating swap caused by switching from LIBOR discounting to OIS discounting.

	Term structure 4 to 6			Term structure 5 flat			Term structure 6 to 4		
Option life, T_0 (Yrs)	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$
1	5.0	1.0	6.0	6.0	0.0	6.0	7.0	–1.0	6.0
3	14.1	2.2	16.3	14.8	0.0	14.8	15.3	–2.1	13.3
5	24.5	3.7	28.2	23.5	0.0	23.5	22.6	–3.4	19.1
7	35.7	5.6	41.3	32.0	0.0	32.0	28.5	–5.1	23.4
10	53.4	9.2	62.6	43.7	0.0	43.7	34.7	–7.9	26.8
20	111.0	27.4	138.4	69.8	0.0	69.8	34.1	–17.9	16.2

The swaps underlying the option pay the option holder a fixed rate equal to the forward swap rate based on LIBOR discounting. Price differences are measured in basis points.

Table 6 The increase in the price of an option to enter into a 5-year pay-fixed swap caused by switching from LIBOR discounting to OIS discounting.

	Term structure 4 to 6			Term structure 5 flat			Term structure 6 to 4		
Option life, T_0 (Yrs)	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$	Discount effect $V_{LO} - V_{LD}$	Fwd. rate effect $V_{OD} - V_{LO}$	Total $V_{OD} - V_{LD}$
1	6.3	–1.2	5.1	6.0	0.0	6.0	5.8	1.2	6.9
3	15.4	–2.9	12.5	14.8	0.0	14.8	14.1	2.7	16.8
5	25.7	–5.3	20.5	23.5	0.0	23.5	21.4	5.0	26.4
7	37.0	–8.5	28.5	32.0	0.0	32.0	27.4	7.9	35.3
10	54.8	–15.0	39.8	43.7	0.0	43.7	33.7	13.4	47.2
20	113.2	–53.7	59.4	69.8	0.0	69.8	33.4	40.4	73.9

The swaps underlying the option require the option holder to pay a fixed rate equal to the forward swap rate calculated using LIBOR discounting. Price differences are measured in basis points.

In practice option pricing models are calibrated to market prices by implying volatilities. This has implications for the implied volatility surface calculated by a dealer for whom discounting practices are not in alignment with market practice. Consider the case of Dealer X who is the first dealer to consider switching from LIBOR to OIS discounting for option pricing purposes. Since all other dealers are using LIBOR discounting observed market prices for swap options are given by V_{LD}. When Dealer X calibrates his swap option pricing model to the observed market prices, the price differences caused by the different approaches to discounting translate into differences in the implied volatility.

To illustrate this, we calculate option prices for options with lives from 1 to 20 years on a five-year swap based on LIBOR discounting, V_{LD}. The fixed rate on the swap is the forward swap rate based on LIBOR discounting plus an offset of between –2% and +2%. The volatility of the forward swap rate is 20% for all strikes and maturities. These prices are then backed through the OIS discounting swap option pricing model to find the implied volatility that set $V_{\text{OD}} = V_{\text{LD}}$.

The results depend on the slope of the term structure and the terms of the options. Figure 1 shows the volatility term structure that Dealer X would calculate for options to enter into a swap that pays fixed that when the term structure of interest rates is upward sloping. The implied volatilities are always less than the market volatility used in the LIBOR discounting option pricing model. The volatility term structure is generally downward sloping except for in-the-money options which exhibit a steeply upward sloping term structure for relatively short-term options.

Figure 2 shows the shape of the volatility smile for the same set of options. The implied volatilities are highest for out-of-the-money options and lowest for in-the-money options. The smile is most pronounced for short maturity options and becomes increasingly flatter as the option maturity lengthens. For at- and out-of-the-money options longer term options have lower implied volatilities. The reverse is true for deep in-the-money options.

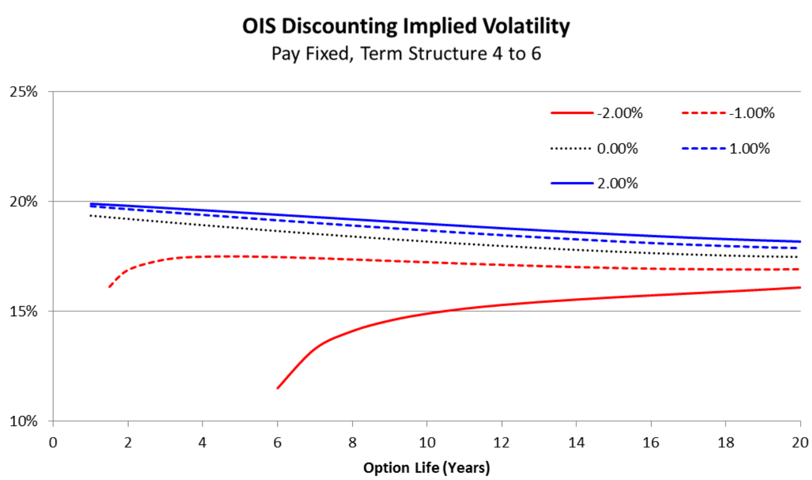

Figure 1 Implied volatilities calculated using OIS discounting when the market uses LIBOR discounting. The volatility used by the market is assumed to be 20% for all strike prices and maturities. Options with lives from 1 to 20 years on a five-year swap in which fixed is paid are considered. The fixed rate on the swap is the forward swap rate based on LIBOR discounting plus offsets of –2%, –1%, 0%, 1%, and 2%.

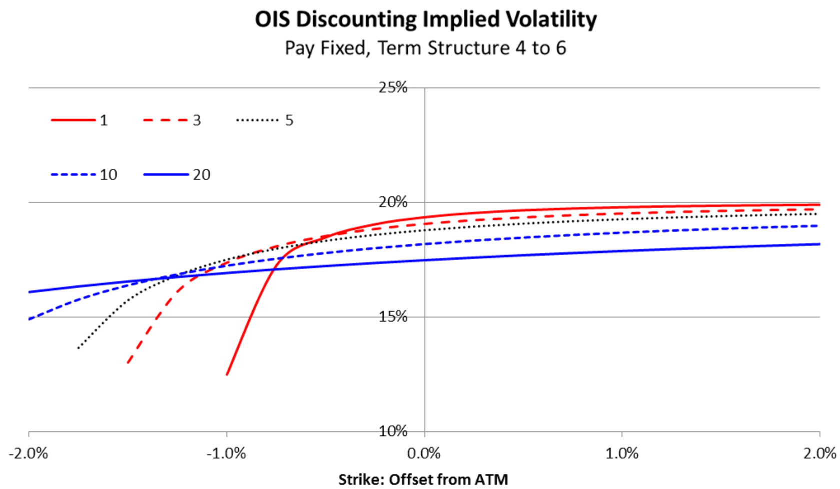

Figure 2 Implied volatilities calculated using OIS discounting when the market uses LIBOR discounting. The volatility used by the market is assumed to be 20% for all strike prices and maturities. Options with maturities of 1, 3, 5, 10, and 20 years on a five-year pay-fixed swap are considered. The fixed rate on the swap is the forward swap rate based on LIBOR discounting plus an offset between –2%, and +2%.

A similar situation applies to a Dealer Y who uses LIBOR discounting when the observed market prices for swap options are given by V_{OD}. When Dealer Y calibrates his swap option pricing model to market prices he will in a similar way observe a volatility surface resulting from the different pricing characteristics of the two option pricing models.

Because both OIS discounting and LIBOR discounting models are calibrated to market data the main effect of the differences in the models will in practice be captured by the implied volatilities. As a result the models will agree perfectly for all options in the calibration set and the price differences for other options will be small.

6 OIS discounting and cap/floor pricing

Consider a caplet, one leg of an interest rate cap, in which the floating rate of interest is capped at a specified fixed rate of interest, K. We use a similar notation to that for an interest rate swap. The notional principal is L, the start date is T_i, and the end or payment date is T_{i+1}. On the start date we observe the LIBOR rate R_i that applies between T_i and T_{i+1}. Then on the end date, T_{i+1}, there is a payoff of $\max(R_i - K, 0)L(T_{i+1} - T_i)$.

With LIBOR discounting the standard market model gives the value of the payment as

$$V_{\text{LD}} = [F_{\text{LD}}(T_i, T_{i+1})N(d_{1,\text{LD}}) - KN(d_{2,\text{LD}})] \times L(T_{i+1} - T_i)P_{\text{LD}}(T_{i+1})$$

where

$$d_{1,\text{LD}} = \frac{\ln[F_{\text{LD}}(T_i, T_{i+1})/K]}{\sigma_{\text{LD}}\sqrt{T_i}} + 0.5\sigma_{\text{LD}}\sqrt{T_i}$$

$$d_{2,\text{LD}} = \frac{\ln[F_{\text{LD}}(T_i, T_{i+1})/K]}{\sigma_{\text{LD}}\sqrt{T_i}} - 0.5\sigma_{\text{LD}}\sqrt{T_i}$$

where σ_{LD} is the volatility of $F_{\text{LD}}(T_i, T_{i+1})$.

With OIS discounting, the standard market model becomes

$$V_{\text{OD}} = [F_{\text{OD}}(T_i, T_{i+1})N(d_{1,\text{OD}}) - KN(d_{2,\text{OD}})]L(T_{i+1} - T_i)P_{\text{OD}}(T_{i+1})$$

where

$$d_{1,\mathrm{OD}} = \frac{\ln[F_{\mathrm{OD}}(T_i, T_{i+1})/K]}{\sigma_{\mathrm{OD}}\sqrt{T_i}} + 0.5\sigma_{\mathrm{OD}}\sqrt{T_i}$$

$$d_{2,\mathrm{OD}} = \frac{\ln[F_{\mathrm{OD}}(T_i, T_{i+1})/K]}{\sigma_{\mathrm{OD}}\sqrt{T_i}} - 0.5\sigma_{\mathrm{OD}}\sqrt{T_i}$$

where σ_{OD} is the volatility applied to $F_{\mathrm{OD}}(T_i, T_{i+1})$.

A standard interest-rate cap is constructed of many of these legs in which the payment date for one leg is the rate reset date for the next leg. The total cap value is the sum of all the individual caplet values. Sometimes the same implied volatility is used for all caplets and sometimes a different volatility is used for each caplet.

A floorlet is an instrument that provides a payoff of $\max(K - R_i, 0)L(T_{i+1} - T_i)$ at time T_{i+1}. The value of the floorlet under LIBOR and OIS discounting are given by

$$\begin{aligned} V_{\mathrm{LD}} = {} & [KN(-d_{2,\mathrm{LD}}) - F_{\mathrm{LD}}(T_i, T_{i+1}) \\ & \times N(-d_{1,\mathrm{LD}})]L(T_{i+1} - T_i)P_{\mathrm{LD}}(T_{i+1}) \end{aligned}$$

and

$$\begin{aligned} V_{\mathrm{OD}} = {} & [KN(-d_{2,\mathrm{OD}}) - F_{\mathrm{OD}}(T_i, T_{i+1}) \\ & \times N(-d_{1,\mathrm{OD}})]L(T_{i+1} - T_i)P_{\mathrm{OD}}(T_{i+1}) \end{aligned}$$

Similar to caps, a standard interest-rate floor is constructed of many of these legs in which the payment date for one leg is the rate reset date for the next leg.

To examine the effect of the switch to OIS discounting on the prices of caps and floors, we consider quarterly reset caps and floors with different lives. In all cases, in accordance with market practice, the first leg of the cap or floor is omitted so the cap starts at time 0.25. The cap or floor strike price is set equal to the corresponding swap rate under LIBOR discounting. The forward rate volatility is 20% and the same three term structures as before are used.

Table 7 shows the value differences for caps for the three term structures of swap rates. Table 8 does the same for floors. The results are similar to those in Tables 5 and 6. The impact of switching from LIBOR discounting to OIS discounting increases as cap maturity increases and the discounting effect is greater than the effect of the change in the forward rate.

As discussed in Section 4, in practice the cap and floor models would be calibrated to observed market prices. As a result, any dealer whose discounting differs from market practice will induce a volatility surface that differs from the market volatility surface.

7 Building trees to value nonstandard transactions

So far this paper has focused on transactions which are handled by the market using standard market models based on Black (1976). As we have shown, it is fortunate that, once forward LIBOR has been correctly calculated, only a small modification to the standard market models is necessary.[11] We now move on to consider instruments which cannot be handled using standard market models. A single interest rate tree describing the evolution of the LIBOR rates is commonly used to value these instruments when LIBOR discounting is used. In this section, we explain that it is still possible to obtain a value using a single tree when OIS discounting is used.

For the sake of definiteness, consider a derivative providing payoffs dependent on three-month LIBOR.[12] The general procedure is as follows:

(1) Calculate the OIS zero curve from swap rates as discussed earlier in this paper.
(2) Calculate a LIBOR zero curve when OIS discounting is used as discussed earlier in this paper.

Table 7 The increase in the price of an approximately at-the-money cap caused by switching from LIBOR discounting to OIS discounting.

Cap life (Yrs)	Term structure 4 to 6: Discount effect $V_{LO} - V_{LD}$	Term structure 4 to 6: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 4 to 6: Total $V_{OD} - V_{LD}$	Term structure 5 flat: Discount effect $V_{LO} - V_{LD}$	Term structure 5 flat: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 5 flat: Total $V_{OD} - V_{LD}$	Term structure 6 to 4: Discount effect $V_{LO} - V_{LD}$	Term structure 6 to 4: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 6 to 4: Total $V_{OD} - V_{LD}$
1	0.1	0.0	0.1	0.2	0.0	0.2	0.2	0.0	0.2
3	2.0	–0.2	1.9	2.2	0.0	2.2	2.4	0.1	2.6
5	7.3	–0.8	6.6	7.5	0.0	7.5	7.6	0.7	8.3
7	17.0	–2.2	14.8	16.1	0.0	16.1	15.6	1.9	17.4
10	41.1	–6.7	34.5	35.6	0.0	35.6	31.5	5.3	36.9
20	217.8	–58.5	159.3	146.6	0.0	146.6	97.7	36.9	134.6

Price differences are measured in basis points.

Table 8 The increase in the price of an approximately at-the-money floor caused by switching from LIBOR discounting to OIS discounting.

Floor life (Yrs)	Term structure 4 to 6: Discount effect $V_{LO} - V_{LD}$	Term structure 4 to 6: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 4 to 6: Total $V_{OD} - V_{LD}$	Term structure 5 flat: Discount effect $V_{LO} - V_{LD}$	Term structure 5 flat: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 5 flat: Total $V_{OD} - V_{LD}$	Term structure 6 to 4: Discount effect $V_{LO} - V_{LD}$	Term structure 6 to 4: Fwd. rate effect $V_{OD} - V_{LO}$	Term structure 6 to 4: Total $V_{OD} - V_{LD}$
1	0.1	0.0	0.1	0.2	0.0	0.2	0.2	0.0	0.2
3	1.9	0.1	2.0	2.2	0.0	2.2	2.6	–0.1	2.5
5	6.4	0.5	6.9	7.5	0.0	7.5	8.5	–0.6	8.0
7	14.2	1.3	15.5	16.1	0.0	16.1	18.3	–1.5	16.8
10	32.3	3.6	35.8	35.6	0.0	35.6	39.8	–4.3	35.6
20	141.6	22.4	164.0	146.6	0.0	146.6	162.4	–32.3	130.1

Price differences are measured in basis points.

(3) Assume a one-factor interest rate model for the evolution of the OIS zero curve and construct a tree for the Δt-period OIS rate. The drift of the short rate will typically include a function of time so that the initial OIS term structure is matched. Procedures for building the tree are described in, for example, Hull and White (1994, 2014).
(4) Roll back through the tree to calculate the value of a three-month bond at each node so that the three-month OIS rate is known at each node.
(5) Assume a one-factor model for the spread between the three-month LIBOR rate and the three-month OIS rate. Similar to the model for the OIS rate, this will include a function of time so that the LIBOR zero curve can be matched.
(6) Assume a correlation between the three-month OIS rate and the spread in step 5.
(7) Advance through the OIS tree one step at a time matching the tree-based value of a forward rate agreement based on three-month LIBOR rates with the analytic value calculated using the procedures described in Section 3. The procedure is as follows: At the time t step choose a trial value for the unconditional expected spread at time t. (As mentioned, the process for the spread incorporates a function of time in the drift.) Calculate the conditional expected spread at each node of the OIS tree at time t based on the unconditional expected spread. The expected three-month LIBOR rate at each node of the OIS tree is then the three-month OIS rate at the node plus the expected spread. The value of the forward rate agreement at each node at time t can be calculated from the three-month LIBOR and OIS rates at the node. Discounting these values back through the tree produces the tree-based estimate of the value of the forward rate agreement. (These present value calculations are simplified by storing the Arrow–Debreu prices at each node when the OIS tree is constructed in step 3.) An iterative procedure is used to determine the unconditional expected spread at time t that matches the tree-based price with the analytic price.

The result of this procedure is a tree that has the OIS rate and the expected LIBOR rate at each node. The tree matches both the OIS swap rates and LIBOR swap rates. It can be used to value derivatives whose payoffs depend on LIBOR when OIS discounting is used.

The potentially difficult part of the procedure is calculating the expected spread at time t at each node of the OIS tree for a trial value of the unconditional expected spread at time t. For some models analytic results are available. In other cases numerical procedures can be developed.[13] The appendix shows that analytic results are available when (a) a mean-reverting Gaussian model is used for both the instantaneous OIS rate and the spread and (b) a mean-reverting Gaussian model is used for both the logarithm of the OIS rate and the logarithm of the spread. The first model can be viewed as a natural extension of Hull and White (1990) interest rate model. The second can be viewed as a natural extension of Black and Karasinski (1991) interest rate model.

When valuing American-style options, there is the complication that the decision to exercise at a node may depend on the value of the spread, s, at the node. This can be accommodated as follows. First we calculate for the node the critical value of s above which early exercise is optimal. We will refer to this as s^*. Define $q = \Pr(s > s^* \mid R)$. The value of the option at the node is

$$(1 - q)U_1 + qU_2$$

where U_1 is the value of the option (calculated from the value at subsequent nodes) if there is no exercise and U_2 is the intrinsic value of the option at the node. The calculation of q is explained in the Appendix for the two models considered.

We will illustrate the model using the extension of Black and Karasinski (1991). Specifically, if r is the short-term OIS rate and $x = \ln(r)$, then

$$dx = [\theta(t) - a_x x]dt + \sigma_x dz_x$$

where a_x and σ_x are positive constants, $\theta(t)$ is a function of time chosen to match the OIS zero curve that is calculated from OIS swap rates, and dz_x is a Wiener process. If s is the three-month LIBOR−OIS spread and $y = \ln(s)$

$$dy = [\psi(t) - a_y y]dt + \sigma_y dz_y$$

where a_y and σ_y are positive constants, $\psi(t)$ is a function of time, and dz_y is a Wiener process. The correlation between dz_x and dz_y is assumed to be a constant, ρ.

We consider a ten-year Bermudan swap option where early exercise can take place on each swap payment date on or after year two. The holder has an option to enter into the swap. The swap underlying the option is a quarterly reset swap in which three-month LIBOR is received and a fixed rate is paid. The fixed rate is set equal to the market rate for a swap that starts in two years and matures in ten years. This ensures that at the first possible exercise date, two years from now, the option is approximately at-the-money. The OIS and LIBOR term structure data is the same as that for the first of the three cases defined in Section 2. The instantaneous OIS rate volatility, σ_x, is 20%, the reversion rate, a_x, is 5%, and the LIBOR–OIS spread reversion rate, a_y, is 5%. We consider a number of different values of σ_y and ρ.

A necessarily prerequisite to valuing the Bermudan swap option is valuing the underlying swap at each node of the tree. This can be calculated by rolling back through the tree in the usual way. The value of the swap at a node at time $i\Delta t$ is the present value of the expected value at the nodes that can be reached at time $(i+1)\Delta t$ plus the present value of any payment based on the expected value of LIBOR at the node.

The value of the option for various values of the LIBOR−OIS spread volatility, σ_y, and the correlation, ρ, between the spread and the OIS rate are shown in Table 9. The value of the option when the spread is constant, $\sigma_y = 0$, is 2.92% of the swap notional. If the correlation is positive the option value increases as the spread volatility increases while if the correlation is negative the option value decreases as the spread volatility increases. The reason for this is that when the OIS rate is high and the correlation is positive the LIBOR−OIS spread is greater than average. Similarly, when the OIS rate is low and the correlation is positive the LIBOR−OIS spread is smaller than average. As a result, the LIBOR rate (i.e., the OIS rate plus the spread) exhibits more variability than the OIS rate. This increases the value of options on the LIBOR rate. If the correlation is negative the reverse is true and the LIBOR rate exhibits less variability than the OIS rate resulting in lower option prices.

The results show that a correlation between the spread and OIS can have an appreciable effect on option prices. For example, when the spread volatility is 15% and the correlation is 0.25 the value of the option increases by about 5%. This result is predicated on the assumption that we know the volatility of OIS and the spread. However, in practice all model parameters are typically implied from the observed market prices for options with different strike prices and maturities. The model is in effect a tool for pricing non-standard interest rate derivatives consistently with standard interest rate derivatives. As a result, the calibrated OIS and spread volatility parameters will be sensitive to the assumed correlation but option prices calculated using the calibrated models will be much less sensitive to this correlation.

Table 9 The value of a Bermudan option to enter into a 10-year swap as a percentage of the swap notional for various values of the LIBOR–OIS spread volatility, σ_y, and the correlations between the spread and the OIS rate, ρ.

	LIBOR−OIS spread volatility, σ_y					
ρ	0.01 (%)	0.05 (%)	0.10 (%)	0.15 (%)	0.20 (%)	0.25 (%)
0.99	2.96	3.12	3.33	3.54	3.74	3.94
0.75	2.95	3.07	3.23	3.39	3.54	3.70
0.50	2.94	3.02	3.12	3.23	3.33	3.44
0.25	2.93	2.97	3.02	3.07	3.12	3.18
0.00	2.92	2.92	2.92	2.92	2.92	2.92
–0.25	2.91	2.87	2.82	2.76	2.71	2.66
–0.50	2.90	2.82	2.71	2.61	2.52	2.42
–0.75	2.89	2.76	2.61	2.47	2.32	2.18
–0.99	2.88	2.72	2.52	2.33	2.15	1.97

The option can be exercised on any payment date on or after year 2. The instantaneous OIS rate volatility is 20% and the reversion rate is 5% ($\sigma_x = 0.20$ and $a_x = 0.05$). The LIBOR–OIS spread has a reversion rate of 5% ($a_y = 0.05$). The 4 to 6 term structure applies.

8 Conclusions

When interest rate derivatives are valued, moving from LIBOR to OIS discounting has two effects. First, forward interest rates and forward swap rates change. Second, the discount rates applied to the cash flows calculated from those forward rates change. The impact of the switch is small for short-dated instruments, but becomes progressively larger as the life of the instrument becomes longer.

One mistake that is sometimes made is to (a) calculate LIBOR zero rates and LIBOR forward rates and by bootstrapping LIBOR forward swap rates in the traditional way (i.e., in a world where LIBOR discounting is employed) and (b) apply OIS discount rates to the cash flows calculated by assuming that the LIBOR forward rates will be realized. Our results show the errors caused by making this mistake. The error is zero for flat term structures, but can be as much as 200 basis points for long-dated transactions and the upward or downward sloping term structures we have considered.

This paper has shown how the standard market models for caps and swap options can be adjusted to accommodate OIS discounting. For more complex instruments we have presented a new procedure for adjusting the way interest rate trees are used to accommodate OIS discounting. Both the OIS rate and the expected LIBOR rate can be modeled using a single tree. A stochastic spread makes LIBOR more (less) volatile than OIS if the correlation is positive (negative). This results in higher (lower) option prices than arise when the spread is deterministic.

Appendix: Analytic results for two one-factor term structure models

In this appendix we explain how the spread conditional on the OIS rate can be calculated analytically for two cases of the model outlined in Section 6.

The Gaussian model

First consider the case where both the instantaneous OIS rate and the three-month spread follow mean-reverting Gaussian processes. Define r as the instantaneous OIS rate and s as the spread between three-month LIBOR and the three-month OIS rate. The processes for r and s are

$$dr = [\theta(t) - a_r r]dt + \sigma_r dz_r$$

and

$$ds = [\psi(t) - a_s s]dt + \sigma_s dz_s$$

where a_r, a_s, σ_r, and σ_s are positive parameters, $\theta(t)$ is a function of time chosen so that the process for the instantaneous OIS rate is consistent with the current OIS zero curve, $\psi(t)$ is a function of time chosen so that the spread is consistent with the LIBOR zero curve, and dz_s and dz_r are Wiener processes. We assume that the correlation between the two Wiener processes is ρ.[14]

It follows that the unconditional distributions of r and s at time t are

$$r(t) \sim \varphi\left(\mu_r(t), \frac{\sigma_r^2}{2a_r}(1 - e^{-2a_r t})\right)$$

$$s(t) \sim \varphi\left(\mu_s(t), \frac{\sigma_s^2}{2a_s}(1 - e^{-2a_s t})\right)$$

where $\varphi(\mu, v)$ denotes a normal distribution with mean μ and variance v and

$$\mu_r(t) = r_0 + e^{-a_r t}\int_0^t \theta(\tau)e^{a_r\tau}d\tau$$

$$\mu_s(t) = s_0 + e^{-a_s t}\int_0^t \psi(\tau)e^{a_s\tau}d\tau$$

The distribution of s conditional on r is

$$\varphi(\alpha(r), \beta)$$

where

$$\alpha(r) = \mu_s + \rho\frac{\sigma_s\sqrt{(1 - e^{-2a_s t})/a_s}}{\sigma_r\sqrt{(1 - e^{-2a_r t})/a_r}}(r - \mu_r)$$

and

$$\beta = \sigma_s^2\frac{1 - e^{-2a_s t}}{2a_s}(1 - \rho^2)$$

The value of $E(s\,|\,r)$ is $\alpha(r)$. As indicated in Section 6 we choose the μ_s iteratively so that forward LIBOR rates are matched. For this model $\Pr(s > s^* \mid R)$ is

$$N\left(\frac{\alpha(r) - s^*}{\sqrt{\beta}}\right)$$

where, as before, N is the cumulative normal distribution function. These results provide the necessary ammunition to value American-style options.

The lognormal model

Consider next the situation where both the logarithm of the instantaneous OIS rate and the logarithm of the three-month spread follow mean-reverting Gaussian processes. Defining $x = \ln(r)$ and $y = \ln(s)$ we assume

$$dx = [\theta(t) - a_x x]dt + \sigma_x dz_x$$

and

$$dy = [\psi(t) - a_y y]dt + \sigma_y dz_y$$

where a_x, a_y, σ_x, and σ_y are positive constants, $\theta(t)$ and $\psi(t)$ are a function of time chosen to match the OIS zero curve and the LIBOR zero curve, and dz_x and dz_y are Wiener process. In this case ρ is the correlation between dz_x and dz_y.

Similar to the Gaussian model case, the distribution of y conditional on x is

$$\varphi(\alpha(x), \beta)$$

where

$$\alpha(x) = \mu_y + \rho\frac{\sigma_y\sqrt{(1 - e^{-2a_y t})/a_y}}{\sigma_x\sqrt{(1 - e^{-2a_x t})/a_x}}(x - \mu_x)$$

and

$$\beta = \sigma_y^2\frac{1 - e^{-2a_y t}}{2a_y}(1 - \rho^2)$$

For the lognormal model, $E(s \mid R) = E(e^y \mid x = \ln R)$. This is

$$\exp\{\alpha(\ln R) + \beta/2\}$$

We choose the μ_y iteratively so that forward LIBOR rates are matched. In this case $\Pr(s > s^* \mid R)$ is

$$N\left(\frac{\alpha(\ln R) - \ln s^*}{\sqrt{\beta}}\right).$$

Notes

1 Johannes and Sundaresan (2007) argued pre-crisis that the prevalence of collateralization in the interest rate swap market means that discounting at LIBOR rates is no longer appropriate.

2 The reason usually given for this is that these transactions are funded by the collateral and cash collateral often earns the effective federal funds rate. The OIS rate is a continually refreshed federal funds rate. Hull and White (2012, 2013a, 2013b) argue that OIS is the best proxy for the risk-free rate and that it should be used when valuing both collateralized and non-collateralized transactions.

3 Mercurio (2009) may have been the first researcher to investigate how the standard market models for caps and swap options can be adapted to accommodate OIS discounting.

4 Our objective is to keep the notation as simple as possible so we do not consider the various day count practices that apply in practice. However, if $T_{i+1} - T_i$ and other time parameters are interpreted as the accrual fraction that applies under the appropriate day count convention, our results are in agreement with industry practice.

5 This is a small simplification. In practice, the interest rate is fixed two days before the start of the period to which it applies.

6 This assumes that the day count convention used for the floating rates in the swap is the same as that used to calculate the forward rates. This is the case for a standard swap.

7 A swap of the federal funds rate for LIBOR involves the arithmetic average of effective federal funds rate for the period being considered, whereas payments in an OIS are calculated from a geometric average of effective federal funds rates. A "convexity adjustment" is in theory necessary to adjust for this. See, for example, Takada (2011).

8 Depending on one's point of view both V_{LD} and V_{OD} may be legitimate value calculations. However, V_{LO} is not a legitimate value calculation. The forward rate $F_{OD}(T_1, T_2)$ is a martingale when the numeraire is $P_{OD}(T_2)$, but the forward rate $F_{LD}(T_1, T_2)$ is not a martingale for this numeraire.

9 See for example Hull (2015), Chapter 29, for a discussion of the standard market models used for interest rate derivatives.

10 Note that σ_{OD} and σ_{LD} are in general not equal.

11 As an alternative to using standard market models, one can value caplets or swap options by integrating over the joint distribution of OIS and the expected spread. This approach is discussed by Mercurio and Xie (2012).

12 The same approach can be used for derivatives dependent on any other reference rate. If the tenor of the reference is τ, a single tree is used to model the OIS Δt-period rate and the expected spread between the τ-period OIS rate and the τ-period reference rate.

13 A general numerical procedure that can be used is as follows. Construct a tree for the Δt-OIS rate, r. Construct a tree step-by-step for the spread, s. At the each time step, determine the unconditional distributions for r and s. (In the case of the s-tree a trial value for the center of the tree is being assumed.) Assume a Gaussian copula (or some other convenient copula) to determine $E(s \mid r)$ at each node of the r-tree. The advantage of the analytic results given in the Appendix for the normal and the lognormal models is that calculations are much faster because it is not necessary to build the s-tree.

14 The parameter ρ is defined as the correlation between the Wiener processes. It can be shown that it is also the correlation between the values of r and s at time t.

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