Journal Of Investment Management, Vol. 8, No. 3, (2010), pp. 93–114

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THE LIBOR/SABR MARKET MODELS: A CRITICAL REVIEW

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This paper reviews the LIBOR market model (LMM) and the LMM-SABR model. While a plethora of interest rate models, such as fundamental models, single-plus models, double-plus models, and triple-plus models, can be used for valuation of plain vanilla derivatives, only a few models such as the LMM and the LMM-SABR have been proposed as models that can hedge plain vanilla derivatives as well as value complex interest rate derivatives. However, given that LMM and LMM-SABR models are triple-plus models, they are calibrated to market prices by allowing time-inhomogeneous volatilities, and by changing numerous model inputs period by period. Changing the model period by period and using time-inhomogeneous volatilities make risk-return analysis impossible under the physical measure. Further, this paper demonstrates that the LMM-SABR model is based on the highly questionable assumption of zero drifts for the volatility processes (under the forward rate specific measures), which has no economic justification, and can lead to explosive behavior for volatilities. We suggest high-dimensional affine and quadratic models that use fast analytical approximations (such as the Fourier inversion method and the cumulant expansion method) for pricing caps and swaptions, as alternatives to the LMM and the LMM-SABR model.

1 Introduction

Over the past few years, the stochastic volatility-based SABR model, proposed by Hagan *et al.* (2002), has become a competing market standard to the LIBOR market model (LMM) for pricing plain vanilla interest rate derivatives such as caps and swaptions. Rebonato and White (2009) and Rebonato *et al.* (2009) generalize the SABR model using a stochastic volatility extension of the LMM with a common numeraire, such that the new model called the LMM-SABR model can be used for pricing a variety of interest rate derivatives. This paper provides a critical review of both the LMM and the LMM-SABR model using the new taxonomy of term structure models given by Nawalkha, Beliaeva, and Soto (NBS) (2007, 2009). Under the new taxonomy, all term structure models (TSMs) are classified into one of the following four categories:

(i) Fundamental TSMs (i.e., preference-dependent time-homogeneous models),

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(ii) Single-plus models (i.e., preference-free time-homogeneous models),
(iii) Double-plus models (i.e., preference-free models with time-inhomogeneous drifts for the state variable processes and time-homogeneous bond return volatilities, that fit the initially observed bond prices),
(iv) Triple-plus models (i.e., preference-free models with time-inhomogeneous drifts for the state variable processes, and time-inhomogeneous bond return volatilities, that exactly fit both the initially observed bond prices and initially observed prices of a chosen set of plain vanilla interest rate derivatives).

Single-plus models have one extra degree of freedom over fundamental models, as the former models are preference-free, while the later models are preference-dependent. Double-plus models have two extra degrees of freedom over fundamental models, as these models are not only preference-free, but also allow a perfect fit with the initially given bond prices. However, moving from single-plus to double-plus exposes these models to some degree of *smoothing* (defined as fitting models to arbitrary time dependencies without any economic rationale), even though this comes with the advantage of perfectly fitting the initially given bond prices. The triple-plus models have three degrees of freedom over fundamental models. These models are preference-free, allow a perfect fit with initially given bond prices, and allow a perfect fit with initially given prices of plain vanilla derivatives. However, triple-plus models are exposed to two sources of time-inhomogeneity, one of them being the time-inhomogeneous evolution of volatilities.

As argued in this paper, the popular versions of the LMM and the LMM-SABR model are triple-plus models, exposing these models to the dangers of smoothing. Though double-plus versions of the LMM and the LMM-SABR model can be derived, Rebonato (2002), Brigo and Mercurio (2006), and Rebonato *et al.* (2009) recommend triple-plus versions of these models to exactly fit the cross-section of plain-vanilla derivatives. Hence, despite becoming industry benchmarks, the triple-plus features of the LMM and LMM-SABR model raise serious concerns about their reliability over time.

This second section of this paper begins with a review of the LMM. We focus mainly on the LFM (i.e., Lognormal forward Libor model) version of the LMM model and show how to derive this model with a single numeraire. The third section of this paper considers various extensions of the LMM that can explain the volatility smile in the caps market, including the CEV and displaced-diffusion extensions for explaining a monotonic smile, and the LLM-SABR extension for explaining the non-monotonic smile.

The fourth section provides a critique of both the LMM and the LMM-SABR model. The critique focuses on two important issues. First, we focus on the assumption of zero drifts for the volatility processes under the LMM-SABR model. This assumption is inconsistent with economic fundamentals and is shown to be the main weakness of the LMM-SABR model. Second, we evaluate the usefulness of the LMM and the LMM-SABR model from the perspective of a partially hedged trader or an investor/speculator who must evaluate risk-return trade-offs. We show that due to the nature of calibrations used under the LMM and the LMM-SABR model, it is virtually impossible to perform risk-return analysis, which requires information from the physical measure.

The final section of this paper recommends high-dimensional affine and quadratic models that use fast analytical approximations (such as the Fourier inversion method and the cumulant expansion method) for pricing caps and swaptions, as alternatives to the LMM and the LMM-SABR model.

2 The LIBOR Market Model

The LMM model was discovered by Brace, Gatarek, and Musiela (1997), and was initially referred to as the BGM model by practitioners. However, Miltersen *et al.* (1997) discovered this model independently, and Jamshidian (1997) also contributed significantly to its initial development. To reflect the contribution of multiple authors, many practitioners including Rebonato (2002) renamed this model as the Libor market model or the LMM in abbreviated form. The two commonly used versions of the LMM are the lognormal forward Libor model (LFM) for pricing caps, and the lognormal forward swap model (LSM) for pricing swaptions. The LFM assumes that the discrete forward Libor rate follows a lognormal distribution under its own numeraire, while the LSM assumes that the discrete forward swap rate follows a lognormal distribution under the swap numeraire. Though the two assumptions are theoretically inconsistent, they lead to small discrepancies in calibrations using realistic parameterizations. This section also derives a joint framework by deriving the LFM using a *single* numeraire, which leads to exact formulas for pricing caps, and approximate formulas for pricing swaptions. A variety of specifications of instantaneous volatilities and correlations are considered for the LFM with a single numeraire, consistent with the double-plus and the triple-plus versions of this model.

2.1 The LFM

Consider the relationship between the discrete Libor rate $L(T_i, T_{i+1})$ for the term $U_i = T_{i+1} - T_i$, and the zero-coupon bond price $P(T_i, T_{i+1})$, given as follows:

$$1 + L(T_i, T_{i+1})\hat{U}_i = \frac{1}{P(T_i, T_{i+1})} \tag{1}$$

where $t \leq T_0 < T_1 < T_2 < \cdots < T_n$ is the timeline, and $\hat{U}_i$ is the accrual factor for the period T_i to T_{i+1}, calculated using actual/360 day-count basis.

The time t discrete forward rate for the term $U_i = T_{i+1} - T_i$, is related to the price ratio of two zero-coupon bonds maturing at times T_i and T_{i+1} as follows:

$$1 + f(t, T_i, T_{i+1})\hat{U}_i = \frac{P(t, T_i)}{P(t, T_{i+1})} \tag{2}$$

The forward rate converges to the future Libor rate at time T_i, or:

$$f(T_i, T_i, T_{i+1}) = L(T_i, T_{i+1}) \tag{3}$$

Equation (2) can be rewritten as follows:

$$f(t, T_i, T_{i+1})P(t, T_{i+1}) = \frac{1}{\hat{U}_i}(P(t, T_i) - P(t, T_{i+1})) \tag{4}$$

The expression $f(t, T_i, T_{i+1})P(t, T_{i+1})$ is equal to a constant times the difference between zero-coupon bond prices maturing at dates T_i and T_{i+1}. Hence, the expression $f(t, T_i, T_{i+1})P(t, T_{i+1})$ gives the price of a *traded asset*. Now, consider a non-dividend paying, positive valued numeraire asset with a price $x(t)$. Under absence of arbitrage, an equivalent martingale measure exists corresponding to the asset $x(t)$, such that the ratio of the price of any traded asset to the price of asset $x(t)$ is a martingale under this measure. Hence, the process,

$$y(t) = \frac{f(t, T_i, T_{i+1})P(t, T_{i+1})}{x(t)} \tag{5}$$

must have a zero-drift under an equivalent martingale measure corresponding to the numeraire asset $x(t)$. If $x(t) = P(t, T_{i+1})$, then $f(t, T_i, T_{i+1}) = [f(t, T_i, T_{i+1})P(t, T_{i+1})]/P(t, T_{i+1})$ is a martingale under the equivalent measure defined with respect to the numeraire $P(t, T_{i+1})$. The equivalent measure with respect to the numeraire $P(t, T_{i+1})$ is also called the *forward measure* defined with respect to the maturity T_{i+1}. Since $f(t, T_i, T_{i+1})$

is a martingale under this forward measure, the stochastic process of $f(t, T_i, T_{i+1})$ has zero drift under this measure. The LFM specifies the following zero-drift stochastic process for $f(t, T_i, T_{i+1})$ under this forward measure:

$$\frac{df(t, T_i, T_{i+1})}{f(t, T_i, T_{i+1})} = \sigma_i(t)dZ_i(t) \tag{6}$$

where $dZ_i(t)$ is a Wiener process under the forward measure $\tilde{\mathcal{P}}^i$ defined with respect to the numeraire asset $P(t, T_{i+1})$, and it $\sigma_i(t)$ measures the volatility of the forward rate process. The volatility can depend upon time, and various types of time-dependent volatility functions that are considered later in this paper. Using Ito's lemma, the stochastic process of the logarithm of the forward rate is given as follows:

$$d\ln f(t, T_i, T_{i+1}) = \frac{-\sigma_i^2(t)}{2}dt + \sigma_i(t)dZ_i(t) \tag{7}$$

The stochastic integral of the above equation can be given as follows. For all $0 \leq t \leq T_i$,

$$\ln f(t, T_i, T_{i+1}) = \ln f(0, T_i, T_{i+1}) - \int_0^t \frac{\sigma_i^2(v)}{2}dv + \int_0^t \sigma_i(v)dZ_i(v) \tag{8}$$

Since the volatility function $\sigma_i(t)$ is deterministic, the logarithm of forward rates is normally distributed, implying that the forward rate is lognormally distributed. For $t = T_i$, the above equation implies that the future Libor rate $L(T_i, T_{i+1}) = f(T_i, T_i, T_{i+1})$, is also lognormally distributed. This explains why this model is called the lognormal forward Libor model, or the LFM. Though each forward rate is lognormally distributed under its own forward measure, it is not lognormally distributed under any arbitrary forward measure. The implications of this observation are addressed later in this paper.

The lognormal forward rate process provides a theoretical justification for the widely used Black formula for caplets. To see this, consider the Black formula for pricing caplets, with the payoff of the ith caplet at time T_{i+1} is defined as follows:

$$\text{Caplet Payoff at } T_{i+1} = F \times \hat{U}_i \times \max\left[L(T_i, T_{i+1}) - K_i, 0\right] \tag{9}$$

where, F is the notional value of the caplet, and K_i is the caplet strike rate. Even though the payment is made at time T_{i+1}, the amount to be paid is known with certainty at time T_i, and hence, the option expires at time T_i. The Black formula for the time t price of the ith caplet is given as follows:

$$P_{\text{Caplet}_i}(t) = F \times \hat{U}_i \times P(t, T_{i+1})\{f(t, T_i, T_{i+1}) \times \mathcal{N}(d_{1,i}) - K_i\mathcal{N}(d_{2,i})\} \tag{10}$$

where,

$$d_{1,i} = \frac{\ln\left(f(t, T_i, T_{i+1})/K_i\right) + \vartheta_i^2(T_i - t)/2}{\vartheta_i\sqrt{T_i - t}}$$

$$d_{2,i} = \frac{\ln\left(f(t, T_i, T_{i+1})/K_i\right) - \vartheta_i^2(T_i - t)/2}{\vartheta_i\sqrt{T_i - t}}$$

where ϑ_i is the Black implied volatility of the caplet, assumed to be a constant. Though the above formula was initially developed using heuristic arguments based upon Black's (1976) option formula, the forward rate process given in equation (6) provides a theoretical justification for using this formula. To see this note that the forward rate $f(t, T_i, T_{i+1})$ follows a martingale under the forward measure (see equation (6)), and hence, the time t price of the caplet can be obtained by solving the expectation of the caplet payoff given by equation (9) as follows:

$$P_{\text{Caplet}_i}(t) = F \times \hat{U}_i \times P(t, T_{i+1}) \times E_t^i(\max\left[L(T_i, T_{i+1}) - K_i, 0\right]) \tag{11}$$

where the expectation is taken under the forward measure associated with the numeraire $P(t, T_{i+1})$. It can be easily verified that solving the above expectation gives the same caplet price defined in equation (10), but with the following definition of the Black implied volatility:

$$\vartheta_i = \sqrt{\frac{1}{T_i - t}\int_t^{T_i} \sigma_i^2(u)du} \tag{12}$$

Hence, the widely used Black caplet price formula is theoretically valid under the LFM. The attractive feature about equation (12) is that it provides traders with a simple and intuitive definition of Black implied volatility, given as the square root of the average variance of the percentage changes in the forward rate, over the period t to T_i (also called the "root mean square volatility" of forward rates).

2.2 *Multiple factor LFM under a single numeraire*

Since individual caplets comprising the cap can be priced using different numeraires, the lognormal assumption can be maintained for different forward rates (see equation (6)) for pricing caps. However, swaptions and coupon bond options represent *options on portfolios* (and not portfolios of options, like caps), and hence the joint stochastic evolution of different forward rates must be modeled for pricing these instruments. Modeling the joint evolution of forward rates requires that all forward rate processes be measured under a single forward measure corresponding to a single numeraire asset. As we show in the following, using a single forward measure allows only a specific forward rate (corresponding to this measure) to be distributed lognormally. All other forward rates are not distributed lognormally and do not have known densities. Hence swaptions must be priced either by using Monte Carlo simulations or by using approximate analytical solutions under the multifactor LFM.

To allow for a single numeraire, redefine the forward rate process given in equation (6), under the forward measure $\tilde{\mathcal{P}}^k$ with a non-zero drift as follows:

$$\frac{df(t, T_i, T_{i+1})}{f(t, T_i, T_{i+1})} = \mu_i^k(t)dt + \sigma_i(t)dZ_i^k(t) \tag{13}$$

The above equation defines Wiener processes $dZ_i^k(t)$ corresponding to the stochastic processes of different forward rates $f(t, T_i, T_{i+1})$, for $i = 0, 1, \ldots, n-1$, under a single numeraire asset $P(t, T_{k+1})$. The generalized notation in the above also redefines the Wiener process $dZ_i(t)$ corresponding to the stochastic process of the ith forward rate in equation (6), as $dZ_i^i(t)$. The relationship between $dZ_i^k(t)$ and $dZ_i^i(t)$ is given as:

$$dZ_i^k(t) = dZ_i^i(t) - (\mu_i^k(t)/\sigma_i(t))dt \tag{14}$$

In general, any zero-coupon bond of maturity T_{k+1}, can serve as the numeraire. However, in order to have the numeraire "alive" for pricing derivatives with all maturities, the zero-coupon bond with the longest maturity date T_n can be chosen.

Though the drift of the ith forward rate process $f(t, T_i, T_{i+1})$ is zero under its own numeraire $P(t, T_{i+1})$ and it is non-zero under the numeraire $P(t, T_{k+1})$, with $i \neq k$. Mathematically, this can be stated as:

$$\begin{aligned} \mu_i^k(t) &\neq 0, \quad \text{for } i \neq k \\ \mu_i^k(t) = \mu_i^i(t) &= 0, \quad \text{for } i = k \end{aligned} \tag{15}$$

The solution for $\mu_i^k(t)$ when $i \neq k$, can be obtained as follows. Let the numeraire asset in equation (5) be given as $x(t) = P(t, T_{k+1})$. Then in absence of arbitrage, an equivalent measure must exist under which $y(t)$ process in equation (5) is a martingale, or:

$$E^k(dy(t)) = 0 \tag{16}$$

where,

$$y(t) = \frac{f(t, T_i, T_{i+1})P(t, T_{i+1})}{P(t, T_{k+1})} \tag{17}$$

The above equation can be simplified under three different cases as follows:

For $i > k$,

$$y(t) = f(t, T_i, T_{i+1}) \times \frac{P(t, T_{k+2})}{P(t, T_{k+1})} \times \frac{P(t, T_{k+3})}{P(t, T_{k+2})} \times \cdots \times \frac{P(t, T_{i+1})}{P(t, T_i)} = \frac{f(t, T_i, T_{i+1})}{\prod_{j=k+1}^{i} (1 + f(t, T_j, T_{j+1})\hat{U}_j)} \quad (18)$$

For $i < k$,

$$y(t) = f(t, T_i, T_{i+1}) \times \frac{P(t, T_{i+1})}{P(t, T_{i+2})} \times \frac{P(t, T_{i+2})}{P(t, T_{i+3})} \times \cdots \times \frac{P(t, T_k)}{P(t, T_{k+1})} = f(t, T_i, T_{i+1}) \prod_{j=i+1}^{k} (1 + f(t, T_j, T_{j+1})\hat{U}_j) \quad (19)$$

For $i = k$,

$$y(t) = f(t, T_i, T_{i+1}) \quad (20)$$

Using Ito's lemma on the above three equations to get the stochastic differential of $y(t)$, substituting equations (6) and (14) (under the generalized notation, $dZ_i(t) = dZ_i^i(t)$), and then equating the expectation of $dy(t)$ to zero using equation (16) gives the solutions of the drift terms under the three different cases as follows:

$$\mu_i^k(t) = \sigma_i(t) \sum_{j=k+1}^{i} \frac{\sigma_j(t)\rho_{ij}(t)f(t, T_j, T_{j+1})\hat{U}_j}{(1 + f(t, T_j, T_{j+1})\hat{U}_j)}, \quad \text{for } i > k$$

$$\mu_i^k(t) = -\sigma_i(t) \sum_{j=i+1}^{k} \frac{\sigma_j(t)\rho_{ij}(t)f(t, T_j, T_{j+1})\hat{U}_j}{(1 + f(t, T_j, T_{j+1})\hat{U}_j)}, \quad \text{for } i < k$$

$$\mu_i^k(t) = \mu_i^i(t) = 0, \quad \text{for } i = k \quad (21)$$

where $\rho_{ij}(t)$ gives the correlation between the changes in the ith and jth forward rates defined as follows:[1]

$$\rho_{ij}(t)dt = dZ_i(t)dZ_j(t) \quad (22)$$

Since the quantity $f(t, T_j, T_{j+1})\hat{U}_j/(1 + f(t, T_j, T_{j+1})\hat{U}_j)$ is always between 0 and 1, and $\sigma_i(t)$ and $\rho_{ij}(t)$ are bounded, the drifts in equation (21) remain bounded. Hence, the change of measure in equation (14) satisfies the Novikov condition of the Girsanov theorem, and all interest rate derivatives can be priced in an arbitrage-free manner using the numeraire $P(t, T_{k+1})$.

However, the drifts under the single numeraire are no longer deterministic and depend upon the current values of the forward rates. This implies that the LFM does not have lognormally distributed forward rates under a single numeraire, even though each forward rate is lognormally distributed under its own numeraire. The drift specification in equation (21) also makes the forward rate process non-Markovian. Further, since this drift specification does not allow a known distribution function for the forward rates, analytical solutions cannot be obtained for interest rate derivatives that require modeling of the joint evolution of multiple forward rate processes under a single numeraire (e.g., swaptions and many exotic interest rate derivatives, except plain-vanilla caps). In absence of analytical solutions, these derivatives may be solved using Monte Carlo simulations.

Finally, Hull and White (1999) and Rebonato (2002) have shown that an approximation of the Black formula for swaptions (known as the LSM model), allows pricing swaptions under the multi-factor LFM with a single numeraire. This approximation is not only easy to compute, but it is also quite accurate. Since most exotic interest rate products are priced off the price curves of caps and swaptions, having a single framework for pricing these vanilla products is quite useful. The details of this approximation are summarized by Nawalkha *et al.* (2007).

2.3 Specifying volatilities and correlations

There are two main problems associated with the specification of forward rate volatilities under the LIBOR market model (LMM) (i.e., the general model inclusive of both LFM and LSM). The first problem is very general and does not depend upon the chosen form of the volatility functions. This problem is inherent to the very structure of LMM, and is a consequence of the inability of the LMM to allow significant humps in the Black implied volatilities, regardless of the choice of forward rate volatility functions. The second problem is specific to the particular functional forms chosen for forward rate volatilities. Since the specification of forward rate volatilities under the LMM is designed to perfectly fit a given set of plan vanilla derivatives—such as a sequence of caps with increasing maturities—this requirement leads to time-inhomogeneous volatilities. We consider both these problems in this section. In the final part of this section, we outline some parameterizations of the correlation structure of forward rate changes given by Schoenmakers and Coffey (2003) and Doust (2007).

2.3.1 Forward rate volatilities: A general problem

Using the time zero definition of Black implied volatilities in equation (12), and rearranging terms we get:

$$\vartheta_i^2 T_i = \int_0^{T_i} \sigma_i^2(u)du \tag{23}$$

Taking the partial derivative of the left-hand side of the above equation we get:

$$\frac{\partial(\vartheta_i^2 T_i)}{\partial T_i} = 2\vartheta_i \frac{\partial \vartheta_i}{\partial T_i} T_i + \vartheta_i^2 \tag{24}$$

From the above equation it follows that:

IF

$$\frac{\partial \vartheta_i}{\partial T_i} < -\frac{\vartheta_i}{2T_i} \tag{25}$$

THEN

$$\frac{\partial(\vartheta_i^2 T_i)}{\partial T_i} < 0 \tag{26}$$

It is well known that Black implied volatility function ϑ_i is humped under normal market conditions, and hence for a range of maturities (generally, somewhere after 1.5 and 2.5 years), the partial derivative $\partial\vartheta_i/\partial T_i$ is less than zero. However, as shown by Rebonato (2002), very often this partial derivative is significantly negative such that the condition in equation (25) is satisfied and hence the inequality in equation (26) holds.

Now, assume that the forward rate volatility is time-homogenous, such that:

$$\sigma_i(t) = h(T_i - t) \tag{27}$$

The time-homogeneity assumption implies that:

$$\begin{aligned}\vartheta_i^2 T_i &= \int_0^{T_i} \sigma_i^2(u)du \\ &= \int_0^{T_i} h^2(T_i - u)du \\ &= \int_{dT_i}^{T_i+dT_i} h^2(T_i + dT_i - u)du \end{aligned} \tag{28}$$

for any infinitesimally small dT_i. Taking the partial derivative of the above equation with respect to T_i, we get:

$$\frac{\partial(\vartheta_i^2 T_i)}{\partial T_i} = \lim_{dT_i \to 0} \frac{\left(\int_0^{T_i+dT_i} h^2(T_i + dT_i - u)du - \int_0^{T_i} h^2(T_i - u)du\right)}{dT_i} \tag{29}$$

Substituting the last equality from equation (28) into the above equation, we get:

$$\begin{aligned}\frac{\partial(\vartheta_i^2 T_i)}{\partial T_i} &= \lim_{dT_i \to 0} \frac{\left(\int_0^{T_i+dT_i} h^2(T_i + dT_i - u)du - \int_{dT_i}^{T_i+dT_i} h^2(T_i + dT_i - u)du\right)}{dT_i} \\ &= \lim_{dT_i \to 0} \frac{\left(\int_0^{T_i+dT_i} h^2(T_i + dT_i - u)du - \int_{dT_i}^{T_i+dT_i} h^2(T_i + dT_i - u)du\right)}{dT_i} \\ &= \lim_{dT_i \to 0} h^2(T_i + dT_i) \\ &= h^2(T_i)\end{aligned} \tag{30}$$

Since the square of any function h has to be positive, the partial derivative of $\vartheta_i^2 T_i$ with respect to T_i can never be negative. Hence, the expression $\vartheta_i^2 T_i$ is a strictly increasing function of T_i under the LLM with a time-homogenous forward rate volatility function. As noted earlier, the *observed* term structure of Black implied volatilities is generally humped, and often the hump is significant enough such that the inequality given in equation (25) holds, making the partial derivative of $\vartheta_i^2 T_i$ with respect to T_i, negative. Thus, the LLM with a time-homogenous forward rate volatility function is frequently inconsistent with the observed implied volatilities of caps.

2.3.2 *Time-inhomogeneous evolution of forward rate volatilities*

The most significant criticism of the LMM is that it allows time-inhomogeneous evolution of forward rate volatilities in order to perfectly fit a chosen set of plain vanilla interest rate derivatives. As shown in the previous section, obtaining time-homogeneous evolution of forward rate volatilities requires that the following equation,

$$\vartheta_i^2 T_i = \int_0^{T_i} \sigma_i^2(u)du \tag{31}$$

is satisfied for every maturity T_i, while $\sigma_i(t)$ remains a function of $T_i - t$, only. As shown in the previous section, this is impossible in some scenarios, and even when it is possible, it will be too much of a coincidence that equation (31) will be satisfied for a set of ϑ_i, that are obtained by perfect calibration of the model to a set of interest rate cap prices, while keeping $\sigma_i(t)$ a function of $T_i - t$, only. In order to force equation (31) to hold, Rebonato (2002) and others suggest the following trick. Define a set of time-inhomogeneous functions $k(T_i)$ which depend upon calendar time T_i, and a time-homogeneous function $h(T_i - t)$ which depends on $T_i - t$, only, such that:

$$\vartheta_i^2 T_i = \int_0^{T_i} \sigma_i^2(u)du = k^2(T_i)\int_0^{T_i} h^2(T_i - u)du \tag{32}$$

Rebonato (2002) suggests an optimization procedure that keeps the functions $k(T_i)$ as close to 1 as possible while keeping the function $h(T_i - t)$ time-homogeneous. However, little economic justification exists for using the function $k(T_i)$ except that it allows obtaining a perfect fit with Black implied volatilities. To understand the implications of using the function $k(T_i)$, first consider the case where forward rate volatility $\sigma_i(t) = k(T_i)$, such that $h(T_i - t)$ equals 1. For this case, the volatility of a given forward rate does not change at all with the passage of time, such that a 10-year forward rate has the same volatility after 9 years, when it becomes a 1-year forward rate. Also, the volatilities of different forward rates with the *same* residual maturities, at different points in time, have different volatilities.

Of course, Rebonato's (2002) optimization procedure minimizes the effect of such economically undesirable implications of using the function $k(T_i)$. Since $\sigma_i(t) = h(T_i - t)k(T_i)$, most of the forward rate volatility is explained away by function $h(T_i - t)$. According to Rebonato (2002), the function $k(T_i)$ is generally "close to unity." However, what is defined as close to unity is left unspecified, in statistical and economic terms by Rebonato. For example, if $k(T_i)$ is really close to unity, it may reflect only trading noise, and so it should be ignored. But since $k(T_i)$ is considered significant in *economic terms* to be included as an essential part of the optimization procedure for calibrating the LLM, it obviously serves as a *smoothing* variable that captures the effects of some systematic factor(s).

For more insight, consider the values of function $k(T_i)$ given in Rebonato (2002, Figure 8.18, p. 242). From this figure, it can be seen that $k(1) = 1.03$ and $k(3) = 0.96$ (approximately) at time 0. After two years, $k(3)$ is still 0.96 (since $k(T_i)$ does not change with the passage of time), but now impacts the volatility the one-year forward rate observed at time 2. Hence, at time 0, the volatility of the one-year forward rate is increased by a factor of 1.03, and at time 2, the volatility of the one-year forward rate is decreased by a factor of 0.96. The difference in how the volatility of one-year forward rate is impacted by the function $k(T_i)$ from time 0 to time 2 changes by a factor of $0.96/1.03 = 0.93$, or by approximately 7%. In other words, 7% difference in implied volatility is explained away by the function $k(T_i)$.

2.4 Instantaneous correlations between forward rate changes

One of the redeeming features about the LMM is that it imposes few restrictions on the instantaneous correlations between changes in forward rates of different maturities. However, obtaining the correlation matrix by fitting to swaption prices (caps do not depend on correlations under the LMM) would leave too many parameters to be estimated, without ensuring whether the correlation matrix is a valid correlation matrix. To ensure that the correlation matrix is valid, Schoenmakers and Coffey (2003) suggest certain conditions that ensure that the correlation matrix admits a Cholesky decomposition. In the following, we review the Schoenmakers and Coffey (2003) method, followed by another simpler and similar method proposed by Doust (2007), that also admits a Cholesky decomposition.

Let the instantaneous correlation between the percentage changes in the forward rates $f(t, T_i, T_{i+1})$ and $f(t, T_j, T_{j+1})$ be given as $\rho_{ij}(t)$. A desirable instantaneous correlation structure should have the following properties. For all i and j equal to $0, 1, 2, \ldots, n-1$:

(1) $\rho_{ii}(t) = 1$.
(2) $\rho_{ij}(t) = \rho_{ji}(t)$.
(3) $-1 \leq \rho_{ij}(t) \leq 1$.
(4) The correlation matrix is positive semi-definite (or all eigenvalues are non-negative).
(5) $\rho_{ij}(t) = f(T_i - t, T_j - t)$.
(6) $\lim_{T_j \to \infty} \rho_{ij}(t) = \rho_\infty > 0$.
(7) $\rho_{j,j+k}(t) > \rho_{i,i+k}(t)$, for $j > i$, and $k > 0$.

The first four properties are mathematical properties of any well-defined correlation matrix. The last three properties are based on economic considerations regarding the LFM. The fifth property requires that the correlations be time-homogenous functions, and depend only on the residual maturities $T_i - t$ and $T_j - t$. The sixth property requires that asymptotic correlation defined as the correlation between the percentage changes of any forward rate and the infinite maturity forward rate is positive. The intuition behind this observation is based upon using long maturity forward rates (e.g., 10–20 years) as a proxy for the infinite maturity forward rate. Economic arguments based on the possibility of riskless arbitrage (see Dybvig *et al.*, 1996) require that infinite maturity forward rate be constant, and hence, the asymptotic correlation must be zero. On the other hand, typical applications of the LLM assume that the asymptotic correlation is positive. We do not take a strong theoretical position on this issue for the exposition of the LLM. The seventh property requires that the correlations between the percentage changes in forward rates with the same difference in maturities, should be higher for longer maturity forward rates. In other words, the correlation between the percentage changes in the 15-year and the 16-year maturity forward rates should be higher than the correlation between the percentage changes in the 1-year and the 2-year maturity forward rates.

Schoenmakers and Coffey (2003) derive a set of parametric functional forms for the correlation function. These authors begin with a finite sequence of positive real numbers given as follows:

$$a_0 < a_1 < a_2 < \cdots < a_{n-1} \tag{33}$$

such that,

$$\frac{a_0}{a_1} < \frac{a_1}{a_2} < \cdots < \frac{a_{n-2}}{a_{n-1}} \tag{34}$$

The correlations between the percentage changes in discrete forward rates are defined as follows:

$$\rho_{i,j}(t) = \frac{a_i}{a_j},$$

where

$$i \leq j, \quad \text{for all } i, j = 0, 1, 2, \ldots, n-1. \tag{35}$$

The above equation only defines the upper triangle of the correlation matrix including the diagonal elements. The elements in the lower triangle of the correlation matrix, excluding the diagonal elements, are defined as:

$$\rho_{i,j}(t) = \rho_{j,i}(t),$$

where

$$i > j, \quad \text{for all } j = 0, 1, 2, \ldots, n-2, \quad \text{and} \quad i = 1, 2, \ldots, n-1 \tag{36}$$

using the second property of a correlation matrix given earlier. The above framework allows the correlations between the percentage changes in forward rates with the same difference in maturities, to be higher for longer maturity forward rates, or $\rho_{j,j+k}(t) > \rho_{i,i+k}(t)$, for $j > i$, and $k > 0$ (see property 7). In general, the above correlations satisfy all seven properties given earlier using n number of parameters $a_0, a_1, a_2, \ldots, a_{n-1}$.

Schoenmakers and Coffey (2003) demonstrate that the above representation of correlation matrix can always be characterized in terms of a finite sequence of non-negative numbers $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$, as follows:

$$\rho_{i,j}(t) = \exp\left(- \sum_{k=i+1}^{j} (k-i)\Delta_K + \sum_{k=j+1}^{n-1} (j-i)\Delta_k \right), \quad \text{for all } i < j \tag{37}$$

where $i = 0, 1, 2, \ldots, n-2$, and $j = 1, 2, \ldots, n-1$, and,

$$\rho_{i,i}(t) = 1, \quad \text{for all } i = 0, 1, 2, \ldots, n-1 \tag{38}$$

The representation given above is neither parametric nor non-parametric. It is not parametric since the number of parameters is of the order $O(n)$, and it increases linearly with the number of forward rates. The representation is also not purely non-parametric, since that would require $O(n^2)$ number of parameters. Hence, Schoenmakers and Coffey (2003) call the above representation as *semi-parametric*. These authors also show that putting additional restrictions on the sequence of non-negative numbers $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$, leads to simple parametric forms of correlation functions as special cases of the semi-parametric correlation matrix.

Doust (2007) provides a similar and yet simpler method that admits a Cholesky decomposition for the correlation matrix. As a simple demonstration of the Doust (2007) method, consider a 4×4 correlation matrix corresponding to changes in four different forward rates of increasing maturities. Define $4 - 1 = 3$ numbers b_1, b_2, and b_3, with the only restriction as follows:

$$-1 \leq b_i \leq 1, \quad i = 1, 2, \text{ and } 3 \tag{39}$$

The correlation matrix is given as:

$$\begin{bmatrix} 1 & b_1 & b_1 b_2 & b_1 b_2 b_3 \\ b_1 & 1 & b_2 & b_2 b_3 \\ b_1 b_2 & b_2 & 1 & b_3 \\ b_1 b_2 b_3 & b_2 b_3 & b_3 & 1 \end{bmatrix} \tag{40}$$

Generalizing the above result, any $N - 1$ real numbers, which lie in between -1 and 1 are sufficient for generating a $N \times N$, positive definite correlation matrix using the Doust method.

By appropriately selecting the values of the parameters b_i, different types of decorrelation patterns can be obtained. For example, a simple one-parameter Doust model is given by the function $b_k = \exp(-\beta/k)$ for each k. If more control is required on the rate of decorrelation as a function of tenor of the forward rate, then a two-parameter Doust model can be used with the function $b_k = \exp(-\beta/k^\gamma)$ for each k.

3 The LIBOR Market Model Smiles, Too

3.1 Explaining the smile: The first approach

In the beginning of mid-1990s, a smile (or smirk) appeared in the pricing of caplets, resulting in a monotonically decreasing Black implied volatility as a function of the strike rate of the caplets. The smile in the interest rate derivative market did not seem to be related to a sudden increase in risk-aversion or "crashophobia" as it was in the equity options market.[2] Further, unless the downward jumps in the interest rates had become more likely to occur after mid-1990s, or the investors' risk-aversion against a sudden drop in interest rates had increased significantly, the caplet smile did not seem to be driven by jump-induced risk aversion/distributional effects. A more likely explanation of the appearance of the smile was the realization among traders that the lognormal forward rate distribution did not capture the forward rate dynamics properly. The lognormal distribution implies a strong dependence of the forward rate volatility on its level. For example, under the lognormal distribution, a given forward rate is twice as volatile when it is at 6% versus when it is at 3%. Though interest rate volatilities do increase with the level of the rates, the increase is not so strong as to be *proportional* to their level. Hence, a more likely explanation of the appearance of the caplet smile beginning mid-1990s is that the LFM was replaced by traders with better models which correct the misspecification of this model. Of course, this view is not shared by all researchers. For example, Jarrow *et al.* (2007) use a model with unspanned stochastic volatility and jumps to explain the caplet smile.

Two models that resolve this type of misspecification of the LFM are:

(i) models that use a general CEV process for capturing the forward rate dynamics, and
(ii) models with a displaced diffusion for the forward rate process.

In the following, we give analytical solutions to caplets under both extensions of the LFM. Before giving these solutions, we would like to make two related observations. First, as shown by Marris (1999), with suitable parameterizations, an almost perfect correspondence exists between the solutions of caplets using the CEV approach and the displaced diffusion approach over a range of strikes. Since the displaced diffusion approach is significantly easier to use analytically, this approach can be used as a numerical approximation of the CEV approach, even if the trader believed the CEV approach to be true. Second, the above two extensions are not the only approaches to fit the caplet smile. As shown by Glasserman and Kou (2000), the caplet smile can also be fitted using a jump-diffusion model with time-dependent jump intensity and jump size distribution parameters. However, the time-dependent jump approach of Glasserman and Kou introduces a highly time-inhomogenous caplet smile, such that future caplet smiles can be markedly different from current caplet smiles.

3.1.1 The CEV extension of the LFM

The CEV extension of the LFM by Andersen and Andreasen (2000) specifies the following zero-drift stochastic process for $f(t, T_i, T_{i+1})$ under the forward measure $\tilde{\mathcal{P}}^i$ defined with respect to the numeraire asset $P(t, T_{i+1})$:

$$df(t, T_i, T_{i+1}) = \sigma_i(t) f(t, T_i, T_{i+1})^{\beta} dZ_i(t), \quad 0 \leq \beta \leq 1 \qquad (41)$$

where $dZ_i(t)$ is a Wiener process under the forward measure $\tilde{\mathcal{P}}^i$, and $\sigma_i(t)$ measures the volatility of the forward rate process. The values of $\beta = 0$ and $\beta = 1$, correspond to the cases of Gaussian and lognormal forward rate dynamics, respectively. For the values of $0 < \beta < 0.5$, the above equation has a unique solution, if $f(t, T_i, T_{i+1}) = 0$ is assumed to be the absorbing barrier. For the case, $0 < \beta \leq 0.5$, the analytical solution of the probability density of $f(T, T_i, T_{i+1})$, conditional on $f(t, T_i, T_{i+1})$ is given as follows:[3]

$$p(x) = 2(1-\beta) k^{1/(2-2\beta)} (uw^{1-4\beta})^{1/(4-4\beta)} \times e^{-(u+w)} I_{1/(2-2\beta)}(2\sqrt{uw}) \qquad (42)$$

where,

$$k = \frac{1}{2v_T^2 (T-t)(1-\beta)^2}$$

$$u = k(f(t, T_i, T_{i+1}))^{2(1-\beta)}$$

$$w = kx^{2(1-\beta)}$$

I_q = the modified Bessel function of the first kind of order q, and

$$v_T = \sqrt{\frac{1}{T-t}\int_t^T \sigma_i^2(u)du}$$

The knowledge of the conditional probability density given above is useful in pricing exotic options using Monte-Carlo simulations. The analytical formula of a caplet can be obtained by using the following formula given in equation (11):

$$P_{\text{Caplet}_i}(t) = F \times \hat{U}_i \times P(t, T_{i+1}) \times E_t^i(\max[L(T_i, T_{i+1}) - K_i, 0]) \qquad (43)$$

where by definition $L(T_i, T_{i+1}) = f(T_i, T_i, T_{i+1})$. Using the conditional density given in equation (42), the expectation in the above equation can be solved to give:

$$P_{\text{Caplet}_i}(t) = F \times \hat{U}_i \times P(t, T_{i+1}) \times \left\{ \begin{array}{l} f(t, T_i, T_{i+1}) \left(1 - \chi^2 \left(2K_i^{1-\beta}; \frac{1}{1-\beta} + 2, 2u\right)\right) \\ -K_i \chi^2 \left(2u; \frac{1}{1-\beta}, 2kK_i^{1-\beta}\right) \end{array} \right\} \qquad (44)$$

where $\chi^2(x; a, b)$ is the cumulative distribution of the non-central chi-squared distribution with a degrees of freedom and parameter of non-centrality equal to b, computed at point x. The above formula implies a monotonically decreasing Black implied volatility as a function of the strike rate of the caplets. The smile becomes steeper with decreasing values of β. Hence, by appropriately choosing the parameter β, the CEV forward rate process can be calibrated to fit the caplet smile. The above formula is valid only for β values between 0 and 0.5. If $0.5 < \beta < 1$, then Monte Carlo simulations must be used to price caplets.

Recall that one of the appealing features of the LFM is that it allows a perfect fit with the at-the-money caplets using the three-step method outlined by Rebonato (2002). Unfortunately, in the presence of a caplet smile, a perfect fit with the caplets of all maturities and all strikes cannot be obtained. Hence, one must choose the β value, which minimizes the deviations of the model caplet prices from the actual caplet prices.

3.1.2 Displaced diffusion extension of the LFM

As mentioned earlier, with suitable parameterizations, an almost perfect correspondence exists between the solutions of caplets using the CEV approach and the displaced diffusion approach over a range of strikes. Hence, the displaced diffusion model can be used as a numerical approximation to the caplet prices even when the trader believe the CEV approach to be true. Using the displaced diffusion framework, the forward rates are defined as follows:

$$f(t, T_i, T_{i+1}) = \delta + Y_i(t) \tag{45}$$

where the stochastic process of the state variable $Y_i(t)$ is given as follows:

$$\frac{dY_i(t)}{Y_i(t)} = \sigma_{yi}(t)dZ_i(t) \tag{46}$$

Using Ito's lemma, the stochastic process of the forward rate is given as follows:

$$df(t, T_i, T_{i+1}) = \sigma_{yi}(t)(f(t, T_i, T_{i+1}) - \delta)dZ_i(t) \tag{47}$$

Taking the stochastic integral of equation (46), and then substituting equation (45), the forward rate at time T (such that $t \leq T < T_i < T_{i+1}$) can be represented as follows:

$$\begin{aligned} f(T, T_i, T_{i+1}) &= \delta + (f(t, T_i, T_{i+1}) \\ &\quad - \delta)e^{-(1/2)\int_t^T \sigma_{yi}^2(v)dv + \int_t^T \sigma_{yi}(v)dZ_i(v)} \end{aligned} \tag{48}$$

Hence, the forward rates follow a shifted lognormal distribution. To solve for caplet prices note that substituting $f(T_i, T_i, T_{i+1}) = L(T_i, T_{i+1}) = \delta + Y_i(T_i)$, in the caplet valuation formula in equation (43), gives:

$$\begin{aligned} P_{\text{Caplet}_i}(t) &= F \times \hat{U}_i \times P(t, T_{i+1}) \\ &\quad \times E_t^i(\max[\delta + Y_i(T_i) - K_i, 0]) \\ &= F \times \hat{U}_i \times P(t, T_{i+1}) \\ &\quad \times E_t^i(\max[Y_i(T_i) - K_{yi}, 0]) \end{aligned} \tag{49}$$

where,

$$K_{yi} = K_i - \delta \tag{50}$$

Since $Y_i(T_i)$ is lognormally distributed, the expectation in equation (49) has the same form of solution as under the LFM, and can be obtained by a simple inspection of equations (10) and (12). Hence, the caplet price is given as follows:

$$\begin{aligned} P_{\text{Caplet}_i}(t) &= \text{Black}_{\text{Caplet}_i}(f(t, T_i, T_{i+1}) \\ &\quad - \delta, K_i - \delta, T_i - t, \vartheta_{yi}) \\ &= F \times \hat{U}_i \times P(t, T_{i+1}) \\ &\quad \times \{(f(t, T_i, T_{i+1}) - \delta)\mathcal{N}(d_{1,i}) \\ &\quad - (K_i - \delta)\mathcal{N}(d_{2,i})\} \end{aligned} \tag{51}$$

where,

$$d_{1,i} = \frac{\ln((f(t, T_i, T_{i+1}) - \delta)/(K_i - \delta)) + \vartheta_{yi}^2(T_i - t)/2}{\vartheta_{yi}\sqrt{T_i - t}}$$

$$d_{2,i} = \frac{\ln\left((f(t, T_i, T_{i+1}) - \delta)/(K_i - \delta)\right) - \vartheta_{yi}^2 (T_i - t)/2}{\vartheta_{yi}\sqrt{T_i - t}}$$

where ϑ_{yi} is defined as follows:

$$\vartheta_{yi} = \sqrt{\frac{1}{T_i - t}\int_t^{T_i} \sigma_{yi}^2(u)du} \qquad (52)$$

The above caplet formula is identical to the corresponding formula under the LFM given in equation (10), except for the term δ, which allows for fitting the caplet smile. In general, negative values of δ allow fitting a monotonically decreasing caplet smile.

A potential criticism of the displaced diffusion model is that negative values of δ allow for the occurrence of negative forward rates. However, for the range of values of δ required for fitting the caplet smile, the probability of occurrence of negative rates is extremely low. Hence, for most practical purposes the displaced diffusion model works well in capturing the caplet smile. Further, since a direct one to one correspondence exists between the displaced diffusion model and the CEV extension of the LFM (see Marris, 1999), the above formula can also be used as an analytical approximation of the latter model, with a suitable parameterization.

In order to allow maximum generality that allows pricing of swaptions also, we model the forward rates using a single numeraire under the displaced diffusion model. Similar to the single numeraire-based LFM (see equations (13) and (21)), the joint dynamics of forward rates under the displaced diffusion model are given by transforming equations (46) and (47), using a change of measure, as follows:

$$\frac{dY_i(t)}{Y_i(t)} = \mu_{yi}^k(t)dt + \sigma_{yi}(t)dZ_i^k(t) \qquad (53)$$

and,

$$df(t, T_i, T_{i+1}) = (f(t, T_i, T_{i+1}) - \delta)(\mu_{yi}^k(t)dt + \sigma_{yi}(t)dZ_i^k(t)) \qquad (54)$$

where,

$$\mu_i^k(t) = \sigma_{yi}(t) \times \sum_{j=k+1}^{i} \frac{\sigma_{yj}(t)\rho_{ij}(f(t, T_j, T_{j+1}) - \delta)\hat{U}_j}{(1 + f(t, T_j, T_{j+1})\hat{U}_j)}, \quad \text{for } i > k$$

$$\mu_i^k(t) = -\sigma_{yi}(t) \times \sum_{j=i+1}^{k} \frac{\sigma_{yj}(t)\rho_{ij}(f(t, T_j, T_{j+1}) - \delta)\hat{U}_j}{(1 + f(t, T_j, T_{j+1})\hat{U}_j)}, \quad \text{for } i < k$$

$$\mu_i^k(t) = \mu_i^i(t) = 0, \quad \text{for } i = k \qquad (55)$$

Since individual caplets comprising the cap can be priced using different numeraires, the lognormal assumption can be maintained for different forward rates (see equations (46) and (47)) for pricing caps. However, swaptions and coupon bond options represent *options on portfolios* (and not portfolios of options, like caps), and hence the joint stochastic evolution of different forward rates must be modeled for pricing these instruments. Modeling the joint evolution of forward rates requires that all forward rate processes are measured under a single forward measure and are given by equations (54) and (55), respectively.

3.2 Capturing the smile using the LMM-SABR model

Both the CEV and displaced-diffusion extensions of the LFM can only allow monotonically decreasing smiles. However, since 1998, the caplet smile has taken more complex shapes, decreasing first for a wide range of strikes and then increasing over a short range of strikes, resembling the "hockey-stick shape." A number of stochastic volatility models have been proposed in the literature to explain the hockey-stick shaped smile, including Andersen and Andreasen (2000), Andersen and Brotherton-Ratcliffe (2001), Hagan *et al.* (2002), Wu and Zhang (2002), Joshi and Rebonato (2003), Piterbarg (2003, 2005), Jarrow *et al.* (2007), Rebonato and Kainth (2004), Rebonato and White (2009), Rebonato and McKay (2009), and Rebonato *et al.* (2009).

For various practical reasons, the SABR model of Hagan, Kumar, Lesniewski, and Woodward (2002) (HKLW) has been adopted by the industry as the new market standard for pricing plain-vanilla derivatives, such as caps and swaptions. The attractive feature of this model is that it allows stochastic volatility factors, without increasing the computational burden significantly. The original version of the SABR model provides an analytical approximation only for pricing caps. Rebonato and White (2009) extend the SABR model using a single numeraire, such that the extended model called the LLM-SABR model provides analytical approximations for swaptions, and links the prices of caps and swaptions in a unified framework.

Under the SABR model of HKLW, the forward rate $f(t, T_i, T_{i+1})$ follows the following stochastic process under its own measure:

$$df(t, T_i, T_{i+1}) = f(t, T_i, T_{i+1})^{\beta_i}\sigma_i(t)dZ_i(t) \quad (56)$$

and the volatility process is given under the same measure as follows:

$$d\sigma_i(t) = \sigma_i(t)v_i dW_i(t) \quad (57)$$

where, $dZ_i(t)dW_i(t) = \rho_i dt$, and the initial values of the forward rate and its volatility are given as $f(0, T_i, T_{i+1})$ and $\sigma_i(0)$, respectively. As shown by HKLW, the price of a caplet under the SABR model can be approximated by the Black formula in equation (10), with Black implied volatility ϑ_i replaced as follows:

$$\vartheta_i = A\left[\frac{x}{\chi(x)}\right]B \quad (58)$$

where,

$$A = \frac{\sigma_i(0)}{(fK)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta_i)^2}{24}\ln^2\left(\frac{f}{K}\right) + \frac{(1-\beta_i)^4}{1920}\ln^4\left(\frac{f}{K}\right) + \cdots\right]} \quad (59)$$

$$B = \left[1 + \left(\frac{(1-\beta_i)^2}{24}\frac{\sigma_i^2(0)}{(fK)^{1-\beta}} + \frac{\rho_i\beta_i v_i\sigma_i(0)}{4(fK)^{(1-\beta)/2}} + \frac{2-3\rho_i^2}{24}v_i^2\right)T_i + \cdots\right] \quad (60)$$

$$x = \frac{v_i}{\sigma_i(0)}(fK)^{(1-\beta_i)/2}\ln\left(\frac{f}{K}\right) \quad (61)$$

$$\chi(x) = \ln\left(\frac{\sqrt{1-2\rho_i x + x^2} + x - \rho_i}{1-\rho_i}\right) \quad (62)$$

and the initial forward rate is redefined as $f = f(0, T_i, T_{i+1})$, and caplet strike price equals K.

The main advantage of the above formula is that it imposes little additional computational burden to price a caplet, than imposed by the deterministic volatility LIBOR market model. The approximation works well as long as one is not valuing caplets with strikes that are too out-of-money, and expiration dates that are not too distant in the future. The formula captures the CEV nature of the forward rate process using the parameter β_i, it allows instantaneous correlation between the forward rate process and its volatility using the parameter ρ_i, and it allows volatility of volatility using the parameter v_i. Allowing stochastic volatility makes this model more versatile in capturing the hockey-stick shaped smile in pricing caps.

Though the SABR model is useful for pricing caps, it does not allow pricing of swaptions, since forwards rates and volatility processes are defined under the forward-rate specific measure, while pricing swaptions require all forward rate processes and volatility processes to be defined under a common measure. Rebonato and White (2009) and Rebonato *et al.* (2009) extend the SABR model to allow a common measure using the LMM framework. The combined model called the LMM-SABR model derives forward rate drift adjustments and volatility drift adjustments as follows.

To allow for a single numeraire, reconsider the forward rate process and the volatility process given in equations (56) and (57), respectively, in a more general form, given under a common forward measure $\tilde{\mathcal{P}}^j$ associated with a single numeraire asset $P(t, T_{j+1})$, as follows:

$$df(t, T_j, T_{j+1}) = \sigma_j(t, \boldsymbol{f}(\boldsymbol{t}))k_j(t)dZ_j(t) \quad (63)$$

$$dk_j(t) = v_j(t, \boldsymbol{k}_t)dW_j(t) \quad (64)$$

where, $\boldsymbol{f}(\boldsymbol{t}) = [f(t, T_0, T_1), f(t, T_1, T_2), \ldots]$, and $\boldsymbol{k}_t = [k_1(t), k_2(t), \ldots]$, both are vectors containing all forward rates and stochastic volatilities, respectively. Equations (63) and (64) are significant generalizations of equations (56) and (57), respectively, and they apply not only to the LLM-SABR model, but also to many other stochastic volatility extensions of the LLM given in the literature. The above processes have zero drifts, because the measure corresponding to them is also the common forward measure $\tilde{\mathcal{P}}^j$. In general, the stochastic processes for all other forward rates, $f(t, T_i, T_{i+1})$, for all $i \neq j$, and all other volatilities, $k_i(t)$, for all $i \neq j$, will have non-zero drifts. The stochastic processes for these forward rates and volatilities, under the common measure $\tilde{\mathcal{P}}^j$ can be given as follows:

$$df(t, T_i, T_{i+1}) = \mu_i(t, \boldsymbol{f}(\boldsymbol{t}), \boldsymbol{k}(\boldsymbol{t}))dt + \sigma_i(t, \boldsymbol{f}(\boldsymbol{t}))k_i(t)dZ_i^j(t) \quad (65)$$

$$dk_i(t) = \eta_i(t, \boldsymbol{f}(\boldsymbol{t}), \boldsymbol{k}(\boldsymbol{t}))dt, + v_i(t, \boldsymbol{k}_t)dW_i^j(t) \quad (66)$$

for all $i = 0, 1, 2, \ldots n$.

where,

$$dZ_p^j dZ_q^j = \psi_{pq}$$
$$dW_p^j dW_q^j = r_{pq}$$
$$dZ_p^j dW_q^j = \rho_{pq}$$

for all $p, q = 0, 1, 2, \ldots, n$.

By applying a change of measure using the Girsanov theorem, the drift specification in equations (65) and (66), can be derived from the stochastic processes followed by these variables *under their own measures* (recall that these variables will have zero drifts under their own measures, but non-zero drifts under the measure $\tilde{\mathcal{P}}^j$), following similar lines of arguments as given for the LFM model in equations (13) through (21). Using this line of reasoning, Rebonato and White (2009) derive the form the drifts given in equations (65) and (66). They also derive an approximate formula for swaptions using the specification of drifts under a common numeraire.

Rebonato, McKay, White (RMW) (2009) provide an extensive analysis of the LMM-SABR model, and show how to calibrate the model to the prices of caps and swaptions. This is not as straightforward as the traditional LMM, since the number of parameters and state variables under the LMM-SABR model are significantly higher. The functional form of the volatilities of volatilities, the correlations between different forward rate changes, the correlations between forward rate changes and volatilities, and the correlations between different volatilities, all enter as additional inputs under the LMM-SABR model. RMW find that LMM-SABR model succeeds in explaining the non-monotonic smiles observed in the caps and swaptions markets. Of course, this is not surprising as given the number of parameters and state variables that enter into the LMM-SABR model.

4 A Critique of the LMM and the LMM-SABR Model

The following critique of the LMM and the LMM-SABR model is divided into two parts. The first part questions the assumption of zero drifts for the volatility processes under the LMM-SABR model. The second part reconsiders the LMM and the LMM-SABR model from the perspective of a partially hedged trader or an investor/speculator who must evaluate risk-return trade-offs. We show that due to the nature of calibrations used under the LMM and the LMM-SABR model, it is virtually impossible to perform risk-return analysis, which requires information from the physical measure.

4.1 Zero drifts for the volatility processes

The development of the simple formula for pricing caps using the SABR model by HKLW (2002) is reminiscent of the development of Black formula for pricing caps in the late 80s and early 90s. In both instances, the formulas were guided by heuristic derivations, and the fuller development of the economic reasoning which made these formulas consistent with absence of arbitrage, were derived later by other researchers. In the case of Black formula for pricing caps, Brace *et al.* (1997), Jamshidian (1997), and Miltersen *et al.* (1997) demonstrated why using a zero drift for the *discrete* forward rate process under its own measure was economically justified, since the discrete forward rate process could be represented as a discounted traded asset. By Martingale valuation theory, absence of arbitrage guarantees the existence of a measure under which the discounted traded asset has a zero drift.

However, HKLW use a zero drift under the SABR model not only for the forward rate process *but also for the volatility process*. But volatility is not a traded asset, and hence, Martingale valuation theory cannot be invoked to allow a zero drift in equation (57). Obviously, this is done solely for the convenience of obtaining the SABR approximation formula for Black implied volatility in equation (58). With a non-zero mean reverting drift for the volatility process, such a simple approximation formula cannot be obtained. Rebonato and White (2009) and Rebonato *et al.* (2009) also assume zero drift for the volatility process in equation (64), and the use this specification to obtain non-zero drifts in equation (66). These authors maintain zero drifts for the volatility process under the forward rate's own measure, to allow the formula in equation (58) to be consistent with the more general framework they present for pricing swaptions and other complex interest rate derivatives using a common numeraire.

Using a zero drift for the volatility process could be considered as a theoretical issue of not much practical significance, except that volatility is the most important input for pricing options, and to make an erroneous assumption about volatility can lead to serious errors in option prices. To appreciate this argument consider Heston's model for equity option pricing. Let $y(t) = S(t)/B(t)$ represent the stock price discounted by the money market account. The stochastic processes for the discounted stock price and the volatility are given under the physical measure as follows:

$$\frac{dy(t)}{y(t)} = (\mu - r)dt + \sqrt{v(t)}dZ_1(t) \tag{67}$$

$$dv(t) = \alpha(m - v(t))\,dt + \sigma\sqrt{v(t)}dZ_2(t) \tag{68}$$

where μ is the physical stock return, and r is the riskless rate. Now, consider the change of measure, such that the stochastic processes under the risk-neutral measure are given as follows:

$$dy(t)/y(t) = \sqrt{v(t)}dZ_1^Q(t) \tag{69}$$

$$dv(t) = \sigma\sqrt{v(t)}dZ_2^Q(t) \tag{70}$$

The transformation of equation (67) to equation (69) is justified by absence of arbitrage, allowing a zero drift for the discounted stock price process under the risk-neutral measure, but the transformation of equation (68) to equation (70) is not justified by absence of arbitrage alone, and forcing this transformation implies a very peculiar form of the market price of volatility risk $\gamma(t)$, given by the change of measure as follows:

$$dZ_2^Q(t) = dZ_2(t) + \gamma(t)dt \tag{71}$$

where

$$\gamma(t) = \frac{\alpha m}{\sigma\sqrt{v(t)}}dt - \frac{\alpha}{\sigma}\sqrt{v(t)}dt$$

Unless there is a valid empirical justification for using the above form of market price of volatility risk, the option prices produced by the Heston model will be wrong. Since volatilities are almost always mean reverting, both under the physical measure and the risk-neutral measure, equation (70) does not have any practical justification. A similar criticism applies to using zero drifts in the volatility processes in equations (57) and (64) for the SABR and the LMM-SABR models, respectively. Unless market prices of volatility risks are of a very peculiar form, zero-drifts in equations (57) and (64) cannot be obtained from the physical processes corresponding to these volatilities.

Again, note that this is not a theoretical issue of little practical significance. The volatility processes used in equations (57) and (64) will either overprice long-term options, or underprice short-term options, since volatilities are non-stationary and do not revert to a long-term mean. Also, as RMW (2009) point out, the expectation of the volatility generated by equation (57) (which is what conceptually, enters the pricing of caps) consists of many "low" paths, and a very few close to explosive paths. When volatility is high, this leads to problems in numerical convergence and stability of the model.

A more serious issue related to using zero-drifts in the volatility processes has to with calibrating the LMM-SABR model using time-homogeneous functions for volatilities. The following quote by RMW (2009, p. 59) is quite revealing in this regard,

> "This is when time-homogeneous pricing models, which by and large we like, get into trouble. By construction they assume that calendar time is irrelevant, and that the only variable that matters to determine volatilities, correlations, and volatilities of volatilities is the residual time to maturity of the forward rate(s). For time-homogeneous models, a one-year option seen as of today will pretty much "look" and behave like a one-year option in five year or ten years' time. But if today we are in an excited state, this means that a time-homogeneous model fitted to short dated-options will propagate the current state of excitation *ad infinitum.* As a consequence the one-year option in ten years' time will be priced as if the current state of turmoil will still be present, say, ten years from now. What in normal times is a virtue, in excited periods therefore becomes a serious shortcoming."

Note that what is considered a serious shortcoming of "time-homogeneous pricing models" by RMW (2009), is actually a serious shortcoming of *using zero drifts in equations* (57) *and* (64). These zero drifts have no economic justification, and if one were to use drifts with mean reversion, which is in general a property of the volatility processes used in other areas of derivatives (e.g., Heston model for pricing equity options), then even if today we are in an excited state, *it will not imply that a time-homogeneous model fitted to short dated-options will propagate the current state of excitation ad infinitum.* In other words, since volatility must revert towards its long term mean (due to mean reversion), the current state of excitation will not be propagated ad infinitum.

4.2 *Risk-return analysis under the LMM and the LMM-SABR model*

To evaluate the usefulness of the LMM and the LMM-SABR model, we consider two types of participants in the interest rate derivatives market. The first participant is a hedged trader or market maker who maintains zero (or close to zero) exposure to the underlying factors driving the interest rate market, and trades to exploit arbitrage opportunities, thereby providing much needed liquidity to this market. The second participant is a partially hedged trader or an investor/speculator, who maintains significant exposure against the underlying interest rate factors by either only partially hedging using interest rate derivatives, or speculating using interest rate derivatives. The interest rate derivative desks of many investment banks and larger universal banks generally play the role of a hedged trader or market maker. On the other hand, most commercial banks, bond funds, fixed income hedge funds, and other corporations generally act like partially hedged traders, and occasionally as investors/speculators in this market.

The usefulness of an interest rate model for both types of participants in this market, can be measured using the following three objectives:

(i) Valuation of plain vanilla interest rate derivatives,
(ii) Hedging plain vanilla derivatives, and valuation and hedging of complex interest rate derivatives,
(iii) Risk-return analysis.

If the only objective is the valuation of plain vanilla interest rate derivatives—for example to do *mark-to-market accounting*—then a variety of fundamental models, single-plus models, double-plus models, and triple-plus models can be used. By either introducing time-dependencies or allowing more factors, a host of these models can value plain vanilla interest rate derivatives with similar level of accuracy, despite the fact that they may make vastly different assumptions about volatilities and correlations.

If both the first two objectives must be satisfied—for example, in the case of a hedged trader or a market maker—then it is important to use only those models, which have realistic dynamics of volatilities and correlations. The LMM and the LMM-SABR model have been proposed as models that can satisfy the first two objectives, even though these models allow some degree of time-inhomogeneous volatilities, and assume zero drifts for the volatility processes in the case of the LMM-SABR model.

However, if all three objectives must be satisfied—for example, in the case of a partially hedged trader or an investor/speculator—then both the LMM and the LMM-SABR model fail, as these models cannot satisfy the third objective. This is because it is virtually impossible to determine the *physical* evolution of the state variables under the LMM and the LMM-SABR model. Due to the nature of calibrations, these models do not distinguish between state variables and parameters of the model, they use time-inhomogeneous volatilities, and they allow model inputs to change period by period. Since estimation is done mainly for the purpose of obtaining risk-neutral parameters that perfectly fit the cross-section of market prices, it is difficult to apply time-series econometric techniques, which require time-homogeneity assumptions, to infer the market prices of interest rate risk and volatility risk. Since the nature of physical processes under these models remain unknown, meeting the third objective of risk/return analysis is virtually impossible under these models.

Hence, an unhedged trader or an investor/speculator who wants to make risk/return decisions cannot rely on the LMM and the LMM-SABR models for making any meaningful analysis. It is interesting to note that many investment banks that routinely sell interest rate derivatives to partially hedged traders and/or investors/speculators, are well-protected using the LMM and the LMM-SABR model, since their interest rate derivatives desks are totally hedged, but their clients who must make risk/return decisions (since they are *not* totally hedged), have little guidance on how to do this using these time-inhomogeneous models with erroneous assumptions about the volatility processes.

To emphasize the importance of the third objective of doing risk/return analysis, note that this was sorely lacking in the recent years in the credit derivatives market. At the height of the credit bubble, fixed income quants touted a range of credit derivative valuation models that used risk-neutral information implied by the market prices of plain vanilla credit derivatives to value other credit derivatives. So, if the value of a credit default swap (CDS) implied an extremely low risk-neutral probability of default, then that became a *valid input* for pricing other credit derivatives related to the security (or securities) underlying the CDS. This type of relative valuation modeling, where market implied parameters are used without any fundamental analysis, is at the heart of not only the current financial crisis, but also other crises in history where analysts price whatever asset is in vogue using parameters implied by valuation of similar assets. Hence, valuation and hedging are only necessary, but not sufficient conditions for a good model. A good model should not only allow valuation and hedging, which can be done under the risk-neutral measure, *but also allow risk/return analysis under the physical measure*. Just in case the U.S. Treasury market is in a bubble right now, such risk/return analysis could prove very useful for investors in the interest rate derivatives market, in the near future.

5 Alternatives to the LMM and the LMM-SABR Model

Given the critique of the LMM and the LMM-SABR model in the previous section, practitioners may consider using TSMs in the affine and quadratic classes, which allow analytical solutions. NBS (2007) derive single-plus and double-plus extensions of virtually all known affine and quadratic models in the term structure literature. They also test the double-plus versions of commonly used three-factor models in the affine class, using caps and swaptions data from 2007, and report good performance of some of these models. NBS do not consider any triple-plus models due to the high degree of smoothing resulting from two sources of time-inhomogeneity under these models.

Moreover, unlike the very high errors for fundamental quadratic models reported by Li and Zhao (2006), NBS find that double-plus versions of quadratic models with even less flexibility (i.e., with orthogonal factors and no interdependencies through the drift terms) have only a fraction of the error of the fundamental quadratic models with more flexibility (i.e., with correlated factors and interdependencies through the drift terms). For example, NBS report average RMSEs of approximately 5 percent for the preference-free $Q_3(3)++$ model, which are significantly lower than the average RMSEs of 44 percent, 31 percent, and 15 percent for the fundamental $Q_3(3)$, $Q_2(3)$, and $Q_1(3)$ models, respectively (computed by averaging the RMSEs in each of the Panels A, B, and C of Table 5 in Li and Zhao (2006)).

Rebonato and Cooper (1995) criticize low-dimensional short rate models such as those in the affine class, because these models cannot capture the realistic decorrelation patterns found in the empirically observed correlation structures of forward rate changes. However, this criticism can be addressed by considering high-dimensional models in the affine and quadratic classes. For example, NBS provide a range of *simple*[4] $A_M(N)++$ models with M square root factors and $N-M$ Gaussian factors, under which Gaussian factors can have arbitrary correlations, and the models allow stochastic volatility by construction. They also provide preference-free $Q_3(N)++$ in the quadratic class with N factors. Under all these models, the prices of caplets can be computed using only one numerical integral using the method of Fourier inversion (regardless of the number of factors), and the prices of swaptions can be computed *without* any numerical integrals using the cumulant expansion method introduced by Collin-Dufresne and Goldstein (2001) and generalized by NBS (2007). Hence, the double-plus models in the affine and quadratic classes which have performed well (i.e., the $A_0(3)++$ extension of Hull and White (1996) and the $Q_3(3)++$ model of NBS (2007)) can be extended to allow many factors, such that they can fit the empirically observed correlation structures. Since fast numerical schemes have been developed to price caplets and swaptions under an arbitrary number of factors, simple $A_M(N)++$ models and $Q_3(N)++$ models represent potential alternatives to the LMM for valuing and hedging these plain-vanilla derivatives. Further, since probability distribution of the state variables are available in semi-analytical form under the simple $A_M(N)++$ models and $Q_3(N)++$ models, fast Monte Carlo methods can be used to value path-dependent options under these high-dimensional models.

6 Conclusions

This paper presented a critical review of the LIBOR market model (LMM) and the LMM-SABR model. Using the new taxonomy of term structure models by NBS (2007), we show the popular versions of the LMM and LMM-SABR model are *triple-plus* models. These models are calibrated to market prices by allowing time-inhomogeneous volatilities, and by changing numerous model inputs, period by period. Changing the model period by period and using time-inhomogeneous volatilities make risk-return analysis impossible under the physical measure. Further, this paper demonstrated that both the SABR model and the LMM-SABR model are based on the highly questionable assumption of zero drifts for the volatility processes (under the forward rate specific measures), which has no economic justification, and can lead to explosive behavior for volatilities. Finally, this paper recommends using high-dimensional affine and quadratic models that allow fast analytical approximations (such as the Fourier inversion method and the cumulant expansion method) for pricing caps and swaptions, as alternatives to the LMM and the LMM-SABR model.

Notes

1 Since the correlation does not depend on the specific numeraire being used, the Wiener processes in the above equation are written without the superscripts.

2 The smile first appeared in the equity options market after the equity market crash of 1987. The shape of the smile became more pronounced in a few years after the crash, which made the implied volatilities of out-of-the-money put options significantly higher than the implied volatilities of at-the-money put options. Increases in the aversion to risk of another equity market crash, or a reassessment of the probabilities around the tail of the stock return distribution (caused by sudden downward jumps in returns in 1987) were likely explanations of the smile in the equity options market.

3 See Brigo and Mercurio (2001).

4 These are defined as "simple" $A_M(N)++$ models by NBS (2007), because they do not allow correlations between the square root processes and the Gaussian processes.

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Keywords: LIBOR model; SABR model; LMM; caps; swaptions; interest rate derivatives