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THE FUNDAMENTAL LAW OF ACTIVE PORTFOLIO MANAGEMENT

Roger Clarke[a], *Harindra de Silva*[a] *and Steven Thorley*[b,*]

The strategic perspectives and terminology of the fundamental law is a common framework in the practice of active portfolio management. For tractability, fundamental law theory depends on the simplifying assumption of a diagonal covariance matrix of security returns, though the matrices supplied to numerical optimizers are fully populated. We extend the fundamental law of active management to allow for a full covariance matrix and show that the resulting ex-ante (expected) and ex-post (realized) return equations are exact in contrast to the approximate equality of previous derivations. The exactness of ex-post equations allows for performance attribution of realized returns that completely decomposes the return. Because the various fundamental law parameters we define incorporate all the information in the covariance matrix, they should also provide better ex-ante insights as to the sources and limitations of risk-adjusted active return. In addition to the generalization of the fundamental law, we describe a full covariance matrix alpha generation process and add some comments to the concept of implied breadth. The mathematics and practical application of the full covariance matrix fundamental law parameters are illustrated using an EAFE benchmarked portfolio with the 21 countries as individual securities.

The fundamental law of active management continues to be developed and explored as a central theme in the science of portfolio management. The fundamental law, first articulated by Grinold (1989) and developed throughout the book by Grinold and Kahn (1994), has seen further refinements by Clarke *et al.* (2002), Qian and Hua (2004), and others.[1] The fundamental law is the basis of numerous research papers on investment management as well as the underlying rationale for many active portfolio strategies.

Grinold (1989) and Clarke *et al.* (2002) (hereafter CST) provide equations for ex-ante (expected) and

[a]Analytic Investors, 500 South Grand Ave., Suite 2300, Los Angeles, CA 90071.

[b]Marriott School, Brigham Young University, Provo, UT 84602, USA.

*Corresponding author. Tel.: +1-801-422-6065; e-mail: steven.thorley@byu.edu

ex-post (realized) portfolio performance under the simplifying assumption of a diagonal covariance matrix. We relax the assumption of uncorrelated residual security returns and produce fundamental law equations that retain their basic form. We also apply full covariance matrix mathematics to the generation of security alphas as an extension of Grinold (1994) and provide some insights into the concept of breadth. For econometricians, the innovations in this paper can be thought of as a generalized-least-squares (GLS) extension of a theoretical framework that previously corrected for heteroskedasticity through weighted-least-squares, but ignored known cross-correlations in the residuals.

We write from the perspective of an investment process where the risk model and signal are already established. The risk model takes the form of an estimated security return covariance matrix while the signal is some proprietary evaluation process the portfolio manager uses to assign forecasted or expected returns to each security. We take as given that the objective of portfolio management is to add value to a passive benchmark index. Although we speak in terms of active return and risk with respect to an equity market index, the mathematics also apply to the special case of a market-neutral long/short strategy where the benchmark is the risk-free rate.

The outline of the paper is as follows. We first specify an active portfolio objective function and find a closed-form solution for the optimal security weights. We then develop the fundamental law theory under the generalized assumption of a full covariance matrix in four stages: ex-ante and ex-post equations for both unconstrained and constrained portfolios. Next, we discuss full covariance matrix implications for the alpha generation process and the implied breadth. Finally, we illustrate various fundamental law concepts with a numerical example based on the EAFE index and its 21 constituent country returns in September 2004.

1 Objective function

Before making some specific comments about terminology and the objective function, we introduce some key algebraic symbols. We use Ω (omega) for the $N \times N$ security return covariance matrix, α (alpha) for the $N \times 1$ forecasted return vector, r for the $N \times 1$ realized return vector, and w for the $N \times 1$ active weight vector. Individual elements in the vectors are shown by the respective vector symbol with an i subscript. Nondiagonal elements of the square covariance matrix Ω are $\sigma_{i,j}$ and diagonal elements (residual return variances) are σ_i^2. Other notational choices and the use of terminology will be explained in the development of the theory.

The standard mean–variance optimization problem in portfolio management is to adjust the choice of security weights to maximize a utility function that increases in expected active return and decreases with active risk, with the tradeoff based on a risk-aversion coefficient, λ;

$$U = E(R_{\mathrm{A}}) - \lambda\sigma_{\mathrm{A}}^2 \tag{1}$$

where $E(R_{\mathrm{A}})$ is the portfolio's expected active return and σ_{A} is the portfolio's ex-ante active risk. Using the previously defined symbols, the matrix-algebra form of Eq. (1) is

$$U = \alpha' w - \lambda w' \Omega w \tag{2}$$

where the apostrophe indicates the transpose function.

Although similar in form to the general mean–variance optimization problem in portfolio theory, several aspects of the objective function in Eq. (2) should be noted. First, the security weights in the optimization are over- or underweights with respect to the benchmark, sometimes referred to as *active* weights. In other words, the elements of the weight vector w sum to zero because they represent the differences between the weights of each security in the managed and benchmark portfolios. Second, the individual security returns of concern in Eq. (2) are residual to the general market return and other factors that the portfolio manager is not attempting to forecast. For example, consider the basic one-factor market model for the cross-section of security returns in a given period,

$$R_i = \beta_i R_M + r_i \tag{3}$$

where R_i is a security return in excess of the risk-free rate, R_M is the market-wide excess return, and β_i is the security beta or sensitivity to the market return.[2] The security returns of interest in our optimization problem, both expected and realized, are the residual returns in Eq. (3). Thus, the $N \times 1$ expected residual security return vector α represents the manager's forecast of the $N \times 1$ residual security return vector r.

The focus on residual security returns means that the expected active portfolio return specified in the objective function, $E(R_A) \equiv \alpha' w$, is not just the difference between the managed and benchmark portfolio returns. Instead, we refer to the simple difference between the managed portfolio return, R_P, and the benchmark portfolio return, R_B, as the portfolio's *relative* return; $\Delta R \equiv R_P - R_B$. The relative and active portfolio returns are related by

$$\Delta R = (\beta_P - \beta_B) R_M + R_A \tag{4}$$

where β_P is the market-beta of the managed portfolio and β_B is the market-beta of the benchmark. As always, portfolio returns and betas are calculated as the market-weighted average of the securities that make up the portfolio. In the special case of a one-factor model where the market and benchmark portfolios are identical, Eq. (4) becomes $\Delta R = (\beta_P - 1) R_B + R_A$ and the active portfolio return is the beta-adjusted return on the managed portfolio, $R_A = R_P - \beta_P R_B$. Alternatively, a more general case of Eq. (3) allows for a multifactor definition of residual security returns where the set of market-wide factors might also include size (exposure to large vs. small capitalization securities), value (exposure to high vs. low book to market ratio securities), or other factors specifically excluded from the manager's forecast of individual security returns.

Consistent with the focus on the active portfolio return, the portfolio risk parameter in the objective function, σ_A, is *active* as opposed to *relative* (i.e., benchmark tracking error) risk. Specifically, the covariance matrix, Ω, in Eq. (2) is for security returns that are residual to the set of unforecasted market-wide factors.[3] For example, under the one-factor market model for security returns shown in Eq. (3), the relative risk or benchmark tracking error of the managed portfolio, $\sigma^2_{\Delta R}$, is related to active risk by

$$\sigma^2_{\Delta R} = (\beta_P - \beta_B)^2 \sigma^2_M + \sigma^2_A \tag{5}$$

where σ^2_M is the estimated variance of the market portfolio. Thus, active risk in the objective function is only equivalent to tracking error if the managed portfolio is constrained to have a market beta equal to the benchmark (i.e., $\beta_P = \beta_B$ in Eq. (5)).

A final comment on the objective function is that we make no general assumptions about the internal process for generating the elements of the alpha vector. Later we mention the implications of using Grinold's prescription (1994) and suggest that if a scoring process is used, a full covariance matrix version of Grinold's prescription is more appropriate. In the full covariance matrix alpha generation process we propose, the vector α is informed by Ω. With these caveats, we assume that the complete set of managerial beliefs about the distributional properties of the residual returns is incorporated in α and Ω. In other words, the distributional parameters of security returns at the beginning of the given period are fixed and assumed to be known by the manager, while the realized returns at the end of the period are random variables. We do not address the actual accuracy of the risk model or the issue of stationarity over time. Although important in practical applications, these issues are outside the scope of this paper.

2 Optimal solution

The manager's optimization problem is to choose active weights that maximize the utility function in Eq. (2). The first-order condition, which involves the differentiation of the quadratic form $w'\Omega w$, is

$$\frac{\partial U}{\partial w} = \alpha - 2\lambda\Omega w = 0 \tag{6}$$

and solving for w gives the optimal active weight vector as

$$w^* = \frac{1}{2\lambda}\Omega^{-1}\alpha \tag{7}$$

which employs the inverse covariance matrix Ω^{-1}. The numerical procedure for inverting an $N \times N$ covariance matrix can be computationally challenging, although the use of a factor-based risk model helps.[4] In addition to the inverse, we will employ the symmetric square-root of the covariance matrix, denoted $\Omega^{1/2}$, and its inverse, $\Omega^{-1/2}$, which also involve computational algorithms.[5] These square root matrixes are unique and have matrix algebra properties analogous to simple scalar algebra, specifically $\Omega^{1/2}\Omega^{1/2} = \Omega$, $\Omega^{-1/2}\Omega^{-1/2} = \Omega^{-1}$, and $\Omega^{1/2}\Omega^{-1/2} = I$.

To provide intuition for the optimal active weight in Eq. (7), we assume for a moment that the covariance matrix is perfectly diagonal (i.e., zero correlation between residual security returns). Under this simplifying assumption, the inverse correlation matrix is also diagonal with elements $1/\sigma_i^2$ and the optimal active weight for each security is intuitively proportional to the expected security return divided by return variance;

$$w_i^* = \frac{1}{2\lambda}\frac{\alpha_i}{\sigma_i^2} \tag{8}$$

We will occasionally return to the simplifying assumption of a diagonal covariance matrix to provide intuition and interpretation of the results, but the key innovation of this paper is to employ a security return covariance matrix, Ω, that is fully populated (i.e., nondiagonal elements need not be zero).

The optimization problem in Eq. (2) does not include a budget constraint that the sum of the active weights is zero. The budget constraint in matrix form is $w'\iota = 0$ where ι (iota) is an $N \times 1$ vector of 1s. We assume that the raw security alphas are shifted by a constant to be "cash neutral" so that the budget condition is met without a formal constraint.[6]

Instead of continuing our analysis using the risk-aversion parameter λ, we assume that the portfolio manager expresses risk aversion in terms of an active risk parameter σ_A. Substituting the optimal weight vector solution in Eq. (7) into the definition $\sigma_A^2 \equiv w'\Omega w$, and some algebra, gives

$$w^* = \frac{\sigma_A}{\sqrt{\alpha'\Omega^{-1}\alpha}}\Omega^{-1}\alpha \tag{9}$$

which is the unconstrained optimization result used in subsequent analysis. The closed-form solution in Eq. (9) also applies to an optimization problem that maximizes expected return subject to a limit on active portfolio risk. In that problem, the scalar parameter σ_A is simply an alternative way of expressing risk aversion, albeit with a hard limit on the allowable active risk in the portfolio.

Having specified the optimization problem, we turn our attention to the development of the fundamental law, given a full covariance matrix. We derive four results of the law representing ex-ante and ex-post expressions with and without constraints. First, we present the ex-ante unconstrained case and identify the additional assumptions required to derive the original Grinold (1989) form of the law. Second, we extend the unconstrained case to an attribution equation for the ex-post or realized active return, which requires the introduction of the realized information coefficient and realized return dispersion. Third, we acknowledge the role of constraints and expand the ex-ante law to include the transfer coefficient developed in CST. Fourth, we present an ex-post attribution equation for the realized active portfolio return under constraints, which introduces an additional parameter, the noise coefficient.

3 Ex-ante fundamental law without constraints

The ex-ante form of the fundamental law is obtained by substitution of the optimal active weight vector w^* in Eq. (9) into the definitional relationship $E(R_A) \equiv \alpha' w$ and some algebra to give

$$\frac{E\,(R_A)}{\sigma_A} = \sqrt{\alpha'\Omega^{-1}\alpha} \tag{10}$$

The ratio of expected active return to active risk in Eq. (10) is commonly referred to as the information ratio. The information ratio is conceptually similar to the better-known Sharpe ratio, but with an emphasis on active versus total portfolio return and risk. For a market-neutral strategy with the risk-free rate as a benchmark, the information and Sharpe ratios are synonymous.

A more familiar expression of the basic fundamental law can be derived using the simplifying assumption of a diagonal covariance matrix and Grinold's (1994) prescription for alpha generation. According to the Grinold (1994) prescription, the elements of the expected residual return vector are calculated by

$$\alpha_i = \text{IC}\sigma_i S_i \tag{11}$$

where the S_i are a set of cross-sectionally zero-mean unit-variance scores on each security, and IC is the information coefficient, the parameter the portfolio manager believes represents the expected correlation coefficient between security rankings and realized returns. By substituting Eq. (11) into Eq. (10), and assuming a diagonal covariance matrix, we obtain

$$\frac{E(R_A)}{\sigma_A} = \text{IC}\sqrt{N} \tag{12}$$

The simple Grinold (1989) fundamental law expressed in Eq. (12) illustrates the key strategic perspective that the ex-ante information ratio is a product of forecasting skill as measured by the information coefficient, and the square-root of what Grinold (1989) defines as breadth. However, the notion of breadth in Eq. (10) using the full covariance matrix is somewhat ambiguous as discussed later. Breadth can only be defined as N in Eq. (12) because of the simplifying assumption of a diagonal security return covariance matrix. Equation (10) represents the most we can say about the expected information ratio without additional assumptions.

4 Ex-post fundamental law without constraints

The fundamental law analysis in the prior section involved expected active return. Indeed, the fundamental law as first presented by Grinold (1989) is strictly an ex-ante concept; the expected information ratio is a function of expected signal success (information coefficient) and breadth of application. If the portfolio is constructed using the optimal weights, w^*, the ex-post or realized active portfolio return for any given period is

$$R_A = r' w^* \tag{13}$$

Substituting the solution for optimal active weights in Eq. (9) into the realized active portfolio return in Eq. (13), and multiplying and dividing by the expression $\sqrt{r'\Omega^{-1}r}$, gives the ex-post active portfolio return as

$$R_A = \frac{r'\Omega^{-1}\alpha}{\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{r'\Omega^{-1}r}}\sqrt{N}\,\sigma_A\sqrt{\frac{r'\Omega^{-1}r}{N}} \tag{14}$$

Note that while the variables used are vectors and matrices, each of the four terms on the right-hand side of Eq. (14) are scalars (i.e., single-valued variables).

We define the first term in Eq. (14) as the realized information coefficient

$$\rho_{\alpha,r} \equiv \frac{r'\Omega^{-1}\alpha}{\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{r'\Omega^{-1}r}} \tag{15}$$

The notation $\rho_{\alpha,r}$ connotes a covariance-adjusted cross-sectional correlation coefficient between elements of the forecasted and realized residual security return vectors α and r. The correlation coefficient is a well-known bivariate statistic where the covariance is divided by each variable's standard deviation. The correlation coefficient nature of Eq. (15) can be seen by splitting the inverse covariance matrix into square root matrices as follows

$$\rho_{\alpha,r} = \frac{(\Omega^{-1/2}r)'(\Omega^{-1/2}\alpha)}{\sqrt{(\Omega^{-1/2}\alpha)'(\Omega^{-1/2}\alpha)}\sqrt{(\Omega^{-1/2}r)'(\Omega^{-1/2}r)}} \tag{16}$$

Equation (16) has the form of a correlation coefficient between the elements of two $N \times 1$ vectors, $\Omega^{-1/2}\alpha$ and $\Omega^{-1/2}r$. For example, under the simplifying assumption of a diagonal covariance matrix, the realized information coefficient is the correlation between the cross-section of risk-adjusted expected and realized security returns,

$$\rho_{\alpha,r} \approx \mathrm{CORR}\left(\frac{\alpha_i}{\sigma_i}, \frac{r_i}{\sigma_i}\right) \tag{17}$$

where CORR( ) is the well-known statistical function. Besides the simplifying assumption of a diagonal covariance matrix, the approximate equality notation is used in Eq. (17) because elements of the vectors $\Omega^{-1/2}\alpha$ and $\Omega^{-1/2}r$ may have small but nonzero means ex-post, while the common statistical covariance and correlation functions are based on deviations from the mean.

An econometric analogy may provide some intuition. The realized information coefficient in Eq. (17) is analogous to the weighted-least-squares procedure in econometrics, where left- and right-hand side variables in a regression are divided by risk estimates as a correction for heteroskedasticity (see Judge *et al.*, p. 359). On the other hand, the full covariance matrix structure in Eq. (15) is analogous to the generalized-least-squares procedure where a regression is estimated using a known covariance matrix (see Judge *et al.*, p. 329). In fact, the inverse square root of the covariance matrix, $\Omega^{-1/2}$, is similar to the "transformation matrix" used to adjust both left- and right-hand side variables in the GLS procedure. Much of what is derived in this paper with respect to both realized and ex-ante fundamental law parameters can be seen as a GLS extension to Clarke *et al.* (2002), followed by a GLS extension of the alpha generation prescription in Grinold (1994).

We define the final term in Eq. (14) as the covariance-adjusted realized return dispersion

$$D \equiv \sqrt{\frac{r'\Omega^{-1}r}{N}} \tag{18}$$

By splitting the inverse correlation matrix, as in Eq. (16), Eq. (18) can be seen as the matrix equivalent of a standard deviation function. For example, under the simplifying assumption of a diagonal covariance matrix we have

$$D \approx \mathrm{STD}\left(\frac{r_i}{\sigma_i}\right) \tag{19}$$

where STD( ) is the well-known statistical function and the equivalence is only approximate because of a potentially nonzero mean in realized risk-adjusted residual returns. Because realized residual returns are normalized by their estimated risk, the expected value of the return dispersion in Eq. (18) is approximately 1.[7]

Substituting the definitions in Eqs. (15) and (18) into Eq. (14) gives a surprisingly simple and conceptually useful decomposition of the realized active portfolio return in the absence of constraints

$$R_A = \rho_{\alpha,r}\sqrt{N}\sigma_A D \quad (20)$$

Performance attribution is facilitated by the decomposition of the realized active return into the success of the ranking process as measured by the covariance-adjusted realized information coefficient, $\rho_{\alpha,r}$, two ex-ante parameters (the number of securities and active risk), and covariance-adjusted realized return dispersion, D. Equation (20) is exact if the full-covariance matrix formulas are used for $\rho_{\alpha,r}$ and D. Finally, the ex-ante expression in Eq. (10) is reconfirmed by taking the expectation of both sides of Eq. (20).[8]

5 Ex-ante fundamental law under constraints

Portfolio managers rarely construct their portfolios using the exact closed-form optimal active weights shown in Eq. (9). Rather, managers feed the expected return vector, α, and covariance matrix, Ω, into a numerical optimizer under various constraints with the specification that active risk should not exceed the ex-ante parameter value σ_A. Formal constraints such as long-only security positions, turnover restrictions, and maximum individual position sizes, as well as minimum trading units and transaction costs, all result in actual active weights, w, that deviate from optimal weights, w^*. The actual active weights do not have a closed-form solution but do have two mathematical restrictions. First, they must sum to zero based on the budget constraint. Second, to allow for a relevant comparison with previous results we assume that the active risk of the constrained portfolio is equal to that of the unconstrained case, σ_A, so that

$$w'\Omega w = w'^*\Omega w^* = \sigma_A^2 \quad (21)$$

Substituting the actual active weights into the expected active portfolio return, $E(R_A) \equiv \alpha' w$, dividing and multiplying by the terms $\sqrt{\alpha'\Omega^{-1}\alpha}$ and $\sqrt{w'\Omega w}$, and using Eq. (21), we have

$$E(R_A) = \frac{\alpha' w}{\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{w'\Omega w}}\sqrt{\alpha'\Omega^{-1}\alpha}\,\sigma_A \quad (22)$$

This motivates an ex-ante parameter that measures the correlation between the expected returns, α, and the actual active weights taken, w. We define the full-covariance matrix transfer coefficient as

$$\mathrm{TC} \equiv \frac{\alpha' w}{\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{w'\Omega w}} \quad (23)$$

Note that the value of the transfer coefficient using unconstrained optimal active weights from Eq. (9) is exactly 1.[9] Following analysis like that used in the covariance-adjusted realized information coefficient in Eq. (16), the correlation coefficient nature of the transfer coefficient is

$$\mathrm{TC} = \frac{(\Omega^{-1/2}\alpha)'(\Omega^{1/2}w)}{\sqrt{(\Omega^{-1/2}\alpha)'(\Omega^{-1/2}\alpha)}\sqrt{(\Omega^{1/2}w)'(\Omega^{1/2}w)}} \quad (24)$$

Equation (24) has the form of a correlation coefficient between the N elements of vectors $\Omega^{-1/2}\alpha$ and $\Omega^{1/2}w$. For example, under the simplifying assumption of a diagonal covariance matrix, the transfer coefficient is the correlation coefficient between the cross-section of risk-adjusted expected security returns and risk-weighted active weights,

$$\mathrm{TC} \approx \mathrm{CORR}\left(\frac{\alpha_i}{\sigma_i}, w_i\sigma_i\right) \quad (25)$$

Substituting the transfer coefficient definition from Eq. (23) into Eq. (22) gives

$$E(R_A) = \mathrm{TC}\sqrt{\alpha'\Omega^{-1}\alpha}\,\sigma_A \quad (26)$$

A comparison of the constrained expected active return in Eq. (26) to the unconstrained value in Eq. (10) shows that the ex-ante impact of constraints is precisely measured by the transfer coefficient, TC, when fundamental law parameters are calculated using the full covariance matrix.

6 Ex-post fundamental law under constraints

The ubiquitous imposition of formal constraints in the optimization routines used by portfolio managers leads to actual security weights, w, that deviate from the optimal weights, w^*, when the constraints are binding. Define the $N \times 1$ weight-not-taken vector as

$$c \equiv w - \mathrm{TC}w^* \quad (27)$$

The weight-not-taken for each security represents the difference between the portfolio's actual weight and the expected weight, given the proportional impact of constraints measured by the transfer coefficient. The correlation between this mismatch and subsequent security returns will result in a type of unexpected noise. The elements of vector c sum to zero because the sets of actual and optimal active weights both sum to zero. We need to determine the value of the quadratic term $c'\Omega c$, which is used in our final proof. The definition in Eq. (27) and some algebra (using Eqs. (9), (21), and (23)) gives

$$c'\Omega c = (1 - \mathrm{TC}^2)\sigma_A^2 \quad (28)$$

The realized active return to the portfolio using actual active weights is $R_A = r'w$. Using the definition of optimized constrained weights-not-taken in Eq. (27) with some algebra (using Eqs. (9), (15), (18), and (28)) gives

$$R_A = r'(TCw^* + c) = \left(\mathrm{TC}\rho_{\alpha,r} + \frac{(1-\mathrm{TC}^2)^{1/2} r'c}{\sqrt{r'\Omega^{-1}r}\sqrt{c'\Omega c}}\right) D\sqrt{N}\sigma_A \quad (29)$$

We define a final parameter called the realized noise coefficient, which measures the unexpected performance noise due to constraints;

$$\rho_{c,r} \equiv \frac{r'c}{\sqrt{r'\Omega^{-1}r}\sqrt{c'\Omega c}} \quad (30)$$

The noise coefficient can be characterized as the covariance adjusted cross-sectional correlation coefficient between weights-not-taken and realized security returns; more formally the N elements of the vectors $\Omega^{1/2}c$ and $\Omega^{-1/2}r$. Substituting Eq. (30) into the final form of Eq. (29), and reordering terms, gives the complete decomposition of the realized active portfolio return under constraints

$$R_A = (\mathrm{TC}\rho_{\alpha,r} + (1-\mathrm{TC}^2)^{1/2}\rho_{c,r})\sqrt{N}D\sigma_A \quad (31)$$

Equation (31) provides a complete decomposition of realized active portfolio return into parameters that measure signal performance, the expected and realized impact of portfolio constraints, and covariance-adjusted realized security return dispersion. In contrast to the ex-post result in CST, Eq. (31) is exact and based on the relaxed assumption of a nondiagonal residual return covariance matrix. For purposes of performance attribution, Eq. (31) can be rearranged into two terms that represent the signal and noise contributions to the realized active portfolio return;

$$\begin{aligned} R_A = {} & \mathrm{TC}\rho_{\alpha,r}\sqrt{N}D\sigma_A \\ & \text{(Signal contribution)} \\ & + (1-\mathrm{TC}^2)^{1/2}\rho_{c,r}\sqrt{N}D\sigma_A \\ & \text{(Noise contribution)} \end{aligned} \quad (32)$$

The portfolio's realized active return under constraints has been exactly decomposed into parameters that have meaning to the portfolio manager, including the transfer coefficient, the covariance-adjusted realized information coefficient and realized return dispersion, the number of securities, and a noise term. The noise term can be thought of as the portion of the realized active return not explained by the signal contribution. The results in Eq. (26), or direct application of the expectations operator in Eq. (32) similar to Footnote 8, indicates that the expected value of the noise term is zero. Besides a performance attribution framework that completely decomposes the actual return, Eq. (32) gives the investor insights about the reduction in potential value added due to constraints as explained in Clarke *et al.* (2005).

7 The breadth parameter and alpha generation

The full-covariance matrix fundamental law equations derived in this paper reflect the returns from N securities in the investment universe, where N is not necessarily equivalent to breadth. While the notion of breadth conveys important intuition in active management, breadth is hard to measure in practice when residual security returns are correlated to each other in complex ways as embodied in the covariance matrix, and when the security ranking system is generally correlated to various risk factors. For example, if industry membership is captured in the covariance matrix of residual returns, then a set of security alphas that tends to place all the securities of a particular industry together has lower breadth than a system where the alphas are cross-sectionally uncorrelated to industry membership. Buckle (2004) provides some intuition on the breadth parameter using simple covariance matrices where all of the off-diagonal elements are equal. Alternatively, one can conduct breadth analysis under the more general assumption of a factor-based risk model. For example, even under a full (i.e. nondiagonal) covariance matrix, it can be shown that breadth is equal to the number of securities if the alpha vector is perfectly orthogonal to each of the risk-model factors.[10] While the number of securities, not breadth, is the parameter employed in the full covariance equations derived in this paper, the concept of breadth is commonly associated with the fundamental law theory and thus suggests some comment. Below we provide a practical approximation of implied breadth and demonstrate its dependence on the alpha generation procedure.

To motivate a more general measure of breadth if one finds that concept helpful, consider Eq. (10), which gives the expected value of the information ratio for the unconstrained portfolio. Grinold (1989) suggests that the expected information ratio equals the expected information coefficient times the square root of breadth. Using this framework and Eq. (10) allows breadth to be defined for any given set of expected returns, α, and risk model, Ω, as

$$\text{Breadth} = \frac{\alpha'\Omega^{-1}\alpha}{\text{IC}^2} \tag{33}$$

where IC is the information coefficient used to generate security alphas. Under the simplifying assumption of a diagonal covariance matrix and Grinold's (1999) alpha process, the breadth parameter implied by Eq. (33) is the number of securities.

As mentioned earlier, the concept of breadth is more ambiguous and perhaps less relevant under the assumption of a full covariance matrix. Further, since implied breadth as calculated in Eq. (33) is based on the unconstrained information ratio of Eq. (10), the practical reality of constraints reduces strategy breadth by a proportion equal to the transfer coefficient TC. To provide some intuition for breadth, consider a simple two security benchmark where the single off-diagonal element of the 2×2 correlation matrix is ρ and the two elements on the diagonal are equal (i.e., two securities with equal residual return variances). The only possible scores with a cross-sectional mean zero and unit variance scores for $N = 2$ are $+1$ and -1. Using Grinold's (1994) alpha prescription in Eq. (11), the procedures for matrix inversion and multiplication gives breadth as defined in Eq. (33) of $2(1+\rho)/(1-\rho)$. Thus, for a pair of highly correlated securities with say $\rho = 0.8$, breadth is 18.0. Breadth is relatively large (nine times the number of securities) because the differential ranking of highly correlated securities is almost an arbitrage. Alternatively, if the off-diagonal element of the correlation matrix is $\rho = -0.8$, then implied breadth is only 0.22. Breadth is now almost zero because the differential ranking between the two securities simply reflects their highly negative correlation as given by the risk model. Note that these breadth inferences rest on the assumption that the manager's forecast of residual security returns are made with an awareness of what the risk model predicts about their correlation structure. If, on the other hand, the manger's ranking process for securities is made without reference to the risk model, one might want to modify the alpha generation process as discussed below.

In practice, many portfolio managers rank securities by some scoring process without reference to the residual correlations predicted by the risk model. In other words, the relative values of the individual elements in α are assigned by the manager without reference to the off-diagonal values assumed in Ω. Although fairly common, this practice seems internally inconsistent if residual return correlations are material. We introduce an alpha generation process analogous to the full covariance matrix definitions of realized information coefficient and transfer coefficient developed in prior sections. In the spirit of Grinold's (1994) prescription in Eq. (11), the alpha vector is generated from raw scores using the square root residual security return covariance matrix as follows

$$\alpha = \mathrm{IC}\Omega^{1/2}S \tag{34}$$

where S is an $N \times 1$ vector of mean-zero unit-variance scores. The alpha generation procedure in Eq. (34) can be thought of as the GSL equivalent to Grinold's (1994) prescription, and is in fact alluded to in Footnote 10 of that paper.

Besides internal consistency, one of the virtues of the full covariance matrix alpha generation process in Eq. (34) is that breadth as defined in Eq. (33) is now equal to the number of securities. Specifically, substitution of the full covariance matrix alpha generation process in Eq. (34) into Eq. (33) gives

$$\begin{aligned}\text{Breadth} &= \frac{(\mathrm{IC}\Omega^{1/2}S)'\Omega^{-1}(\mathrm{IC}\Omega^{1/2}S)}{\mathrm{IC}^2} = S'S \\ &= \sum_{i=1}^{N} s_i^2 = N \end{aligned} \tag{35}$$

where the last step in Eq. (35) assumes that individual security scores, s_i , have a cross-sectional mean of zero and unit variance. Conceptually, when alphas are constructed using all of the information embedded in the covariance matrix, the breadth of the strategy is by definition equal to the number of securities. Note, however, that using the breadth parameter for assessing the potential value added by an active management strategy as described by Grinold (1989) also requires the information coefficient, and the value of the information coefficient is by definition dependent on the alpha generation procedure. Thus, breadth and the information coefficient are codependent in the alpha generation process, which lessens their separate strategic insights in a full covariance matrix context. We note that the development of the full covariance matrix fundamental law equation in the prior sections of this paper does not assume any specific alpha generation procedure and as a result the notion of breadth is somewhat ambiguous unless more is assumed about the alpha generating process.

8 Numerical example using the EAFE index

As a numerical example of the full covariance matrix fundamental law theory, we employ the EAFE international equity index with the 21 member countries representing individual securities. We use as a proxy for the general market portfolio the MSCI World Index that includes all 23 countries (adding the US and Canada). We choose September 2004 as a recent month with typical realized return results. For purposes of illustration, the market risk model is simply the covariance matrix of the historical dollar-based excess returns for the prior 60 months (September 1999 to August 2004). The residual return covariance matrix is then calculated using the market betas from the total return covariance matrix as specified in Footnote 2. In practice, simple historical covariance calculations are generally considered inadequate predictive risk models and portfolio managers employ more sophisticated factor-based matrices with GARCH estimators and other econometric enhancements. The forecast signals for this simple example are a set of standard normal scores that are randomly assigned to the 21 countries in the EAFE benchmark. In practice, a portfolio manager would use some proprietary process for ranking securities within the investable set. In illustrating the calculations in this section we take as given the forecasts and the calculations are not dependent on the precise forecasting procedure used.

To illustrate the impact of constraints, we optimize a long-only (i.e., no-short-sell constrained) portfolio with the additional restriction that the absolute active weight on any single country does not exceed 20 percent. The EAFE benchmark weights are prior month-end (end of August 2004) and the realized returns are the actual returns in excess of the risk-free rate for each country as reported by MSCI during September 2004. Using a small (21-country) numerical example allows us to report data on the entire set of countries, including the forecasted and realized residual returns and most of the parameters in the risk model. We optimize under an active risk constraint of 1.00 percent monthly, and an assumed IC of 0.100. While not essential to the illustration, we also impose a constraint that the managed portfolio has a global market beta equal to the EAFE benchmark so that the net active market beta is zero. Under this constraint, the active risk and tracking error of the managed portfolio are identical, as shown in Eq. (4). Calculations are preformed in Excel, including matrix inversion and numerical optimization (i.e. Excel Solver) except for the matrix square root calculations, which employ a characteristic root procedure in SAS.

Table 1 shows the main optimization results, with risk model details given in Table 2 and forecasted and realized return details given in Table 3. All of the return and risk parameters in the tables are nonannualized values for the one-month investment horizon. The first column in the lower half of Table 1 shows the 21 countries in the EAFE index sorted by their benchmark weights in the next column. While the number of countries is small, the concentration of the benchmark weights in the larger countries is indicative of most capitalization-weighted equity indexes. For example, almost half of the total benchmark is in the two largest countries, the United Kingdom and Japan, and several of the smaller countries have less than a 1 percent benchmark weight. The next two columns in Table 1 show the constrained optimal active weight and resulting total managed portfolio weight for each country. Notice that none of the managed portfolio weights are negative due to the no-short-sell constraint and thus none of the smaller countries have materially negative active weights. This illustrates the typical problem of a small-capitalization bias in short-sell constrained portfolios that are not simultaneously constrained to be market-cap neutral with respect to the benchmark.

The next column in Table 1 shows the alpha (expected residual return) vector, with details on the alpha generation process shown in Table 3. Raw alphas are generated from a set of normally distributed scores that are randomly assigned to each country and then calculated using the full covariance technique shown in Eq. (34). We later consider the same case using the Grinold (1994) prescription for alpha generation and discuss the breadth implications. Table 3 shows that the final alphas are cash neutral (as explained in Footnote 6) by applying a constant shift of about eight basis points to the raw alphas. Table 3 also gives the details on calculating realized residual returns from the MSCI returns for September 2004. As specified in Eq. (3), the risk-free rate (about 15 basis points in September 2004) is subtracted first, and then the returns are adjusted for each country's market-wide beta as given in Table 2. The final three columns in Table 1 are the unconstrained optimal active weights (Eq. (9)),

Table 1 Fundamental law example EAFE September 2004.

Constraint set and IC parameter		*Realized active return calculation*	
1) Long only constraint		Portfolio return	1.21%
2) Country active weight constraint	20%	Benchmark return	0.80%
		Active return	0.41%
3) Net market beta exposure	0.000		
4) Active risk limit/month	1.00%	*Fundamental law parameters*	
Assumed information coefficient	0.100	Realized information coefficient	0.068
		Transfer coefficient	0.671
General calculations		Realized noise coefficient	0.058
Optimal information ratio	0.450	Number of securities	21.0
Implied breadth	20.3	Realized dispersion	1.005
Ex-ante information ratio		*Active return decomposition*	
Expected active return/month	0.30%	Signal contribution	0.21%
Ex-ante active risk/month	1.00%	Noise contribution	0.20%
Information ratio	0.302	Explained active return	0.41%

	Constrained				Unconstrained optimized		
Country	Benchmark weight (%)	Active weight (%)	Portfolio weight (%)	Expected alpha (%)	Active weight (%)	Weight-not-taken	Realized return
United Kingdom	25.4	−0.2	25.2	0.13	21.2	−14.4	1.55
Japan	23.5	−5.2	18.3	−0.05	1.5	−6.2	−3.77
France	9.2	−9.2	0.0	−0.23	−32.8	12.8	1.18
Switzerland	7.0	−7.0	0.0	−0.25	−9.0	−0.9	1.07
Germany	6.6	−6.6	0.0	−0.23	−6.5	−2.2	2.51
Australia	5.1	−5.1	0.0	−0.68	−19.2	7.8	5.08
Netherlands	4.7	16.2	20.9	0.12	20.9	2.2	−0.09
Italy	3.8	−3.8	0.0	0.11	3.9	−6.4	3.93
Spain	3.6	−0.1	3.5	0.18	3.7	−2.5	1.83
Sweden	2.4	−2.4	0.0	−0.26	3.0	−4.4	4.70
Hong Kong	1.8	−1.8	0.0	−0.54	−6.3	2.5	−0.09
Finland	1.3	2.4	3.7	−0.01	0.7	1.9	7.87
Belgium	1.2	−1.2	0.0	−0.11	−7.5	3.8	5.78
Singapore	0.9	−0.9	0.0	−0.61	0.3	−1.1	3.29
Ireland	0.8	7.3	8.1	0.28	4.4	4.3	3.13
Denmark	0.8	−0.8	0.0	0.13	−1.5	0.2	4.41
Norway	0.5	−0.5	0.0	−0.02	9.2	−6.7	9.47
Greece	0.5	−0.5	0.0	−0.84	−2.5	1.2	2.65
Portugal	0.4	20.0	20.4	0.75	14.0	10.6	4.52
Austria	0.3	−0.3	0.0	−0.05	−2.3	1.2	3.75
New Zealand	0.2	−0.2	0.0	−0.42	5.0	−3.6	5.17
Total	100.0	0.0	100.0		0.0	0.0	

Table 2 Risk assumptions.

Country	Market weight (%)	Total return Std Dev (%)	Market beta	Residual return Std Dev (%)
United Kingdom	10.75	4.40	0.857	2.01
Japan	9.94	5.70	0.728	4.63
France	3.90	6.09	1.164	2.97
Switzerland	2.96	4.68	0.737	3.26
Germany	2.79	7.94	1.477	4.20
Australia	2.16	5.12	0.813	3.53
Netherlands	1.99	6.23	1.181	3.12
Italy	1.59	6.03	0.910	4.37
Spain	1.51	6.35	1.105	3.85
Sweden	1.01	9.36	1.669	5.43
Hong Kong	0.74	6.47	0.940	4.84
Finland	0.55	12.70	1.751	9.86
Belgium	0.53	6.39	0.924	4.80
Singapore	0.37	6.93	0.921	5.51
Ireland	0.35	5.83	0.897	4.15
Denmark	0.33	5.65	0.953	3.59
Norway	0.23	6.12	1.085	3.58
Greece	0.20	8.44	0.794	7.62
Portugal	0.15	6.36	0.856	5.01
Austria	0.13	4.99	0.486	4.47
New Zealand	0.10	6.09	0.730	5.09
United States	55.00	4.86	1.036	1.12
Canada	2.73	6.01	1.107	3.25
Market (World)	100.00	4.57	1.000	

Residual return correlation matrix (first six countries only)

	UK	Japan	France	Switzerland	Germany	Australia	...
UK	1.000	−0.290	0.350	0.481	0.189	−0.120	...
Japan	−0.290	1.000	−0.340	−0.072	−0.418	0.204	...
France	0.350	−0.340	1.000	0.406	0.750	0.060	...
Switzerland	0.481	−0.072	0.406	1.000	0.186	0.058	...
Germany	0.189	−0.418	0.750	0.186	1.000	−0.003	...
Australia	−0.120	0.204	0.060	0.058	−0.003	1.000	...
...	...	...	...	...	...	...	...

Table 3 September 2004 forecasted and realized returns.

Country	Score	Raw alpha (%)	Cash neutral alpha (%)	Total return (%)	Excess return (%)	Residual return (%)
United Kingdom	1.51	0.21	0.13	3.22	3.07	1.55
Japan	0.38	0.04	−0.05	−2.33	−2.48	−3.77
France	−1.51	−0.14	−0.23	3.40	3.25	1.18
Switzerland	−0.82	−0.17	−0.25	2.53	2.38	1.07
Germany	−0.66	−0.14	−0.23	5.28	5.13	2.51
Australia	−2.04	−0.60	−0.68	6.67	6.52	5.08
Netherlands	1.22	0.20	0.12	2.16	2.01	−0.09
Italy	0.51	0.19	0.11	5.69	5.54	3.93
Spain	0.82	0.26	0.18	3.94	3.79	1.83
Sweden	−0.25	−0.18	−0.26	7.81	7.66	4.70
Hong Kong	−1.00	−0.46	−0.54	1.72	1.57	−0.09
Finland	0.12	0.07	−0.01	11.13	10.98	7.87
Belgium	−0.51	−0.03	−0.11	7.57	7.42	5.78
Singapore	−0.38	−0.53	−0.61	5.07	4.92	3.29
Ireland	1.00	0.36	0.28	4.87	4.72	3.13
Denmark	0.25	0.21	0.13	6.25	6.10	4.41
Norway	0.66	0.07	−0.02	11.55	11.40	9.47
Greece	−1.22	−0.76	−0.84	4.21	4.06	2.65
Portugal	2.04	0.84	0.75	6.19	6.04	4.52
Austria	0.00	0.03	−0.05	4.76	4.61	3.75
New Zealand	−0.12	−0.34	−0.42	6.62	6.47	5.17
Average	0.00	−0.04	−0.12	5.16	5.01	3.24

the optimized weight-not-taken due to constraints (Eq. (27)), and the realized residual returns for each country from Table 3.

The top half of Table 1 provides key results on the fundamental law calculations, portfolio optimization, and realized return decomposition. The expected information ratio of the portfolio when unconstrained is 0.450 and is calculated using Eq. (10) using the assumptions in the illustration. The implied breadth of 20.3, calculated using Eq. (33), is close to the number of securities because the raw alphas are generated using the full-covariance matrix procedure given in Eq. (34). Implied breadth is not exactly 21.0 because of the cash-neutralization shift in alphas. The expected active return of the portfolio when constrained is 30 basis points, and when divided by the estimated active risk (limited to 100 basis points by the optimizer) the expected information ratio is 0.302 (displayed results induce some rounding error). Thus, the reduction in the expected information ratio due to portfolio constraints implies a transfer coefficient of about

0.302/0.450 = 0.671. The full-covariance transfer coefficient, calculated directly from Eq. (23), is 0.671 as shown in the upper right-hand corner of Table 1. This comparison validates the *ex-ante* fundamental law mathematics summarized in Eq. (26).

The upper right-hand corner of Table 1 contains the details of the realized active return decomposition. The managed portfolio realized return of 121 basis points minus the benchmark return of 80 basis points gives an active portfolio return of 41 basis points. The full-covariance realized information coefficient for September 2004 is 0.068, slightly below the expected value (IC parameter) of 0.100. In this particular month, the noise coefficient happened to be positive at 0.058, compared with an expected value of zero. A key implication of the ex-post fundamental law is that even for a relatively high TC value of 0.671, the impact of any given realized noise coefficient is greater than the impact of the signal as measured by the information coefficient. As shown in Eq. (31), the information coefficient multiplier is TC = 0.671, while the multiplier for the noise correlation coefficient is $(1 - TC^2)^{1/2} = 0.741$. Thus, while the absolute magnitude of the information coefficient in September 2004 is materially greater than the realized noise correlation coefficient, the signal contribution of 21 basis points is only slightly greater than the noise contribution of 20 basis points. The sum of the signal and noise contributions in the attribution system is 41 basis points, which exactly matches the direct calculation (managed portfolio return minus the benchmark return). This comparison validates the *ex-post* fundamental law mathematics of the attribution system summarized in Eq. (32).

For comparison purposes, we also calculate the fundamental law parameters for September 2004 using *simple* fundamental law definitions as described in Grinold (1989) and CST. Under the assumptions used in previous developments of the fundamental law, the ex-ante and ex-post equations are approximate for two reasons. First, well-known statistical functions (e.g., standard deviation) are based on deviations from mean cross-sectional security values, which may not be exactly zero ex-post. The "zero-means" assumption is particularly damaging in the EAFE example because of the small number of countries (i.e. 21) in the investable universe. As explained in CST, the assumption of zero means is not as damaging for larger universes (e.g. S&P 500 benchmark). Second, the fundamental law parameters in previous developments are based on an assumed diagonal covariance matrix when in fact numerical optimization is generally conducted using the full covariance matrix. As shown in the risk-model summary in Table 2, this assumption is particularly invalid for the countries in the EAFE index. In fact, the rich structure of the cross-sectional correlations among countries was the primary motivation for choosing the EAFE benchmark with countries acting as securities as an illustration of the full covariance matrix fundamental law mathematics. Our concern in this paper is a generalization of the second concern above, and so we calculate the various fundamental law parameters using an assumed diagonal covariance matrix but with statistical functions that do not require zero means.[11]

The transfer coefficient calculated based on a diagonal covariance matrix assumption is 0.620 compared with the full-matrix value of 0.671 shown in Table 1. The full-matrix TC is intuitively larger because it acknowledges the nondiagonal residual return correlations in the risk model supplied to the optimizer. This specific comparison is dependent on the assignment of scores and realized return used in the base-case example, but experiments with other random assignments of the 21 scores to countries verify that the full-covariance matrix TC is generally higher than a diagonal-matrix calculation. We also calculated other fundamental law

Table 4 Alternative cases of implied breadth.

Score Assignment	Base case		Base case		Euro/Pacific dichotomy	
Alpha Generation	Full Covariance		Diagonal Covariance		Diagonal Covariance	
Country	Score	alpha (%)	Score	alpha (%)	Score	alpha (%)
United Kingdom	1.51	0.13	1.51	0.10	0.56	0.22
Japan	0.38	−0.05	0.38	−0.02	−1.79	−0.72
France	−1.51	−0.23	−1.51	−0.65	0.56	0.28
Switzerland	−0.82	−0.25	−0.82	−0.46	0.56	0.29
Germany	−0.66	−0.23	−0.66	−0.47	0.56	0.35
Australia	−2.04	−0.68	−2.04	−0.92	−1.79	−0.52
Netherlands	1.22	0.12	1.22	0.18	0.56	0.29
Italy	0.51	0.11	0.51	0.03	0.56	0.36
Spain	0.82	0.18	0.82	0.12	0.56	0.33
Sweden	−0.25	−0.26	−0.25	−0.33	0.56	0.41
Hong Kong	−1.00	−0.54	−1.00	−0.68	−1.79	−0.76
Finland	0.12	−0.01	0.12	−0.08	0.56	0.66
Belgium	−0.51	−0.11	−0.51	−0.44	0.56	0.38
Singapore	−0.38	−0.61	−0.38	−0.41	−1.79	−0.87
Ireland	1.00	0.28	1.00	0.21	0.56	0.34
Denmark	0.25	0.13	0.25	−0.11	0.56	0.31
Norway	0.66	−0.02	0.66	0.04	0.56	0.31
Greece	−1.22	−0.84	−1.22	−1.13	0.56	0.54
Portugal	2.04	0.75	2.04	0.82	0.56	0.39
Austria	0.00	−0.05	0.00	−0.20	0.56	0.36
New Zealand	−0.12	−0.42	−0.12	−0.26	−1.79	−0.80
Average	0.00	−0.12	0.00	−0.22	0.00	0.10
Standard Deviation	1.00	0.36	1.00	0.42	1.00	0.48
Optimal IR		0.450		0.796		0.374
Assumed IC		0.100		0.100		0.100
Implied breadth		20.3		63.4		14.0

parameters (e.g., the realized information coefficient) using the diagonal-matrix assumption and found material discrepancies. While inaccurate, no directional bias in these parameters was noted in alternative assignments of the base-case scores we examined.

Table 4 summarizes the results of experiments with alternative score vectors to illustrate the concept of implied breadth. The alternative cases are based on the same risk model shown in Table 2, but with different score vectors or alpha generation procedures. The first case in Table 4 is the base case discussed above. The implied breadth of 20.3 using Eq. (33) is quite close to the number of countries because the raw alpha vector is generated using the full covariance matrix. Table 4 next shows results using the Grinold (1994) alpha generation procedure in Eq. (11) but using the same standardized scores as in the base case. The implied breadth of 63.4 is the breadth of the strategy under the assumption that the scores were established by the portfolio manager with an understanding of the large residual return correlations in the risk model. In other words, 63.4 is the implied breadth of the base case signal if the inherent correlations between residual security returns had already been accounted for in developing the standardized scores.

The concept of implied breadth is illustrated again in the third case in Table 4. The scores are assigned to countries based on a dichotomous signal where the 16 European countries are given a single higher rank and the 5 Pacific counties are given a single lower rank. The two score values (0.56 and −1.79) are set so that the mean is 0 and the variance is 1 for the sample size of 21. This case represents a relatively narrow forecast that the European region will outperform the Pacific region over the next month. An examination of the risk model's residual return correlation matrix in Table 2 indicates positive correlations within each region. For example, the residual correlation between the United Kingdom and France is 0.350 and the correlation between Japan and Australia is 0.204. In contrast, the correlations between countries in different regions are often negative, for example the United Kingdom to Japan correlation is −0.290.[12] Geographical region appears to be one of the major drivers in the covariance matrix of residual returns, and a dichotomous signal based on region leads to the fairly low implied breath of 14.0 as shown in Table 4. The implied breadth of the dichotomous region signal is still greater than 1 because region is only one of the many factors that explain the covariance structure between countries. In contrast, if the dichotomous scores are assigned arbitrarily (i.e., the UK and every 5th country thereafter gets a negative score), the implied breadth is much higher at 58.6 (not shown) because the scores do not align with any implicit risk factor. In other words, the low implied breadth based on region is not simply an artifact of a dichotomous score. These examples illustrate that the concept of implied breadth is somewhat ambiguous and is linked to the assumptions underlying the process used to develop forecasts for the residual returns.

9 Conclusion

We have generalized the fundamental law of active management to allow for a fully populated covariance matrix and have shown that the parsimony of the ex-ante (expected) and ex-post (realized) active return equations in previous studies is preserved. In fact, the resulting equations are more exact in that fewer simplifying assumptions are required in the derivations. Additionally, the equations we derive employ the number of securities rather than the concept of breadth, which is ambiguous when residual returns are materially correlated. Transfer coefficients, realized information coefficients, and other fundamental law parameters calculated using all of the security correlation information in the risk model should provide portfolio managers with better insights about the potential for value added and more precise attributions of realized performance. For example, the ex-ante transfer coefficient calculated without the off-diagonal elements of the covariance matrix tends to understate the degree to which the signal is translated into active positions. Similarly, an ex-post realized information coefficient that ignores the cross-sectional correlation structure of the security returns can understate or overstate the success of the signal. As a result the proportion of actual return attributed to the success of the forecasting process in contrast to the residual impact of the constraint set will be distorted.

In addition to the generalization of the fundamental law, we describe an alpha generation procedure that incorporates the full covariance matrix. Alphas generated from standard normal scores by this procedure are conceptually superior to alphas that only account for heteroskedasticity across securities. The procedure is particularly relevant when the process for scoring and ranking securities is conducted without the perspective of the security correlations given in the risk model. Finally, we illustrate the concept of implied breadth in the traditional fundamental law and show that breadth is definitionally equal to the number of securities when alphas are generated using the full covariance matrix. Otherwise, the concept of implied breadth is more ambiguous.

The practical significance of the full covariance enhancements to existing fundamental law theory depends on the materiality of the off-diagonal elements in the residual return covariance matrix. For example, while cross-correlations between US stock returns remain after accounting for market beta, they will be less material after also removing exposures to size, value, industry, and other factors. Thus, for our numerical illustration we choose the EAFE index with 21 countries and a simple one-factor global market model that has large positive and negative cross-correlations between country residual returns. The numerical example verifies the matrix mathematics behind the ex-ante and ex-post equations we develop and provides an interesting illustration of both the strategic and performance attribution applications of the full covariance matrix fundamental law.

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Notes

1 Other recent extensions and applications of the fundamental law framework include Buckle (2004) who analyzes breadth, Sorensen *et al.* (2004) who consider multiple sources of alpha, and Clarke *et al.* (2005) who discuss performance attribution.

2 The $M \times 1$ vector of security betas with respect the general market are calculated from a total return covariance matrix, Ω_R, by the equation $\beta = \Omega_R w_M / w_M' \Omega_R w_M$, where w_M is an $M \times 1$ vector of market portfolio weights. Note that the general market may have more securities than the benchmark; $M \geq N$. Security betas calculated in this manner have a market-weighted average value of exactly 1 by definition.

3 Under the one-factor market model, the residual return covariance matrix is computed from the total return covariance matrix, Ω_R, by the matrix equation $\Omega = \Omega_R - (\Omega_R w_M w_M' \Omega_R / w_M' \Omega_R w_M)$, where w_M is the vector of security weights in the market portfolio. In the special case where the benchmark and market portfolios are identical, the residual covariance matrix Ω has rank $N-1$ and is not invertible. Specifically, the benchmark-weighted average residual security return is exactly zero in both expectation and realization, so the Nth residual is a linear function of the other $N - 1$ residuals.

4 In general, inverting an $N \times N$ covariance matrix for large security sets (e.g. 500) is computationally demanding, and for non–factor-based risk models there is no assurance that the matrix inverse exists. A factor-based risk model has the form $\Omega = XFX' + \Delta$, where X is an $N \times K$ security-to-factor exposure matrix, F is a $K \times K$ factor covariance matrix, and Δ is an $N \times N$ diagonal idiosyncratic risk matrix. The general form of the inverse covariance matrix is $\Omega^{-1} = \Delta^{-1} + \Delta^{-1} X (F^{-1} + X' \Delta^{-1} X)^{-1} X' \Delta^{-1}$, which requires the less computationally demanding inverse of two $K \times K$ matrices and the trivial inverse of the diagonal matrix Δ.

5 The symmetric square-root of Ω is found by decomposing it into an orthogonal eigenvector matrix E (i.e. $E^{-1} = E'$) and associated diagonal eigenvalue matrix Λ such that $\Omega = E \Lambda E'$. Taking the square root of the elements of Λ gives $\Lambda^{1/2}$, and then by direct computation $\Omega^{1/2} = E \Lambda^{1/2} E'$.

6 Cash neutral alphas which ensure that the budget constraint is met (i.e., that the closed-form optimal active weights sum to zero) are calculated by a constant shift in the elements of the raw alpha vector given by $\alpha = \alpha_{RAW} - (\iota' \Omega^{-1} \alpha_{RAW} / \iota' \Omega^{-1} \iota) \iota$. As noted in Grinold and Kahn

(1994), page 418, this shift is based on the mathematics of the minimum variance portfolio.

7 The expectation is not exactly 1 for various technical reasons related to the fact that an $N-1$ divisor rather than N is required for a sample standard deviation to be an unbiased forecast of the population parameter.

8 Taking the expectation of both sides of Eq. (20) gives $E(R_A) = E(\rho_{\alpha,r}D)\sqrt{N}\sigma_A = (E(r'\Omega^{-1}\alpha)/\sqrt{\alpha'\Omega^{-1}\alpha})\sigma_A = \sqrt{\alpha'\Omega^{-1}\alpha}\sigma_A$. Note that the expectation operator cannot in general be distributed across the product of ex-post parameters $\rho_{\alpha,r}$ and D because they are not independent random variables.

9 Using the optimal active weights in Eq. (9), we have $TC = \alpha'w^*/\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{w'^*\Omega w^*} = \alpha'\Omega^{-1}\alpha/\sqrt{\alpha'\Omega^{-1}\alpha}\sqrt{\alpha'\Omega^{-1}\Omega\Omega^{-1}\alpha} = 1$.

10 Given a K-factor risk-model; $\Omega = XFX' + \Delta$, the inverse covariance matrix is $\Omega^{-1} = \Delta^{-1} + \Delta^{-1}X(F^{-1} + X'\Delta^{-1}X)^{-1}X'\Delta^{-1}$ and the information ratio squared in Eq. (12) is $\alpha'\Omega^{-1}\alpha = \alpha'\Delta^{-1}\alpha + \alpha'\Delta^{-1}X(F^{-1} + X'\Delta^{-1}X)^{-1}X'\Delta^{-1}\alpha$. Under Grinold's (1994) alpha prescription, the first term is $\alpha'\Delta^{-1}\alpha = IC^2N$. The second term can be written as $A'G^{-1}A$, where G is some square matrix, and A is a $K \times 1$ vector with elements $a_k = \sum X_{k,i}\alpha_i/\sigma_i^2$. If the N risk-adjusted alphas are orthogonal to each of the K factor-exposure vectors, then each element of A is zero.

11 For example, instead of employing the standard deviation function, STD( ), we calculate the square root of the average squared deviations from zero. Thus, the attribution system adds up in terms of the total realized active return, but the fundamental law parameters are distorted by not considering the off-diagonal elements of the residual return covariance matrix.

12 The large negative correlations in Table 3 are for market-model *residual* returns. The *total* return correlations (not reported) for most country pairs is positive. For example, the total return correlation between the United Kingdom and Japan is 0.411.

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