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LONG-RUN INVESTMENT MANAGEMENT FEE INCENTIVES AND DISCRIMINATING BETWEEN TALENTED AND UNTALENTED MANAGERS

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Ferguson and Leistikow [(1997). Journal of Financial Engineering ***6****, 1–30] (FLa) was the first long-run risk-neutral analysis of the performance volatility incentives created by investment management fee structures. This paper extends FLa in six ways. It allows the portfolio's value to change, incorporates expected investment performance, and addresses expenses and distributions. It also shows the impact of paying investment management performance fees from the portfolio, and determines if the contract renewal structure and fee arrangements discriminate effectively among talented and untalented managers. Finally, it introduces a volatility-dependent contract renewal structure that provides good discrimination and strongly motivates manager behavior consistent with client preferences.*

1 Introduction

Investment management fee structures create incentives for investment managers that may not motivate managers to act in their clients' best interest. This paper assumes that clients seek active management and want neither closet indexing nor excessive volatility, relative to a benchmark.

The most common fee structures used in the industry and discussed in the literature are flat fees, call performance fees, and bull-spread performance fees.[1]

The authors surveyed major investment managers and clients in January 1996. Survey respondents represented approximately $573 billion of funds under management. The median amount under management (per respondent) was about $7 billion. Flat fees, call performance fees, and bull-spread performance fees were each used by at least 45% of the respondents. The survey showed that investment practitioners disagree about whether flat fee, call performance fee, and bull-spread performance fee arrangements motivate portfolio managers to

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manage actively rather than closet index. They also disagree about what performance volatility (volatility relative to a benchmark) these fee structures motivate.[2]

Contract renewal issues may motivate portfolio managers to vary intraperiod performance volatility. It pays to reduce performance volatility when performance is good (to assure contract renewal) and increase it when performance is bad (in the hope that a virtually certain loss of business may be saved by luck). A lenient client is unlikely to fire the manager unless he lags the benchmark moderately for several periods or suffers a large negative performance in one period. For these managers, closet indexing then assures contract renewal for several periods while large performance volatility makes contract renewal only a 50–50 proposition.

Many authors have considered the incentives created by investment management fee structures.[3] Most use a single period context, which rules out incorporating incentives associated with contract renewal. Two exceptions are Heinkel and Stoughton (1994) (henceforth HS) and Ferguson and Leistikow (1997) (henceforth FLa).

HS present a two period model that addresses contract renewal issues, contracting issues, and allows managers' effectiveness to change with effort. Unfortunately, it sacrifices too much realism in order to obtain tractability. For example, it allows all active managers to provide positive expected performance which is inconsistent with the fact that market weighted performance must be zero.

FLa was the first long-run risk-neutral analysis of the performance volatility incentives created by investment management fees. This paper extends FLa. It removes the requirement that distributions less contributions equal investment return. It treats the portfolio as untraded (in a limited risk-neutral valuation sense), which is required to make expected investment performance relevant. It incorporates expenses, distributions, and investment management performance fees. It determines if the contract renewal structure and fee arrangements discriminate effectively among talented and untalented managers. Finally, it introduces a volatility-dependent contract renewal structure that provides good discrimination and strongly motivates manager behavior consistent with client preferences.

This paper shows that talented and untalented managers often have different incentives and that some bull-spread and flat fee arrangements effectively discriminate among talented and untalented managers. It also shows that the volatility-dependent contract renewal structures introduced in this paper provide superior results to the volatility-independent structures of FLa. Managers can be strongly motivated to choose a performance volatility that reflects client preferences.

The paper is organized as follows. Section 2 defines important terms and describes the analytical context. Section 3 provides one-period values for generic call, bull-spread, and flat fees. Section 4 derives the long-run value of any investment management fee arrangement that can be represented as an end of period payoff pattern. Section 5 introduces the concept of discrimination and shows how to improve it by changing the performance measurement period. Section 6 introduces volatility-dependent contract renewal strategies and contrasts them with the volatility-independent ones addressed in FLa. The behavior of the portion of a fee structure's value common to all fees, the capitalization factor, is discussed in Section 7. Section 8 compares the incentives created by one-period and long-run call, bull-spread, and flat fees. Section 9 expands the results to determine intraperiod incentives. Section 10 is the conclusion.

2 Definitions and context

A summary of definitions is given at the end of the paper.

Arithmetic performance return over an interval, R_{PB}, is the portfolio's arithmetic return over an interval measured relative to a benchmark. Equation (1) defines arithmetic performance return:

$$1 + R_{PB} \equiv \frac{1 + R_P}{1 + R_B} \tag{1}$$

where $R_P \equiv$ the portfolio's arithmetic return over an interval, $R_B \equiv$ the portfolio's arithmetic return over an interval if it had been invested in the benchmark, and $(1 + R_{PB}) \equiv$ the arithmetic performance return factor. $(1+R_{PB})$ is proportional to the ratio of the portfolio's price to the benchmark's. No generality is lost by thinking of it as the portfolio's price relative to the benchmark's price, i.e. as a relative price.

Continuous performance return over an interval, r_{PB}, is defined as $\ln(1 + R_{PB})$.

The following stochastic processes for $(1 + R_P)$ and $(1 + R_B)$ are common in the literature. Processes of this form with constant parameters imply lognormality over finite intervals.[4] Hence, both quantities can be thought of as standardized asset prices.

$$\frac{d(1 + R_i)}{(1 + R_i)} = \mu_i\, dt + \sigma_i\, dz_i \qquad i = P, B \tag{2}$$

where $\mu_i \equiv$ a drift rate. This is the one-period expected return of R_i expressed in continuous form,[5] $\sigma_i \equiv$ a volatility rate, and $dz_i \equiv$ a standardized Brownian motion disturbance term. The processes for the portfolio and the benchmark imply the stochastic process for $(1 + R_{PB})$.

$$\frac{d(1 + R_{PB})}{(1 + R_{PB})} = [(\mu_P - \mu_B) + (\sigma_B^2 - \rho_{PB}\sigma_P\sigma_B)]dt + \sqrt{\sigma_P^2 - 2\rho_{PB}\sigma_P\sigma_B + \sigma_B^2}\, dz_{(1+R_{PB})} \tag{3}$$

Equation (3) shows that the arithmetic performance return factor and the portfolio's price relative to the benchmark's are also lognormal over finite intervals. Hence, both can be thought of as assets. In particular, $(1+R_{PB})$ can be thought of as the "performance asset." The performance asset's price depends on the portfolio and benchmark prices. Thus, it is a derivative and can be valued as one.

The definition of arithmetic performance in Eq. (1) makes the drift rate (the term in brackets) in Eq. (3) performance. However, $(\mu_P - \mu_B)$ reflects both talent and the portfolio's differential systematic risk, measured relative to the benchmark.[6] $(\sigma_B^2 - \rho_{PB}\sigma_P\sigma_B)$ is a risk adjustment term equal to $(1-\beta_{PB})\sigma_B^2$, where β_{PB} is the coefficient in a regression of portfolio return against benchmark return. Thus, the arithmetic performance factor's drift rate is risk-adjusted performance, not performance. The risk adjustment is zero if the portfolio's beta, measured relative to the benchmark, is one. A beta of one is assumed in the rest of the paper, to simplify the exposition.[7] Thus, $(\mu_P - \mu_B)$ can be thought of as the portfolio's risk-adjusted performance.

The variance rate (the term in the square root) in Eq. (3) corresponds to the variance of the difference between the portfolio and benchmark returns. We denote it by σ_{P-B}^2.

Denote the expected value of by R_{PB} by $\mu_{R_{PB}}$. This is termed the arithmetic performance rate. Then, as made plausible above, the $(1 + R_{PB})$ process's drift rate is $\ln(1 + \mu_{R_{PB}})$. This is called the performance rate (as opposed to the continuous performance rate, which is the drift rate of r_{PB}).[8]

$$\frac{d(1+R_{PB})}{(1+R_{PB})} = \ln(1+\mu_{R_{PB}})dt + \sigma_{P-B}\, dz_{(1+R_{PB})} \tag{4}$$

Denote the arithmetic interest rate by R_F and the continuous interest rate, $\ln(1+R_F)$, by r_F.

Assume risk-neutral valuation is valid for the portfolio and the benchmark. Then, the portfolio and the benchmark have the following risk-neutral processes:

$$\begin{aligned}\frac{d(1+R_i)}{(1+R_i)} &= \ln(1+R_F)\, dt + \sigma_i\, dz_i \\ &= r_F\, dt + \sigma_i\, dz_i, \qquad i = P, B\end{aligned} \tag{5}$$

The performance asset's risk-neutral drift rate is obtained by replacing the portfolio and benchmark drift rates in Eq. (3) by their risk-neutral counterparts. Both their risk-neutral drift rates are the interest rate. This eliminates the $(\mu_P - \mu_B)$ term in the drift rate in Eq. (3); hence, there is no expected performance, and it appears that the manager's talent does not matter.[9]

The easiest way to bring in the portfolio's expected performance is to treat the portfolio as untraded, in a limited sense. It is assumed that the portfolio's price is known, that it can be bought or sold, but that its current or future holdings are unknown (e.g. as is the case for mutual funds). The last assumption prevents hedging the portfolio with its own securities. This precludes constructing the usual Black–Scholes hedge and hence, no longer requires that the portfolio's risk-neutral drift rate be the interest rate. However, since the portfolio's price is known, it is still possible to define traded derivatives on it.[10]

With these assumptions, Ferguson and Leistikow (2001) (henceforth FL) show that the risk-neutral process for the portfolio can be put in the following form:

$$\begin{aligned}\frac{d(1+R_P)}{(1+R_P)} &= \ln[(1+R_F)(1+A)]dt + \sigma_P\, dz_P \\ &= (r_F + a)dt + \sigma_P\, dz_P\end{aligned} \tag{6}$$

Here, A is the portfolio's arithmetic performance rate and a is its performance rate. Roughly speaking, a is the difference between the portfolio's and benchmark's drift rates.

We denote the risk-neutral arithmetic performance rate by $\mu_{R_{PBRN}}$. Then, the risk-neutral performance rate is $\ln(1+\mu_{R_{PBRN}})$.

Equations (6), (4), and (3) imply the following risk-neutral drift rate for the performance asset:

$$\ln(1+\mu_{R_{PBRN}}) = a \tag{7}$$

Now, talent matters. Positive talent increases and negative talent decreases the performance rate.

This paper treats an investment management contract as a traded derivative defined on the portfolio and the benchmark.[11]

3 One-period investment management fees

3.1 Call performance fees

The generic one-period proportionate call performance fee is

$$F_C = \max[0, f(R_{PB} - R_T)] \tag{8}$$

Here, $F_C \equiv$ the call performance fee, as a proportion of the end of period value of an investment in the benchmark,[12] $R_T \equiv$ a threshold level of performance, expressed as an arithmetic performance return, and $f \equiv$ the proportion of performance above the threshold level (excess performance) paid as a performance fee. This is the fee rate.

The value of this performance fee arrangement to managers, C_F, is a special case of Margrabe (1978), an option to exchange one asset for another.

Margrabe treats both assets as traded. This makes their risk-neutral drift rates the interest rate. Thus, there is no risk-neutral expected performance and expected performance is irrelevant. Margrabe is easily modified for underlying assets that have distributions (e.g. dividends). The technique is the same one used for modifying the Black–Scholes call model. The assets' prices are replaced by their prices less the present value of their distributions over the option's life. For example, if an asset's price and arithmetic yield are S and Y, respectively, then the asset's price is replaced by $(1 - Y)S$ (for a one-period option). The significance of this paper's limited untraded portfolio assumption is that expected performance now appears as a negative distribution. Thus, the portfolio's value is multiplied by $(1 + \mu_{R_{PB}})$.

This modification changes the value of C_F (the performance call). The new result is obtained from Margrabe (1978) by "reducing" the initial value of the performance asset, 1.0, by the present value factor of its "dividends." Appendix A summarizes the required development of Margrabe.

$$C_F = f\left[(1 + \mu_{R_{PB}})N(d_1) - (1 + R_T)N(d_2)\right] \tag{9}$$

$$d_1 \equiv \frac{\ln\left[(1 + \mu_{R_{PB}})/(1 + R_T)\right] + (\sigma^2_{r_{PB}}/2)}{\sigma_{r_{PB}}} \tag{10}$$

$$d_2 \equiv d_1 - \sigma_{r_{PB}} \tag{11}$$

where $\sigma_{r_{PB}} \equiv$ the standard deviation of $\ln(1+R_{PB})$.

Differential (portfolio less benchmark) expense and distribution rates are treated similarly. To simplify the presentation, it is assumed below that the portfolio and benchmark have the same expense and distribution rates. To relax this assumption, define E_{PB} and Y_{PB} as the performance asset's expense and distribution rates. These reflect the differential expense and distribution rates between the portfolio and the benchmark. Then, replace $(1 + \mu_{R_{PB}})$ in the formulas with $(1 + \mu_{R_{PB}})(1 - E_{PB} - Y_{PB})$.

3.2 *Bull-spread performance fees*

With bull-spread performance fees, managers keep a fraction of the excess performance up to a cap, R_C. The bull-spread is equivalent to being long a performance call with a strike price of $(1 + R_T)$ and short a performance call with a strike price of $(1 + R_C)$. The value of this fee structure is the difference in the values of the two performance calls that comprise it.[13]

3.3 *Flat fees*

The current one-period value of a flat fee is simply the promised proportion of the portfolio's current value.

4 Long-run investment management fees: description and valuation

Let us make the following assumptions. The client will renew the one-period management contract as long as the manager's most recent one-period arithmetic performance return exceeds a minimum level, R_M. Each period is identical, in that the investment management fee arrangement, the manager's expected performance, and the manager's beginning of period performance volatility do not change.[14] Performance is independent across periods.

Let V_{LR} denote the (risk-neutral) long-run contract value, as a proportion of the portfolio's beginning value, and let there be a terminal date T periods in the future. Let V_{OP} represent the proportionate risk-neutral value of the one-period investment management contract. This depends only on the contract terms and performance parameters, not the portfolio's or benchmark's paths (or values). Then, it is shown in Appendix B that

$$V_{\mathrm{LR}} = \left[\frac{1-[q(1+\mu_{\mathrm{RNGD}})]^T}{1-[q(1+\mu_{\mathrm{RNGD}})]}\right] V_{\mathrm{OP}} \qquad (12)$$

Here, $q \equiv$ a pseudo-risk-neutral contract renewal probability (akin to the risk-neutral probability of an up move in the binomial option pricing model). The pseudo-risk-neutral contract renewal probability does not depend on the portfolio's beginning of period value and is constant from period to period unless the contract renewal terms change. If they do, then it still does not depend on the portfolio's beginning of period value, but does vary from period to period. $\mu_{\mathrm{RNGD}} \equiv (1+\mu_{R_{\mathrm{PB}}})(1-E-Y)-1$. The portfolio's "discounted" risk-neutral expected growth rate. It incorporates performance, expenses, and distributions, but not the interest rate. $E \equiv$ the portfolio's expense ratio, i.e. its expenses for a period divided by its beginning of period value. $Y \equiv$ the portfolio's distribution ratio, i.e. its distributions for a period divided by its beginning of period value.

Appendix B also shows that the pseudo-risk-neutral contract renewal probability depends on two risk-neutral expected growth rates. The first, μ_{RNG}, is the portfolio's risk-neutral expected growth rate. This incorporates the interest rate, performance, expenses, and distributions.

$$1+\mu_{\mathrm{RNG}} = (1+R_{\mathrm{F}})(1+\mu_{R_{\mathrm{PB}}})(1-E-Y) \qquad (13)$$

The second, μ_{PRNG}, is the portfolio's effective risk-neutral expected growth rate. This is lower than its actual risk-neutral expected growth rate, μ_{RNG}, because it reflects the probability of non-renewal.

$$q = \left(\frac{1+\mu_{\mathrm{PRNG}}}{1+\mu_{\mathrm{RNG}}}\right) \qquad (14)$$

The paper's examples assume that performance is independent of the portfolio's risk-neutral return. Appendix B shows that, then,

$$V_{\mathrm{LR}} = \left[\frac{1-[(1-N(z))(1+\mu_{R_{\mathrm{PBRN}}})(1-E-Y)]^T}{1-[(1-N(z))(1+\mu_{R_{\mathrm{PBRN}}})(1-E-Y)]}\right] V_{\mathrm{OP}} \qquad (15)$$

$$z = \frac{r_{\mathrm{M}} - \left[\ln(1+\mu_{R_{\mathrm{PBRN}}}) - \frac{\sigma^2_{r_{\mathrm{PB}}}}{2}\right]}{\sigma_{r_{\mathrm{PB}}}} \qquad (16)$$

$$q = q_* = 1 - N(z) \qquad (17)$$

Here, $N(.) \equiv$ the cumulative normal distribution, $\mu_{R_{\mathrm{PBRN}}} \equiv$ the portfolio's risk-neutral arithmetic performance rate (defined in Eq. (7)), and $r_{\mathrm{M}} \equiv \ln(1+R_{\mathrm{M}})$. This is the minimum performance required for contract renewal, expressed in continuous form.

With convergence, as $T \to \infty$, Eq. (15) becomes

$$V_{\mathrm{LR}} = \left[\frac{1}{1-[(1-N(z))(1+\mu_{R_{\mathrm{PBRN}}})(1-E-Y)]}\right] V_{\mathrm{OP}} \qquad (18)$$

The equations for V_{LR} show that there is a positive marginal return to talent and negative marginal returns to differential expenses and distributions. The examples below show that these marginal returns can be substantial.

Equation (16) shows that since increasing (reducing) the performance rate increases (reduces) $\mu_{R_{\mathrm{PBRN}}}$ and its continuous analog, $(1+\mu_{R_{\mathrm{PBRN}}})$, which is deducted from r_{M}, the effective minimum performance required for contract renewal is reduced (increased). This makes contract renewal more (less) likely. However, the performance rate must be zero if performance volatility is, since the only way to make performance volatility zero is to invest in the benchmark. Also, a reasonable expectation is that the performance rate increases with performance volatility over a moderate range for talented managers and decreases for untalented managers. The paper's methodology works with any sensible function. For illustrative purposes, the performance rate is assumed proportional to performance volatility, $\sigma_{r_{PB}}$. The constant of proportionality, called the performance ratio, is denoted by γ.

A constant performance ratio is somewhat unrealistic. γ should decline beyond a moderate performance volatility. Furthermore, high performance volatility probably is achievable only through leverage. In this case, there is a nonzero probability of a total loss. For these reasons, the authors urge caution in interpreting the examples. However, it is comforting that even with a constant performance ratio the risk-neutral contract renewal probability approaches zero as performance volatility approaches infinity, as shown below.

With the constant performance ratio assumption, Eq. (16) becomes

$$z = -\gamma + \frac{r_{\mathrm{M}}}{\sigma_{r_{\mathrm{PB}}}} + \frac{\sigma_{r_{\mathrm{PB}}}}{2} \qquad (19)$$

5 Discriminating among talented and untalented managers

A desirable feature of a contract renewal structure and fee arrangement is that talented managers have significantly higher risk-neutral contract renewal probabilities than untalented ones. This is defined as effective discrimination among talented and untalented managers.

In principle, the contract renewal structure provides two mechanisms for achieving discrimination. Equations (16), (17), and (19) show that the risk-neutral contract renewal probability can be considered a function of performance volatility and portrayed as a risk-neutral contract renewal probability curve.

Suppose two managers have the same risk-neutral contract renewal probability curve. If it is monotonic and if the two managers are led to adopt different performance volatilities, then they will have different risk-neutral contract renewal probabilities. Alternatively, if the two managers have different risk-neutral contract renewal probability curves, then different heights at their respective performance volatilities also produces different risk-neutral contract renewal probabilities. Only the latter case is relevant. The only way for two managers to have the same risk-neutral contract renewal probability curve is for them to have the same performance ratio. Hence, only in the latter case is there a discrimination.

Discrimination is possible even when managers do not know their performance ratios. Consider two managers who decide to act as if their performance ratios are zero. The fee arrangement then leads them to adopt the same performance volatility. Then, Eq. (19) implies a lower z value for the manager with the best performance ratio. Thus, the contract renewal structure implies a higher risk-neutral contract renewal probability for the manager with the highest (unknown) performance ratio. This is true whenever two managers choose the same performance volatility.

One way to improve the discrimination is to increase the performance measurement period. Denote the one-period z value by z_1 and that for t periods by z_t. Then, both Eqs. (16) and (19) imply that:

$$z_t = z_1\sqrt{t} \qquad (20)$$

z_t approaches 0 as t does. This implies a risk-neutral contract renewal probability of 0.5 for all managers, regardless of their talent. In effect, contract renewal for short periods is determined by noise. As t gets large, z_t becomes large and positive (negative) if z_1 is positive (negative). This implies that the risk-neutral contract renewal probability for sufficiently talented managers ($z_1 < 0$) approaches 1 and that for less talented and poor managers ($z_1 > 0$) approaches 0. This corresponds to determining contract renewal by talent. Note that the manager's performance ratio must exceed $(r_M/\sigma_{r_{PB}}) + (\sigma_{r_{PB}}/2)$ to make $z_1 < 0$. This requires a positive performance ratio unless r_M is sufficiently negative. Under realistic circumstances, z_1 can be positive for talented managers with demanding clients ($r_M > 0$). For these managers, the risk-neutral contract renewal probability will approach 0 as t approaches infinity.

6 Volatility-dependent contract renewal strategies

Typical industry practice is to choose a minimum performance required for contract renewal, r_M, that is unrelated to the manager's choice of performance volatility, $\sigma_{r_{PB}}$. This is defined as a volatility-independent contract renewal strategy. A volatility-dependent contract renewal strategy is defined as one where r_M depends on $\sigma_{r_{PB}}$. Volatility-dependent contract renewal strategies can provide better discrimination and more effectively motivate managers to act in a manner consistent with client preferences.

For illustrative purposes, suppose[15]

$$r_M \equiv -\frac{\sigma_{r_{PB}}^2}{2} \tag{21}$$

Then, in the case of Eq. (16),

$$z = -\left[\frac{\ln(1+\mu_{R_{PBRN}})}{\sigma_{r_{PB}}}\right] \tag{22}$$

$$z_t = -\left[\frac{\ln(1+\mu_{R_{PBRN}})}{\sigma_{r_{PB}}}\right]\sqrt{t} \tag{23}$$

In the case of Eq. (19),

$$z = -\gamma \tag{24}$$

$$z_t = -\gamma\sqrt{t} \tag{25}$$

This assures that the risk-neutral contract renewal probability for talented, neutral, and untalented managers is greater than, equal to, and less than 0.5, respectively. It also implies that the risk-neutral contract renewal probability for talented, neutral, and untalented managers increases, stays the same, and decreases, respectively, with increases in the performance measurement period. In the limit as the performance measurement period approaches infinity, talented, neutral, and untalented managers have a risk-neutral contract renewal probability of 1.0, 0.5, and 0.0, respectively. Thus, this rule assures excellent discrimination as t approaches infinity. On the other hand, unless the performance rate ultimately declines to zero at a rate faster than $1/\sqrt{t}$ there is no penalty for managers who adopt unlimited performance volatility.

It is always possible to find a suitable relationship that provides both a discrimination that improves with the length of the performance measurement period and a penalty for deviating from target performance volatility. Suppose, for example, that the client has a target performance volatility of $\sigma^*_{r_{PB}}$. Let the minimum performance for contract renewal be related to *ex-post* performance volatility as follows.[16]

$$r_M = \alpha\sigma_{r_{PB}}(\sigma_{r_{PB}} - \sigma^*_{r_{PB}})^2 - \frac{\sigma_{r_{PB}}^2}{2} \tag{26}$$

Then,

$$z = -\gamma + \alpha(\sigma_{r_{PB}} - \sigma^*_{r_{PB}})^2 \tag{27}$$

Since $z = -\gamma$ at the target performance volatility, there is a discrimination that improves as the performance measurement period increases, as in the previous case. Performance volatilities that deviate from target increase z and, hence, decrease the risk-neutral contract renewal probability, all else equal. The penalty for deviating from target, controlled by α, provides the manager with a tradeoff between the performance volatility the client is comfortable with and the possible benefit of a better performance rate (or possibly even a better performance ratio) at another performance volatility.[17] If the performance ratio is invariant with performance volatility, then, since z depends on the performance ratio, not the performance rate, the benefit shows up in the performance asset's risk-neutral drift rate, not the risk-neutral contract renewal probability. The choice of a particular volatility-dependent strategy should reflect client preferences.

Consider the risk-neutral contract renewal probability's behavior for various volatility-independent minimum performances and the volatility-dependent strategy. z approaches infinity as performance volatility does. This makes q approach 0. The convergence tends to be slow, except for a volatility-dependent strategy. The limit of q_* as performance volatility approaches 0 depends on r_M for volatility-independent strategies. The limits of z for $r_M > 0$, $r_M = 0$, and $r_M < 0$ are ∞, $-\gamma$, and $-\infty$, respectively. The corresponding limits for q_* are 0, roughly 0.5, and 1, respectively. This makes sense. Closet indexing results in a risk-neutral contract renewal probability of 1.0 when the minimum performance return for contract renewal is negative. It assures no renewal when the minimum performance return for contract renewal is positive.

Other than at zero performance volatility, volatility-independent strategies have higher contract renewal probabilities for higher performance ratios. Consequently, volatility-independent strategies provide good discrimination beyond a threshold performance volatility. In addition, these strategies have substantial risk-neutral contract renewal probabilities at excessive performance volatilities. Managers facing negative minimum performance maximize their risk-neutral contract renewal probability by closet indexing. The motivation is less strong for talented managers than untalented. Still, this is not in the client's interest. Managers facing positive minimum performance maximize their risk-neutral contract renewal probability by adopting positive, and for the most part excessive, performance volatility. The maximums, typically, are reasonably steep on the low side and excessively shallow on the high side. This does not provide strong motivation to restrict performance volatility.

Volatility-dependent strategies are superior. They provide good discrimination, except at excessive performance volatility where the risk-neutral contract renewal probability is effectively 0. They strongly maximize the manager's risk-neutral contract renewal probability at the client's target performance volatility. Note that if the performance ratio depends on performance volatility, then the manager's risk-neutral contract renewal probability will be maximized by a performance volatility somewhat different from the client's target. However, clients will be better off as long as the volatility-dependent strategy reflects their preferences.

Except where noted, the examples in the remainder of the paper presume a constant performance ratio, a realistic yield, an expense ratio of V_{OP}, and a reasonable fee rate of 0.2. This assures convergence of the geometric series over the range of performance volatilities examined, yet allows the portfolio to grow, unless performance is too negative. The expense ratio assumption makes the expense ratio the average investment management fee (proportionate fees for performance based fee structures vary from period to period). This provides approximately the same result as paying the fee from the portfolio.

7 The capitalization factor

Equation (15) shows that the value of the long-run investment management fee arrangement is the product of a capitalization factor (CF) and the one-period value. When there is convergence, CF can be written as follows:

$$\mathrm{CF} = \frac{1}{1 - [(1 - N(z))(1 + \mu_{R_{\mathrm{PBRN}}})(1 - E - Y)]} \tag{28}$$

This CF's partial derivative with respect to performance volatility is

$$\frac{\partial \mathrm{CF}}{\partial \sigma_{r_{\mathrm{PB}}}} = (\mathrm{CF})^2 (1 + \mu_{R_{\mathrm{PB}}})(1 - E - Y) \left[(1 - N(z))\gamma + n(z)\left(\frac{r_{\mathrm{M}}}{\sigma^2_{r_{\mathrm{PB}}}} - \frac{1}{2}\right)\right] \tag{29}$$

z decreases with increase in the performance ratio; hence, $N(z)$ decreases and CF increases. CF does not depend on the interest rate because the interest rate is in both the portfolio's risk-neutral drift rate and the discount factor and hence drops out.

CF approaches 1 as performance volatility approaches infinity, because z approaches infinity (this forces the contract renewal probability, q_*, to 0, implying that only one fee is received). The convergence typically is slow for volatility-independent strategies and fast for volatility-dependent ones.

Equation (29) shows that, for volatility-independent strategies, CF's behavior as performance volatility approaches 0 depends on r_{M} and γ.

When $r_{\mathrm{M}} < 0$

$$\lim_{\sigma_{r_{\mathrm{PB}}} \to 0} \mathrm{CF} = \frac{1}{E + Y} \tag{30}$$

$$\lim_{\sigma_{r_{\mathrm{PB}}} \to 0} \frac{\partial \mathrm{CF}}{\partial \sigma_{r_{\mathrm{PB}}}} = (\mathrm{CF})^2 (1 - E - Y)\gamma \tag{31}$$

The capitalization factor's initial slope is positive, zero, or negative as the performance ratio is.

$N(z)$ increases monotonically from 0 to 1 as performance volatility increases from 0. This makes the capitalization factor $1/(E + Y)$ at 0 performance volatility. As a practical matter, this exceeds 1. Thus, when $\gamma < 0$ the capitalization factor decreases monotonically to 1 and when $\gamma > 0$ the capitalization factor's slope is initially positive, reaches a maximum at a moderate performance volatility and then decreases to 1.

For volatility-independent strategies, the CF motivates closet indexing for lenient clients ($r_{\mathrm{M}} < 0$) unless the performance ratio is positive. A positive performance ratio motivates reasonable performance volatility for lenient clients. The rationale is that closet indexing assures contract renewal when $r_{\mathrm{M}} < 0$. However, adding performance volatility when the performance ratio is positive leads to a higher risk-neutral expected growth rate. With talent, the probability of contract renewal does not drop enough to provide an offset. The result is higher risk-neutral expected fees at positive performance volatility.

When $r_{\mathrm{M}} > 0$,

$$\lim_{\sigma_{r_{\mathrm{PB}}} \to 0} \mathrm{CF} = 1 \tag{32}$$

$$\lim_{\sigma_{r_{\mathrm{PB}}} \to 0} \frac{\partial \mathrm{CF}}{\partial \sigma_{r_{\mathrm{PB}}}} = 0 \tag{33}$$

$N(z)$ is 1 when performance volatility is 0, decreases to a minimum at a performance volatility of $\sqrt{2r_{\mathrm{M}}}$, and then increases to 1 at infinite performance volatility. The minimum value of $N(z)$ is, typically, about 0.5 because $z = \sqrt{2r_{\mathrm{M}}} - \gamma$ at the minimum, which is usually close to 0. Thus, the capitalization factor is 1 when performance volatility is 0, increases to a maximum of roughly $2/(1 + E + Y)$, or about 2, at performance volatility of $\sqrt{2r_{\mathrm{M}}}$, and then decreases to 1 with further increases in performance volatility.

An $r_{\mathrm{M}} > 0$ induces performance volatility of approximately $\sqrt{2r_{\mathrm{M}}}$, but weakly, due to a broad maximum. Here, closet indexing assures the contract will not be renewed. Adding performance volatility increases the chance of getting future fees.

When $r_M = 0$,

$$\lim_{\sigma_{r_{PB}} \to 0} \mathrm{CF} \approx \frac{2}{1 + E + Y} \tag{34}$$

$$\lim_{\sigma_{r_{PB}} \to 0} \frac{\partial \mathrm{CF}}{\partial \sigma_{r_{PB}}} \approx (\mathrm{CF})^2 (1 - E - Y) \left(\frac{\gamma}{2} - 0.2 \right) \tag{35}$$

$N(z)$ increases monotonically with performance volatility from $N(-\gamma) \approx 0.5$ to 1. This makes the capitalization factor about $2/(1 + E + Y)$ at a performance volatility of 0, which approximates to 2 for typical portfolios. As performance volatility increases, the capitalization factor ultimately decreases to 1.

8 One-period versus long-run investment management fee incentives

The following three sections analyze the long-run investment management fee incentives for the call, bull-spread, and flat fee structures.

In the examples that follow:

- Positive and negative threshold performance are 0.02 and −0.02, respectively.
- Positive and negative talent represent performance ratios of 0.3 and −0.3, respectively.[18]
- Lenient and demanding clients represent minimum performance for contract renewal that is negative and positive, respectively.
- Cap performance for the bull-spread structure is 0.08.
- The volatility-dependent contract renewal strategy is that in Eq. (26) with $\sigma^*_{r_{PB}} = 0.1$ and $\alpha = 100$.
- The fee rate is 0.2.

Tables 1–3 summarize the typical characteristics for the call, bull-spread, and flat fee structures, respectively, when the minimum return for contract renewal is volatility independent.

As an example, consider the call case of positive threshold performance (0.02) and positive talent (performance ratio of 0.3). The first row of Table 1 shows that talented managers with lenient clients (negative minimum performance for contract renewal) maximize the value of their fees by taking infinite performance volatility, although there is a local maximum at moderate performance volatility. Infinite performance volatility also maximizes the value of their fees with demanding clients (positive minimum performance for contract renewal). However, there is no local maximum in this case.

The second row of Table 1 shows that the contract renewal probability for demanding clients (corresponding to large performance volatility) is about 0. Two characterizations are given for lenient clients, about 0 for the case of infinite (large) performance volatility and "high" for the case of the moderate performance volatility corresponding to the local maximum.

The other rows of Table 1 and the rows of Tables 2 and 3 are interpreted similarly.

The tables show that, roughly speaking, call performance fees generate unproductive incentives, while some bull-spread performance fees and flat fees provide better (but not entirely desirable) incentives. Detailed analyses for each fee structure follow.

8.1 Call performance fees

The one-period call performance fee's value approaches

$$f(1 + \mu_{R_{PB}}) = f\, \mathrm{e}^{\ln(1+\mu_{R_{PB}})} = f\, \mathrm{e}^{\gamma \sigma_{r_{PB}}} \tag{36}$$

Table 1 Typical characteristics of value maximized call performance fees when the minimum return for contract renewal is volatility independent.

Threshold performance	Talent	Characteristic	Client type: Lenient	Client type: Demanding
Positive	Positive	Performance volatility	Infinity at the global fee maximum or moderate at the local fee maximum	Infinity
		Contract renewal probability	About 0 at the global fee maximum or high at the local fee maximum	About 0
	Negative	Performance volatility	Infinity at the global fee maximum or moderate at the local fee maximum	Infinity
		Contract renewal probability	About 0 at the global fee maximum or high at the local fee maximum	About 0
Negative	Positive	Performance volatility	Infinity at the global fee maximum or moderate at the local fee maximum	Infinity
		Contract renewal probability	About 0 at the global fee maximum or high at the local fee maximum	About 0
	Negative	Performance volatility	Infinity at the global maximum or closet index at the local maximum	Infinity
		Contract renewal probability	About 0 at the global fee maximum or about 0 at the local fee maximum	About 0
All	NA	Discrimination between talented and untalented managers	Poor	Poor

as performance volatility approaches infinity. The limits are infinity for positive performance ratios, f for a performance ratio of 0.0, and 0.0 for negative performance ratios. Consequently, a negative performance ratio never leads to unlimited performance volatility.

Consider what happens when $R_T > 0$ (about 0.02) for various levels of r_M.

Volatility-independent contract renewal strategies motivate managers to maximize performance volatility when $r_M > 0$. This clearly is undesirable

Table 2 Typical characteristics of value maximized bull-spread performance fees when the minimum return for contract renewal is volatility independent.

Threshold performance	Talent	Characteristic	Client type	
			Lenient	Demanding
Positive	Positive	Performance volatility	Moderate	Finite but excessive
		Contract renewal probability	High	Moderate
	Negative	Performance volatility	Moderate	Finite but excessive
		Contract renewal probability	High	Low
Negative	Positive	Performance volatility	Moderate	Finite but excessive
		Contract renewal probability	High	Moderate
	Negative	Performance volatility	Closet Index	Finite but excessive
		Contract renewal probability	About 1	Low
All	NA	Discrimination between talented and untalented managers	Poor	Good

Table 3 Typical characteristics of value maximized flat performance fees when the minimum return for contract renewal is volatility independent.

Threshold performance	Talent	Characteristic	Client type	
			Lenient	Demanding
NA	Positive	Performance volatility	Moderate	Finite but excessive
		Contract renewal probability	High	Moderate
	Negative	Performance volatility	Closet index	Finite but excessive
		Contract renewal probability	About 1	Low
NA	NA	Discrimination between talented and untalented managers	Poor	Good

when $\gamma \leq 0$. It should be viewed with suspicion when $\gamma > 0$ because the constant performance ratio assumption is untenable at large performance volatility. Furthermore, unlimited performance volatility provides no discrimination among talented and untalented managers, because the risk-neutral contract renewal probability approaches 0. It is as if all managers are fired after one period. Probably it is best to view the performance volatility incentives when $r_M > 0$ as unproductive for all the scenarios. The same conclusions apply, for all practical purposes, when $r_M = 0$.

When $r_M < 0$, managers are rewarded for sufficiently positive performance and are fired only when their performance is sufficiently negative. There are local maximums at a reasonable performance volatility. Practical considerations (being fired for adopting obviously excessive performance volatility) could motivate managers to choose a performance volatility corresponding to these local maximums.[19] The quid pro quo is that the combination of negative minimum performance and small performance volatility implies a risk-neutral contract renewal probability close to 1. This is good if the manager is talented and bad if the manager is untalented. The risk-neutral contract renewal probabilities for the local maximums for a positive performance ratio of 0.3, a negative performance ratio of −0.3, and $r_M = -0.1$ are about 0.9989 and 0.985, respectively. There is virtually no discrimination among talented and untalented managers.

A volatility-dependent strategy does not necessarily provide any better incentives. However, a local maximum of desirable shape can be achieved with a suitable choice of minimum performance and parameters. Thus, for practical purposes, where investors' tolerance for performance volatility is limited, volatility-dependent strategies can work even with call performance fees.

The theoretical problem with both the volatility-independent and volatility-dependent strategies is the call performance fee's unbounded one-period value. The solution is to avoid call performance fees.

Next, suppose that $R_T < 0$ (about −0.02). Now closet indexers make money. When $r_M < 0$ there is a local maximum. For sufficiently negative r_M, the local maximum is high enough so that impracticably large performance volatility is required to exceed it. This motivates managers to control performance volatility rather than maximize it.[20] For all practical purposes, negative performance ratios lead to closet indexing and positive performance ratios lead to reasonable performance volatility. Unfortunately, the risk-neutral contract renewal probabilities for performance ratios of 0.3 and −0.3 are indistinguishable and virtually 1 at the maximums when $r_M = -0.1$.

8.2 Bull-spread performance fees

The one-period value of all bull-spread performance fees approach zero as performance volatility approaches infinity. This is because both performance calls approach $f(1 + \mu_{R_{PB}})$; hence their difference approaches zero.[21] Consequently, the long-run values of all bull-spread performance fees also approach zero as performance volatility approaches infinity. Thus, bull-spread performance fees never motivate investment managers to maintain unlimited performance volatility.

Consider the characteristics when $R_T > 0$ (about 0.02) and $R_C > 0$ (about 0.08). Closet indexing provides no value.

When r_M is sufficiently negative, volatility-independent strategies have large and narrow maximums located at reasonable volatilities. They are located at higher performance volatilities as r_M becomes more negative. Managers are unlikely to be fired, and there is virtually no discrimination between talented and untalented managers. Thus, these strategies are unproductive.

When r_M is not so negative, talented managers are motivated to take less risk. Untalented managers are motivated to take much more risk. The fees' values for talented managers are about six times those for untalented managers.

When $r_M \geq 0$, long-run bull-spreads have broad maximums that occur at larger performance volatilities as r_M rises. Managers are motivated to take on substantial, and often excessive, performance volatility unless r_M is small. The optimum performance volatilities for positive and negative performance ratio scenarios (0.3 and −0.3) when $r_M = 0.075$ are 0.44 and 0.31, respectively. The corresponding figures for $r_M = 0.025$ are 0.35 and 0.27. Talented managers are motivated to take on more performance volatility than untalented ones. The risk-neutral contract renewal probabilities for $r_M = 0.075$ are 0.46 and 0.24 for talented and untalented managers, respectively. Those for $r_M = 0.025$ are 0.52 and 0.30. This fee arrangement discriminates effectively among talented and untalented managers, but motivates excessive performance volatility. For all these strategies, the talented manager's fee values are about three times the untalented manager's. This is typical for bull-spread strategy structures with these kinds of parameters.

A volatility-dependent strategy with the assumed parameters provides superior results. The maximums for both talented and untalented managers are sharper than for a corresponding volatility-independent strategy. The talented manager is motivated to take on a performance volatility of 0.12 and has a risk-neutral contract renewal probability of 0.60. The untalented manager is motivated to take on a performance volatility of 0.13 and has a risk-neutral contract renewal probability of 0.35. The talented manager's fee value is about three times the untalented manager's. This strategy provides reasonable performance volatility and good discrimination.

Next, suppose that threshold performance is negative, $R_T < 0$ (about −0.02), and $R_C = 0.08$. Now, closet indexing pays. The optimum performance volatility and risk-neutral contract renewal probability analysis yields about the same results as before. There is discrimination when $r_M \geq 0$, but not when $r_M < 0$. For example, when $r_M = 0$, the optimum performance volatilities for the positive and negative performance ratios are 0.22 and 0.18. Their corresponding risk-neutral contract renewal probabilities are 0.58 and 0.35. Again, the volatility-dependent strategy is superior. Its optimum performance volatilities for the talented and untalented managers are both 0.11. The corresponding risk-neutral contract renewal probabilities are 0.61 and 0.38. An interesting feature of these volatility-independent scenarios is that, when $r_M < 0$, an untalented manager is motivated to closet index but a talented one is not.

There are no substantive changes when the threshold return is 0.

Bull-spread performance fees strongly motivate managers to maintain a specific positive performance volatility if they incorporate sufficiently negative minimum performance. However, the risk-neutral contract renewal probability for the relevant range of minimum performance is approximately 1 regardless of manager talent. In contrast, non-negative minimum performance provides effective discrimination between talented and untalented managers, although the optimum performance volatilities may be more than clients can bear. A volatility-dependent strategy overcomes all the problems with volatility-independent ones.

What if $R_T = 0$, $R_C = 0.01$, and r_M is small positive for volatility-independent strategies. Volatility-independent strategies' maximums have a shape and location that more strongly motivate managers to maintain reasonable performance volatility. If $r_M = 0.01$, for example, the optimum performance volatilities for the talented and untalented managers are 0.17 and 0.11. The corresponding risk-neutral contract renewal probabilities are 0.56 and 0.33.

The volatility-dependent strategy is far better at motivating managers to adopt a target performance volatility. The risk-neutral contract renewal probabilities for the talented and untalented managers are 0.62 and 0.38, respectively.

An important feature in these examples is that the long-run contract value is about three times greater for the talented manager compared with the untalented one. This shows there is a substantial marginal return to talent. This is true of all the fee structures.

Suppose the talented and untalented managers in the above volatility-independent example ($r_M = 0.01$) do not know their performance ratio and decide to act as if it is 0. Then, both of them will choose a performance volatility of 0.12. The managers' new risk-neutral contract renewal probabilities are essentially unchanged at 0.56 and 0.33. This shows that the contract renewal structure and fee arrangement work even when managers do not know if they are talented. However, the same conclusion is true for a volatility-dependent strategy, and it provides much stronger motivation for a reasonable performance volatility and as good discrimination.

An interesting feature of these fee arrangements is their risk-neutral mean time (number of periods) to the manager's firing, μ_T. The formula for computing μ_T is given below.[22]

$$\mu_T = \frac{1}{1-q} = \frac{1}{N(z)} \tag{37}$$

The talented and untalented managers in the above volatility-independent strategy example have mean relationship lives of about 2.3 and 1.5 periods, respectively. These numbers reflect a performance ratio that is reasonable for a one-year period. Longer periods have higher performance ratios for talented managers and lower ones for untalented managers.[23] This typically leads to a higher risk-neutral contract renewal probability for talented managers and a lower one for untalented managers, per period. Similar conclusions apply to a volatility-dependent strategy.

The bull-spread fees for the latter examples are characterized by limited maximum one-period fees (because the assumed performance cap is close to threshold performance), relatively small probabilities of contract renewal (because minimum performance is positive), and low long-run values (because the fee cannot be large in a single period and the manager is likely to be fired before long).

8.3 *Flat fees*

The relation between long-run flat fees and performance volatility for various levels of r_M and for the volatility-dependent strategy is the same as for the capitalization factor. The only difference is their magnitude, due to multiplication by V_{OP}.

Flat fee curves were examined using a flat fee rate of 0.45%. For volatility-independent strategies, there is a very broad maximum at a performance volatility of $\sqrt{2r_M}$ when $r_M \geq 0$. The shape of the maximum does not strongly motivate managers to maintain specific performance volatility, because it is so broad. However, managers are motivated to take on positive performance volatility (because value initially increases at a significant rate) and are weakly motivated not to take on performance volatility beyond $\sqrt{2r_M}$ (because value decreases subsequently). Using $r_M = 0.025$ as an example, the talented and untalented managers' optimal performance volatilities are 0.45 and 0.18. The corresponding risk-neutral contract renewal probabilities are 0.51 and 0.30. The volatility-dependent strategy strongly motivates a performance volatility of 0.10 for both managers and risk-neutral contract renewal probabilities of 0.62 and 0.38 for the talented and untalented managers, respectively.

For the volatility-independent strategies, untalented managers are motivated to closet index when $r_M < 0$ and talented ones are motivated to adopt reasonable performance volatility. The familiar quid pro quo applies, there is an extremely high risk-neutral contract renewal probability and virtually no discrimination.

9 Intraperiod incentives

The same procedure used to obtain Eq. (15) also works when the first-period fee arrangement and its life differ from subsequent ones. Equation (15) must be modified as follows. The analysis parallels that in FLa.

$$V_{LR} = V_{OP1} + q_1(1+\mu_{RNGD})^{t_1} \times \left[\frac{1-[q(1+\mu_{RNGD})]^{T-1}}{1-[q(1+\mu_{RNGD})]}\right] V_{OP} \tag{38}$$

where $V_{OP1} \equiv$ the first period fee arrangement's one-period value, $q_1 \equiv$ the pseudo-risk-neutral contract renewal probability at the end of period 1, and $t_1 \equiv$ the length of period 1, as a proportion of the length of subsequent periods.

The risk-neutral contract renewal probability at the end of period 1 depends on the first period's fee arrangement and performance volatility. It differs from subsequent periods' contract renewal probability. Maximizing long-run value requires choosing a first-period performance volatility and one for all subsequent periods. First, choose a performance volatility for periods 2 and beyond that maximizes

$$\left[\frac{1-[q(1+\mu_{RNGD})]^{T-1}}{1-[q(1+\mu_{RNGD})]}\right] V_{OP}. \tag{39}$$

Then, choose a first period performance volatility that maximizes V_{LR}.

The intraperiod problem can be viewed as a first-period context that differs from subsequent periods. Thus, Eq. (38) explains why it makes sense for managers to alter performance volatility intraperiod as performance unfolds, and provides a useful mechanism for determining how it can be done.[24] Since each subsequent period eventually becomes a first period, this shows that realized performance volatility can be expected to change continuously during any contract period. However, this does not imply that the beginning of contract period volatility varies from one period to another. As long as the contract periods' terms are identical, beginning of contract period volatilities are, as well.

10 Conclusion

Investment management fee structures create incentives for investment managers. It is important to know if these structures motivate managers to act in their clients' best interest. This paper obtained the performance volatility incentives created by investment management fee structures for a materially more realistic context than heretofore available. In particular, the portfolio's value was allowed to change, the portfolio was treated as untraded (in a limited risk-neutral valuation sense) to make expected investment performance relevant, expenses and distributions were incorporated, paying investment management performance fees from the portfolio was addressed, a framework was developed for analyzing how the contract renewal structure and fee arrangements discriminate among talented and untalented managers, and a volatility-dependent contract renewal structure that provides good discrimination and strongly motivates manager behavior consistent with client preferences was introduced.

The impact of expected investment performance was shown to be important and the analysis provided a way to determine the marginal return to talent. The results suggest a relatively large marginal return to talent and significant changes in optimum performance volatility.

The paper also showed that allowing the portfolio's value to change, incorporating expenses and distributions, and paying management fees from the portfolio significantly changes a management fee's long-run value and optimum performance volatility.

For the most part, volatility-independent contract renewal strategies have important deficiencies. They may encourage closet indexing or fail to discourage excessive performance volatility. They may offer strong motivation to adopt reasonable performance volatility, but only when managers are almost never fired, even untalented ones. They may fail to discriminate among talented and untalented managers, particularly for small performance volatilities. All of the volatility-independent contract renewal strategies and performance fee arrangements examined here suffer from these problems.

Volatility-dependent contract renewal strategies can overcome the limitations of volatility-independent ones. The simple volatility-dependent strategy illustrated in this paper proved satisfactory for all the fee arrangements considered, except for the call performance fee.

The model predicts intraperiod performance volatility behavior consistent with that noted in the literature (e.g. Brown *et al.* (1996)).

Appendix A

The performance call is a call on the better of two assets. Margrabe (1978) was the first to value this kind of option. Using the current paper's notation, Margrabe's expiration date dollar payoff is $\max[0, (1+R_{\mathrm{P}}) - (1+R_{\mathrm{T}})(1+R_{\mathrm{B}})]$. His corresponding dollar call value, w, is

$$w = (1+R_{\mathrm{P0}})N(d_1) - (1+R_{\mathrm{T}})(1+R_{\mathrm{B0}})N(d_2) \qquad \text{(A.1)}$$

$$d_1 = \frac{\ln((1+R_{\mathrm{P0}})/(1+R_{\mathrm{T}})(1+R_{\mathrm{B0}})) + \sigma_{r_{\mathrm{PB}}}^2 t/2}{\sigma_{r_{\mathrm{PB}}}\sqrt{t}} \qquad \text{(A.2)}$$

$$d_2 = d_1 - \sigma_{r_{\mathrm{PB}}}\sqrt{t} \qquad \text{(A.3)}$$

The subscript zero denotes time zero.

Divide the first equation by $(1+R_{\mathrm{B0}})$. Then, substitute $(1+R_{\mathrm{PB}}) = (1+R_{\mathrm{P0}})/(1+R_{\mathrm{B0}})$ throughout. Now, note that $(1+R_{\mathrm{P0}})/(1+R_{\mathrm{B0}}) = 1$, $1+R_{\mathrm{B0}} = 1$, and $t = 1$ at the beginning of the period. Finally, reduce the performance asset's "price" by its "yield rate."

Appendix B

Let $V_{\$\mathrm{LR},t}$ denote the (risk-neutral) long-run contract dollar value at time t and let there be a terminal date, T. The value of the long-run contract one period before the terminal date is the value of a one-period contract.[25]

$$V_{\$\mathrm{LR},(T-1)} = V_{\mathrm{OP}} V_{\mathrm{B},(T-1)} \qquad \text{(B.1)}$$

Here, $V_{\mathrm{B},(T-1)} \equiv$ the risk-neutral value of an investment in the benchmark at the beginning of contract period T.

The value of an investment in the benchmark at the beginning of the contract period is used to be consistent with Margrabe (1978). However, each contract period represents a fresh start, so the benchmark's proper beginning of period risk-neutral value is the portfolio's risk-neutral expected value at $(T-1)$, denoted by $V_{P,(T-1)}$. Thus, $V_{B,(T-i)} \equiv V_{P,(T-i)}$ for all periods.

Two periods before the terminal date, the risk-neutral value of the long-run contract is

$$V_{\$LR,(T-2)} = V_{OP} V_{P,(T-2)} + \frac{E_{PRN}(V_{\$LR,(T-1)})}{1+R_F} \tag{B.2}$$

$E_{PRN}(V_{\$LR,(T-1)})$ is the risk-neutral expectation of $V_{\$LR,(T-1)}$ over the portion of the probability space for period $(T-2)$ where the contract is renewed. The "P" in the expectation operator's subscript is a reminder that this is a partial expectation.

The evaluation of $E_{PRN}(V_{\$LR,(T-1)})$ proceeds as follows:

$$E_{PRN}(V_{\$LR,(T-1)}) = V_{OP} E_{PRN}(V_{P,(T-1)}) \tag{B.3}$$

$$\begin{aligned} E_{PRN}(V_{\$LR,(T-1)}) &= V_{OP} \iint_{V_{P,(T-1)} \geq (1+R_M) V_{B,(T-1)}} V_{P,(T-1)} \\ &\quad \times g_V(V_{P,(T-1)}, V_{B,(T-1)}) \mathrm{d}V_{B,(T-1)}\, \mathrm{d}V_{P,(T-1)} \end{aligned} \tag{B.4}$$

Here, $g_V(\cdot)$ is the joint probability density of $(V_{P,(T-1)}, V_{B,(T-1)})$.

Note that the usage of $V_{B,(T-1)}$ in Eq. (B.4) is with respect to realizations during period $(T-2)$, based on a beginning value for V_B of $V_{B,(T-2)} \equiv V_{P,(T-2)}$. This is clearer if Eq. (B.4) is stated in terms of returns.

Denote the portfolio's and benchmark's risk-neutral returns by R_{PRN} and R_{BRN}, respectively. Then,

$$\begin{aligned} &E_{PRN}(V_{\$LR,(T-1)}) \\ &\quad = V_{OP} V_{P,(T-2)} \iint_{\substack{1+R_{PRN,(T-2)} \\ \geq (1+R_M)(1+R_{BRN,(T-2)})}} (1+R_{PRN,(T-2)}) \\ &\qquad \times g_R(R_{PRN,(T-2)}, R_{BRN,(T-2)}) \\ &\qquad \times \mathrm{d}R_{BRN,(T-2)}\, \mathrm{d}R_{PRN,(T-2)} \end{aligned} \tag{B.5}$$

Here, $g_R(\cdot)$ is the joint probability density of $(R_{PRN,(T-2)}, R_{BRN,(T-2)})$. Denoting continuous returns by r and the corresponding density by $g_r(\cdot)$, Eq. (B.5) can be restated as[26]

$$\begin{aligned} &E_{PRN}(V_{\$LR,(T-1)}) \\ &\quad = V_{OP} V_{P,(T-2)} \iint_{\substack{r_{PRN,(T-2)} \\ \geq r_{BRN,(T-2)} + r_M}} e^{r_{PRN,(T-2)}} \\ &\qquad \times g_r(r_{PRN,(T-2)}, r_{BRN,(T-2)}) \\ &\qquad \times \mathrm{d}r_{BRN,(T-2)}\, \mathrm{d}r_{PRN,(T-2)} \end{aligned} \tag{B.6}$$

Due to the problem's structure, the double integral on the right-hand side of Eq. (B.5) does not depend on the portfolio's beginning of period value and is constant from period to period unless the contract renewal terms change. If they do, then it still does not depend on the portfolio's beginning of period value, but does vary from period to period. The double integral's value is a partial risk-neutral expected growth factor for the portfolio. Denote it by $(1+\mu_{PRNG})$. Then,

$$E_{PRN}(V_{\$LR,(T-1)}) = V_{OP}(1+\mu_{PRNG}) V_{P,(T-2)} \tag{B.7}$$

One explanation of Eq. (B.7) is to view the portfolio's effective risk-neutral expected growth rate as reduced to μ_{PRNG} from its actual risk-neutral expected growth rate, μ_{RNG}. Another approach is to define a pseudo-risk-neutral contract renewal probability, q, such that $E_{\text{PRN}}(V_{\text{P},(T-1)}) = q(1 + \mu_{\text{RNG}})V_{\text{P},(T-2)}$. The portfolio's actual risk-neutral expected growth rate then appears explicitly.

$$E_{\text{PRN}}(V_{\$\text{LR},(T-1)}) = V_{\text{OP}}(1 + \mu_{\text{PRNG}})V_{\text{P},(T-2)}$$

$$\equiv qV_{\text{OP}}V_{\text{P},(T-1)} \quad \text{(B.8)}$$

$$q = (1 + \mu_{\text{PRNG}})\left(\frac{V_{\text{P},(T-2)}}{V_{\text{P},(T-1)}}\right) \quad \text{(B.9)}$$

$$V_{\text{P},(T-1)} = (1 + \mu_{\text{RNG}})V_{\text{P},(T-2)} \quad \text{(B.10)}$$

$$q = \left(\frac{1 + \mu_{\text{PRNG}}}{1 + \mu_{\text{RNG}}}\right) \quad \text{(B.11)}$$

$$q(1 + \mu_{\text{RNG}}) = (1 + \mu_{\text{PRNG}}) \quad \text{(B.12)}$$

The pseudo-risk-neutral contract renewal probability also does not depend on the portfolio's beginning of period value and is constant from period to period unless the contract renewal terms change. If they do, then the pseudo-risk-neutral contract renewal probability still does not depend on the portfolio's beginning of period value, but does vary from period to period.

If performance is independent of the portfolio's risk-neutral return, the double integral in Eq. (B.5) reduces to $q_*(1 + \mu_{\text{RNG}})$, where q_* is the risk-neutral probability that the next period's performance will exceed R_{M} (i.e. q_* is the risk-neutral contract renewal probability).[27] Equation (B.5) then implies that $q = q_*$. If performance is positively correlated with the portfolio's risk-neutral return, then, q typically exceeds the risk-neutral contract renewal probability, if the latter is not 1.

Equation (B.2) now can be rewritten as follows:

$$V_{\$\text{LR},(T-2)} = V_{\text{OP}}V_{\text{P},(T-2)} + \left(\frac{qV_{\$\text{LR},(T-1)}}{1 + R_{\text{F}}}\right) \quad \text{(B.13)}$$

$$V_{\$\text{LR},(T-2)} = V_{\text{OP}}V_{\text{P},(T-2)} + V_{\text{OP}}q\left(\frac{1 + \mu_{\text{RNG}}}{1 + R_{\text{F}}}\right)V_{\text{P},(T-2)} \quad \text{(B.14)}$$

Define the portfolio's expense ratio, E, as its expenses for a period divided by its beginning of period value. Define the portfolio's distribution ratio (yield), Y, similarly. Assume that both expenses and distributions are paid at period end. Then, FL shows that the portfolio's risk-neutral expected growth factor is

$$1 + \mu_{\text{RNG}} = (1 + R_{\text{F}})(1 + \mu_{R_{\text{PB}}})(1 - E - Y) \quad \text{(B.15)}$$

Define the portfolio's discounted risk-neutral expected growth factor, $1 + \mu_{\text{RNGD}}$.

$$1 + \mu_{\text{RNGD}} \equiv (1 + \mu_{R_{\text{PB}}})(1 - E - Y) \quad \text{(B.16)}$$

Equation (B.14) now can be rewritten as

$$V_{\$\text{LR},(T-2)} = V_{\text{OP}}[1 + q(1 + \mu_{\text{RNGD}})]V_{\text{P},(T-2)} \quad \text{(B.17)}$$

Three periods before the terminal date, the long-run contract's value is

$$V_{\$\text{LR},(T-3)} = V_{\text{OP}}\{1 + q(1 + \mu_{\text{RNGD}}) + [q(1 + \mu_{\text{RNGD}})]^2\}V_{\text{P},(T-3)} \quad \text{(B.18)}$$

Working backwards to the current period, the present value of the long-run contract, $V_{\$\text{LR}}$, can be represented as the sum of a geometric series:

$$V_{\$\text{LR}} = \left[\frac{1 - [q(1 + \mu_{\text{RNGD}})]^T}{1 - [q(1 + \mu_{\text{RNGD}})]}\right]V_{\text{OP}}V_{\text{P}} \quad \text{(B.19)}$$

This implies the following proportionate long-run value, V_{LR}:

$$V_{LR} = \left[\frac{1-[q(1+\mu_{RNGD})]^T}{1-[q(1+\mu_{RNGD})]}\right] V_{OP} \quad (B.20)$$

Convergence of the geometric series as $T \to \infty$ requires that $q(1+\mu_{RNGD}) < 1$.[28]

Equation (B.20) can also be written without using the pseudo-risk-neutral contract renewal probability. Define the portfolio's partial discounted risk-neutral expected growth factor, $(1+\mu_{PRNGD})$, as was done to obtain $(1+\mu_{RNGD})$, and substitute from Eq. (B.12).[29]

$$V_{LR} = \left[\frac{1-(1+\mu_{PRNGD})^T}{1-(1+\mu_{PRNGD})}\right] V_{OP} \quad (B.21)$$

The evaluation of the risk-neutral contract renewal probability, q_*, proceeds as in FLa, except that the performance asset's risk-neutral drift rate is modified to include the mispricing related term.

Recall that

$$r_{PB} = \ln(1+R_{PB}). \quad (B.22)$$

Since $(1+R_{PB})$ is a lognormal variate, r_{PB} is normally distributed. Thus, the relationship between the moments of the two variates is

$$1+\mu_{R_{PB}} = e^{\mu_{r_{PB}}+(\sigma^2_{r_{PB}}/2)} \quad (B.23)$$

$$\mu_{r_{PB}} = \ln(1+\mu_{R_{PB}}) - \tfrac{1}{2}\sigma^2_{r_{PB}} \quad (B.24)$$

The risk-neutral value of $\mu_{r_{PB}}$, denoted by $\mu_{r_{PBRN}}$, is

$$\mu_{r_{PBRN}} = \ln(1+\mu_{R_{PBRN}}) - \tfrac{1}{2}\sigma^2_{r_{PB}}. \quad (B.25)$$

Denote the continuously compounded minimum performance required for contract renewal by $r_M \equiv \ln(1+R_M)$. Define z as the number of standard deviations by which r_M exceeds $\mu_{r_{PBRN}}$. The risk-neutral contract renewal probability depends on z:

$$z = \frac{r_M - [\ln(1+\mu_{R_{PBRN}}) - (\sigma^2_{r_{PB}}/2)]}{\sigma_{r_{PB}}} \quad (B.26)$$

Suppose the performance rate is proportional to performance volatility, $\sigma_{r_{PB}}$. Denote the constant of proportionality by γ.

With the constant performance ratio assumption, Eq. (B.26) becomes

$$z = -\gamma + \frac{r_M}{\sigma_{r_{PB}}} + \frac{\sigma_{r_{PB}}}{2} \quad (B.27)$$

Equation (B.28) provides the risk-neutral contract renewal probability, q_* for the case of independence. Independence is assumed to illustrate the paradigm:

$$q_* = 1 - N(z) \quad (B.28)$$

Equations (B.27) and (B.28) imply that the risk-neutral contract renewal probability approaches 0 as performance volatility approaches infinity.

Substituting from Eqs. (B.16) and (B.28) into Eq. (B.20):[30]

$$V_{LR} = \left[\frac{1-[(1-N(z))(1+\mu_{R_{PBRN}})(1-E-Y)]^T}{1-[(1-N(z))(1+\mu_{R_{PBRN}})(1-E-Y)]}\right] V_{OP} \quad (B.29)$$

With convergence, Eq. (B.29) reduces to

$$V_{LR} = \left[\frac{1}{1-[(1-N(z))(1+\mu_{R_{PBRN}})(1-E-Y)]}\right] V_{OP} \quad (B.30)$$

Appendix C

Equation (3)

$$f \equiv \frac{1+R_P}{1+R_B} \equiv \frac{x_1}{x_2} \quad (C.1)$$

$$dx_i = \mu_i x_i\, dt + \sigma_i x_i\, dz_i \tag{C.2}$$

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_j\, dx_k \tag{C.3}$$

$$df = \left(\sum_j \frac{\partial f}{\partial x_j} \mu_j x_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 f}{\partial x_j \partial x_k} \rho_{jk} \sigma_j \sigma_k x_j x_k \right) dt + \sum_j \frac{\partial f}{\partial x_j} \sigma_j x_j\, dz_j \tag{C.4}$$

$$\frac{\partial f}{\partial x_1} = \frac{1}{x_2} \qquad \frac{\partial^2 f}{\partial x_1^2} = 0$$
$$\frac{\partial f}{\partial x_2} = -\frac{x_1}{x_2} \qquad \frac{\partial^2 f}{\partial x_2^2} = 2\frac{x_1}{x_2^3} \tag{C.5}$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}$$

$$df = \left\{ \mu_1 \left(\frac{x_1}{x_2} \right) - \mu_2 \left(\frac{x_1}{x_2} \right) + \frac{1}{2} \left[-2\rho_{12}\sigma_1\sigma_2 \left(\frac{x_1}{x_2} \right) + 2\sigma_2^2 \left(\frac{x_1}{x_2} \right) \right] \right\} dt + \left[\sigma_1 \left(\frac{x_1}{x_2} \right) dz_1 - \sigma_2 \left(\frac{x_1}{x_2} \right) dz_2 \right] \tag{C.6}$$

$$\frac{df}{f} = [(\mu_1 - \mu_2) + (\sigma_2^2 - \rho_{12}\sigma_1\sigma_2)]dt + [\sigma_1 dz_1 - \sigma_2 dz_2] \tag{C.7}$$

$$\sigma_f\, dz_f \equiv \sigma_1\, dz_1 - \sigma_2\, dz_2 \tag{C.8}$$

$$\sigma_f^2\, dz_f^2 = \sigma_1^2\, dz_1^2 - 2\sigma_1\sigma_2\, dz_1\, dz_2 + \sigma_2^2\, dz_2^2 \tag{C.9}$$

$$\sigma_f^2 = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2 \tag{C.10}$$

$$\frac{df}{f} = [(\mu_1 - \mu_2) + (\sigma_2^2 - \rho_{12}\sigma_1\sigma_2)]dt + \sigma_f dz_f \tag{C.11}$$

$$\beta_{12} = \rho_{12}\frac{\sigma_1}{\sigma_2} \tag{C.12}$$

$$\frac{df}{f} = [(\mu_1 - \mu_2) + (1 - \beta_{12})\sigma_2^2]dt + \sigma_f\, dz_f \tag{C.13}$$

Partial derivative of the capitalization factor with respect to risk-neutral performance volatility and its limit as performance volatility approaches zero

The formula presumes a constant performance ratio.

Starting with Eq. (18),

$$V_{\mathrm{LR}} = \left[\frac{1}{1 - [(1 - N(z))(1 + \mu_{R_{\mathrm{PBRN}}})(1 - E - Y)]} \right] V_{\mathrm{OP}} \tag{C.14}$$

$$\mathrm{CF} \equiv \frac{1}{1 - [(1 - N(z))(1 + \mu_{R_{\mathrm{PBRN}}})(1 - E - Y)]} \equiv \frac{1}{x} \tag{C.15}$$

$$\frac{\partial \mathrm{CF}}{\partial \sigma_{r_{\mathrm{PB}}}} = \frac{\partial \mathrm{CF}}{\partial x} \frac{\partial x}{\partial \sigma_{r_{\mathrm{PB}}}} = -\frac{1}{x^2} \frac{\partial x}{\partial \sigma_{r_{\mathrm{PB}}}} = -(\mathrm{CF})^2 \frac{\partial x}{\partial \sigma_{r_{\mathrm{PB}}}} \tag{C.16}$$

$$\frac{\partial x}{\partial \sigma_{r_{\mathrm{PB}}}} = -\frac{\partial(1 + \mu_{R_{\mathrm{PBRN}}})}{\partial \sigma_{r_{\mathrm{PB}}}}(1 - E - Y) + N(z)\frac{\partial(1 + \mu_{R_{\mathrm{PBRN}}})}{\partial \sigma_{r_{\mathrm{PB}}}}(1 - E - Y) + n(z)(1 + \mu_{R_{\mathrm{PBRN}}})(1 - E - Y)\frac{\partial z}{\partial \sigma_{r_{\mathrm{PB}}}} \tag{C.17}$$

$$\ln(1+\mu_{R_{\text{PBRN}}}) \equiv \gamma\sigma_{r_{\text{PB}}}$$
$$(1+\mu_{R_{\text{PBRN}}}) = e^{\gamma\sigma_{r_{\text{PB}}}} \quad \text{(C.18)}$$
$$\frac{\partial(1+\mu_{R_{\text{PBRN}}})}{\partial\sigma_{r_{\text{PB}}}} = \gamma(1+\mu_{R_{\text{PBRN}}})$$

$$z \equiv -\gamma + \frac{r_{\text{M}}}{\sigma_{r_{\text{PB}}}} + \frac{\sigma_{r_{\text{PB}}}}{2} \quad \text{(C.19)}$$
$$\frac{\partial z}{\partial\sigma_{r_{\text{PB}}}} = -\frac{r_{\text{M}}}{\sigma^2_{r_{\text{PB}}}} + \frac{1}{2}$$

$$\frac{\partial \text{CF}}{\partial\sigma_{r_{\text{PB}}}} = (\text{CF})^2(1+\mu_{R_{PB}})(1-E-Y) \times \left[(1-N(z))\gamma + n(z)\left(\frac{r_{\text{M}}}{\sigma^2_{r_{\text{PB}}}} - \frac{1}{2}\right)\right] \quad \text{(C.20)}$$

If $r_{\text{M}} < 0$:

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} z = -\infty$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} N(z) = 0$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} n(z) = 0 \quad \text{(C.21)}$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{n(z)}{\sigma^2_{r_{\text{PB}}}} = 0$$

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{\partial \text{CF}}{\partial\sigma_{r_{\text{PB}}}} = (\text{CF})^2(1-E-Y)\gamma \quad \text{(C.22)}$$

If $r_{\text{M}} > 0$:

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} z = \infty$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} N(z) = 1$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} n(z) = 0 \quad \text{(C.23)}$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{n(z)}{\sigma^2_{r_{\text{PB}}}} = 0$$

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{\partial \text{CF}}{\partial\sigma_{r_{\text{PB}}}} = 0 \quad \text{(C.24)}$$

If $r_{\text{M}} = 0$:

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} z = -\gamma \approx 0$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} N(z) = N(-\gamma) \approx 0.5$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} n(z) = n(-\gamma) \approx 0.4 \quad \text{(C.25)}$$
$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{n(z)r_{\text{M}}}{\sigma^2_{r_{\text{PB}}}} = 0$$

$$\lim_{\sigma_{r_{\text{PB}}}\to 0} \frac{\partial \text{CF}}{\partial\sigma_{r_{\text{PB}}}} \approx (\text{CF})^2(1-E-Y)\left(\frac{\gamma}{2} - 0.2\right) \quad \text{(C.26)}$$

Definitions

$A \equiv$ Arithmetic performance rate.
$a \equiv$ Performance rate.
$\mathrm{d}t \equiv$ An infinitesimal time interval.
$\mathrm{d}z \equiv$ A Brownian motion disturbance term. This can be thought of as the product of $\mathrm{d}t$ and the value of a random draw from a standard normal density.
$E \equiv$ The portfolio's expense ratio, i.e. its expenses for a period divided by its beginning of period value.
$F_{\text{C}} \equiv$ The call performance fee, as a proportion of the end of period value of an investment in the benchmark.
$f \equiv$ The proportion of performance above the threshold level (excess performance) paid as a performance fee. This is the fee rate.
$\gamma \equiv$ A drift rate in the context of a continuous (logarithmic) return process. The performance ratio (performance rate/performance volatility) in the context of management contract renewal.

$\mu \equiv$ An expected value in the context of an ordinary random variable. A drift rate in the context of an arithmetic return factor process.

$\mu_{\text{PRNG}} \equiv$ The portfolio's effective risk-neutral expected growth rate. This is lower than its actual risk-neutral expected growth rate because it reflects the probability of non-renewal.

$\mu_{\text{RNG}} \equiv$ The portfolio's risk-neutral expected growth rate. It incorporates the interest rate, performance, expenses, and distributions.

$\mu_{\text{RNGD}} \equiv$ The portfolio's "discounted" risk-neutral expected growth rate. It incorporates performance, expenses, and distributions, but not the interest rate.

$q \equiv$ A pseudo-risk-neutral contract renewal probability (akin to the risk-neutral probability of an up move in the binomial option pricing model).

$R \equiv$ Arithmetic return.

$R_{\text{PB}} \equiv$ Arithmetic performance return of P with respect to B.

$1 + R_{\text{PB}} \equiv$ Arithmetic performance return factor.

$R_{\text{C}} \equiv$ An arithmetic performance return cap applicable to bull-spread performance fees. Managers do not share in performance beyond this level.

$R_{\text{M}} \equiv$ The minimum level of one-period performance required for the manager's contract to be renewed.

$R_{\text{T}} \equiv$ A threshold level of performance, expressed as an arithmetic performance return. Managers do not share in performance below this level.

$r \equiv$ Continuous (logarithmic) return.

$r_{\text{PB}} \equiv$ Continuous performance return.

$\rho \equiv$ Correlation coefficient.

$\sigma^2 \equiv$ A variance in the context of an ordinary random variable. A variance rate in the context of a Brownian motion process. The corresponding standard deviation is called "volatility."

$V_{\text{LR}} \equiv$ The risk-neutral long-run contract value, as a proportion of the portfolio's beginning value.

$V_{\text{OP}} \equiv$ The proportionate risk-neutral value of the one-period investment management contract.

$Y \equiv$ The portfolio's distribution ratio, i.e. its distributions for a period divided by its beginning of period value.

Notes

1 Examples include Davanzo and Nesbitt (1987), Grinblatt and Titman (1989), and Kritzman (1987).

2 Only 59% of the respondents agreed that flat fee arrangements encourage active management (30% disagreed). About 59% of the respondents thought that flat fees motivate managers to take on less than 5% annualized performance volatility. The rest thought that they motivate performance volatility between 5% and 15%.

Almost 90% of the respondents either strongly agreed or agreed that call performance fee arrangements encourage managers to actively manage. The other respondents did not know (5%), or disagreed (5%). About 29%, 59%, 6%, and 6% of the respondents thought that call performance fee arrangements motivate performance volatility below 5%, between 5% and 15%, between 15% and 25%, and above 25%, respectively.

About 95% of the respondents either strongly agreed or agreed that bull-spread performance fees encourage managers to actively manage, while the other respondents did not know. Nearly 33% of the respondents thought that bull-spread performance fees encourage portfolio managers to take on annualized performance volatility below 5%, 60% thought they encourage performance volatility between 5% and 15%, and 7% thought they encourage performance volatility over 25%.

3 Examples include Admati and Pfleiderer (1997), Brown *et al.* (1996), Davanzo and Nesbitt (1987), Ferguson and Leistikow (1997), Grinblatt and Titman (1989), Heinkel and Stoughton (1994), Kritzman (1987), and Starks (1987).

4 Most of the paper presumes constant parameters for ease of exposition. The results apply more generally.

5 To make this plausible, consider the process $dx/x = \mu\,dt + \sigma\,dz$. The solution when $\sigma = 0$ is $x = x_0 e^{\mu t}$. This implies that $\mu = \ln\left[(x_t/x_0)^{(1/t)}\right] = \ln(1 + \mu_{R_x})$.

6 For example, a high beta passive portfolio ought to outperform a low beta passive portfolio over time.

7 There is no loss of generality.

8 The drift rate of r_{PB} is $(\gamma_P - \gamma_B)$, where $\gamma_i \equiv \mu_i - (\sigma_i^2/2)$.

9 Note that when beta is not one, the risk-adjustment term in the drift rate of Eq. (3) remains.

10 See, for example, Hull, *Options, Futures, and Other Derivatives*, 3rd edn., pp. 305–307.

11 Four possible reasons for accepting this approach are as follows.

Risk-neutral valuation is appropriate when one or more underlying assets are untraded if the traded assets, and derivatives defined on the untraded assets, permit hedging away all the underlying assets' risks. In this case, the investment management contract would have to be traded and one other traded derivative defined on the portfolio would have to exist.

Risk-neutral valuation is justified if investors value securities, even untraded ones, at risk-neutral values. For example, the trend in the accounting profession is to value restricted stock options at their risk-neutral values, although they are not traded. Furthermore, security analysts accept these values in estimating securities' fair prices.

Risk-neutral valuation is justified if its predictions are consistent with investors' and investment managers' behavior. As shown below, the risk-neutral valuation of investment management contracts implies behavior that is in good agreement with what practitioners say happens.

Merton (1998) justifies using risk-neutral valuation in his Nobel lecture. He writes: "… the equilibrium price for the derivative security is … the same formula 'as if' the underlying asset is traded continuously. And as a consequence, the Black–Scholes formula would apply even in those applications in which the underlying asset is not traded."

12 The end of period value of an investment in the benchmark is used, to be consistent with Margrabe (1978).

13 The "fulcrum" fee discussed in the literature (e.g. Admati and Pfleiderer, 1997; Starks, 1987) is a negative base fee plus a bull-spread fee where R_T and R_C are -1 and positive infinity, respectively. The negative base fee offsets the positive fee received from the bull-spread fee for performance equal to some specified level. Thus, fulcrum fees are symmetric about this performance level.

14 There is an optimum performance volatility at the beginning of a period that is the same for all periods. This does not preclude changing performance volatility during a period to reflect developing performance. A method for determining the optimal change in performance volatility during a period is presented later.

15 It seems sensible for r_M to be a decreasing function of $\sigma^2_{r_{PB}}$ to prevent the contract renewal decision from depending increasingly on luck.

16 This implies that the investor requires more expected performance in order to take on more performance volatility. Thus, it can be thought of as motivated by a preference function defined on performance return. Contract renewal then can be characterized as occurring when the investment manager provides sufficient "realized performance utility."

17 This approach is very flexible. For example, if the exponent in the error term is 3, instead of 2, then the manager is docked for performance volatility above the target and rewarded for performance volatility below the target. In this case, the target could be interpreted as a soft upper limit.

18 A performance ratio of 0.3 implies a performance rate of 0.03 at performance volatility of 0.1. This would be considered excellent performance by most investment practitioners. The hypothesized performance ratio is about the same as the S&P500's Sharpe ratio.

19 The multiple increases toward its limit of $1/(E + Y)$ and the value of the one-period performance call approaches 0 as performance volatility approaches 0. The multiple increases faster earlier when the minimum performance is highly negative. Thus, the net effect can be a large local maximum at a small volatility for a sufficiently negative minimum performance.

20 When threshold performance is negative and performance volatility is 0, the value of the one-period contract is positive and the multiple is large. The result is a long-run contract value of $-[1/(E+Y)]f\,R_T$. But since the expense ratio is V_{OP}, this becomes $-[1/(Y - R_T)]f\,R_T$.

21 This is a general result that follows from Eq. (9).

22 μ_T is obtained from the following equation:

$$\mu_T = (1 - q) + 2q(1 - q) + 3q^2(1 - q) + \cdots$$

23 Because means are proportional to time and standard deviations are proportional to the square root of time.

24 See Brown *et al.* (1996) for evidence that managers do alter performance volatility as performance unfolds.

25 The following approach works because V_{OP} is unrelated to the portfolio's and benchmark's paths and values, and

because all the flows are proportions of the portfolio's value.

26 With the usual assumptions, $g_r(\cdot)$ is bivariate normal.

27 This is a good approximation when performance is alpha (the portfolio's excess return less beta times the benchmark's). It is a good approximation for this paper's definition of performance when the portfolio's beta, measured relative to the benchmark is 1.0.

28 See Ferguson and Leistikow (2001) for a discussion of convergence issues.

29 The paper uses the pseudo-risk-neutral contract renewal probability to make contract renewal issues clearer.

30 The independence assumption allows q_* to replace q.

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