www.joim.com

Journal Of Investment Management, Vol. 1, No. 3, (2003), pp. 36–47

TIME DIVERSIFICATION

Jack L. Treynor

To maintain constant dollar risk, an investor concerned with his terminal wealth must sell when the stock market rises and buy when it falls. Although an asset with constant dollar risk doesn't exist in nature, it can be approximated with actual investment positions.

Many investors are primarily concerned with their wealth at the end of their careers. Yet most of our theory is concerned with the current year's investment choices. How does each year's investment result affect the investor's terminal wealth? How do the gains and losses from the early years interact with the gains and losses from the later years? In particular, do they add or multiply?

1 A parable

Suppose you personally had the following experience:

At the beginning of a 50-year investment career, you borrowed \$1.00 and invested it. Fifty years later, you pay off the loan. Assume the riskless rate of return is zero.

Over 50 years, the borrowed dollar appreciated to \$117.39. So the accounting at the end of your career is

Gross wealth	\$117.39
Pay off loan	\$1.00
Net wealth	\$116.39

Now, suppose that instead of borrowing you received a \$1.00 bequest from your late, lamented Aunt Matilda. Then, you could account for the terminal impact of the bequest as follows:

Net wealth with own dollar	\$117.39
Net wealth with borrowed dollar	\$116.39
Terminal impact of inheritance	\$1.00

If you took the same dollar investment risk with or without the bequest, your terminal wealth differed by the original dollar, appreciated at the riskless rate of zero. Was the dollar worth \$117.39 50 years later? Or merely \$1? If the latter, then the remaining \$116.39 was the reward for taking 50 years of risk.

As the parable suggests, it is not obvious how their wealth and risk-taking interact to determine the investors' wealth at retirement.

Let

$$u = \text{market's rate of return}$$

$$v = \text{investor's rate of return}$$

$$r = \text{riskless rate}$$

$$h = \text{dollars currently invested}$$
$$w = \text{initial wealth}$$
$$\beta = \text{level of relative (systematic) risk}$$
$$h\beta = \text{level of dollar (systematic) risk}$$

If u and v are rates of return, then $u-r$ and $v-r$ are rates of *excess* return—rates of return to risk taking. For a perfectly diversified asset, beta (β) is of course the ratio of its excess return to the market's excess return. In other words

$$\beta = \frac{v-r}{u-r}.$$

Transposing, we have the so-called "market model":

$$v - r = \beta(u - r),$$
$$v = \beta(u - r) + r.$$

The dollar gain or loss to an investor who invests an amount h in the risky asset is

$$hv = h\beta(u - r) + hr.$$

If he had wealth w, then his dollar investment in the riskless asset was

$$w - h$$

for a riskless gain of

$$r(w - h)$$

and a total dollar gain/loss of

$$h\beta(u - r) + hr + wr - hr = h\beta(u - r) + wr.$$

We see that the investor's dollar gain or loss consists of two terms: one that does not depend on his risk and one that does not depend on his wealth.

2 The buy-and-hold investor

Many finance scholars (Ibbotson-Sinquefield; Cornell; Dimson, Marsh and Staunton) believe the risk in the US stock market's rate of return is roughly stationary across time. At the end of this paper, we offer some evidence. But of course if the risk in rate of return is stationary, then the dollar risk is proportional to the market level.

Now consider a buy-and-hold investor, who invests his/her wealth in the stock market and then lets it ride as the market level fluctuates: he/she will experience constant relative risk. But this means that the *dollar* risk—the risk of his/her dollar gain or loss from the market's excess return—will fluctuate with his/her wealth.

Buy-and-hold investors do not lever. If they did, they would be constantly buying and selling in order to offset the effects of market fluctuations on their desired leverage. But when the market level fluctuates, the beta of a diversified asset does not change. So, for buy-and-hold investors, the only thing that changes is the value of their portfolio. Over a short time period (a year, say) the market model holds: investors get the riskless return on their current wealth, plus a risky excess return equal to their constant beta times their current wealth times the excess return on the market. Restating the model in terms of the investor's wealth at times t and $t-1$ we have

$$W_t - W_{t-1} = h_t\beta_t(u_t - r) + rW_{t-1},$$
$$W_t = h_t\beta_t(u_t - r) + (1 + r)W_{t-1}.$$

Under constant relative risk, each period's exposure to stock market risk is proportional to that period's beginning wealth. We then have

$$W_t = W_{t-1}\beta(u_t - r) + (1 + r)W_{t-1},$$
$$W_t = W_{t-1}[\beta(u_t - r) + (1 + r)].$$

Letting

$$q_t = \beta(u_t - r) + (1 + r),$$

we have

$$W_t = W_{t-1}q_t, \quad W_{t-1} = W_{t-2}q_{t-1},$$

$$W_t = q_t q_{t-1} W_{t-2},$$
$$W_T = q_T q_{T-1} \cdots q_1 W_0.$$

Under buy-and-hold investing, the growth factors for the individual years multiply. So a bad year—a 40% loss, say, in any one year—means a 40% loss in terminal wealth.

When the market level is high investors, being richer, feel more able to bear the higher dollar risk. So, they may feel comfortable focusing on relative risk. But this special case tends to obscure the more general truth that terminal wealth depends on the dollar gains and losses in the individual years of the investor's career.

3 Time diversification

We had for the general case

$$W_t - W_{t-1} = h_t \beta_t (u_t - r) + r W_{t-1}.$$

Gains or losses from past risk-taking affect this year's beginning wealth. But it appreciates at the riskless rate. This year's reward to risk depends only on this year's risk.

Let the dollar gain or loss from risk taking in year t be

$$z_t = h_t \beta_t (u_t - r).$$

Then, the investor's wealth W_T satisfies

$$\begin{aligned} W_t - W_{t-1} &= z_t + r W_{t-1}, \\ W_t &= z_t + (1+r) W_{t-1}, \\ W_{t-1} &= z_{t-1} + (1+r) W_{t-2}, \\ &\vdots \\ W_1 &= z_1 + (1+r) W_0. \end{aligned}$$

The terminal wealth W_T equals

$$z_T + (1+r) z_{T-1} + (1+r)^2 z_{T-2} + \cdots + (1+r)^T W_0.$$

Let Z_t be the gain or loss in year t on investing \$1.00 in the stock market. Then, we have

$$z_t = h_t \beta_t Z_t.$$

Unless he plans to market time, the investor will want each of the individual years to have the same potential impact on his terminal wealth "portfolio." Optimal balance requires

$$W_T - W_0 (1+r)^T = \sum_0^T (1+r)^{T-t} h_t \beta_t Z_t = \sum_0^T Z_t.$$

In order to have the same dollar impact on terminal wealth, each year's Z must have the same weight. But, unless the riskless rate of return r is zero, the terminal impact of one year's gain or loss depends on the time lag separating it from the terminal year. In order for each of the Z_t, with presumably equal risks, to have the same potential impact on the risky portion of the investor's terminal wealth (the expression on right-hand side), the current-dollar risk $h_i B_t$ must vary enough over time to offset this effect. So, we have

$$h_i \beta_t = \frac{1}{(1+r)^{T-t}} = (1+r)^{t-T}.$$

Note that, if the effective riskless rate is positive, the investor's dollar risk $h_t \beta_t$ should actually increase as he ages.[1]

We have seen that for the buy-and-hold investor there is no such thing as time diversification. But, if investors make whatever trades are necessary to sever next year's bet from last year's outcome, then, their gains and losses from each individual year add (algebraically) rather than multiply. Impacts from the individual years on their terminal wealth are

1. cross sectionally diversified, so that all their risk bearing is fully compensated (under the CAPM);
2. mutually uncorrelated.

Unless investors are rash enough to predict that the prospects for next year are different from the prospects for last year, they should be making roughly the same dollar bet on both years. In order to do so, however, they will need to sell every time the market rises and buy every time it falls. They will need to do a lot of buying and selling.

On the one hand, the potential for time diversification is there, even if the buy-and-hold investor cannot realize it. On the other, the cost of trading back to a constant level of dollar risk every time the stock market rises or falls may be daunting. Is this why hardly anyone has tried time diversification?

4 Risk and reward

Consider one year's rate of return on the US stock market. It has a certain distribution, with a certain standard deviation and a certain mean. Even if that distribution is indeed roughly stationary across time, we can measure only the actual rates of return for past years. The investors' probability of terminal loss—of arriving at the end of their career with less wealth than they started out with—depends on both the market risk and the market premium, the expected reward for taking this risk. Because its error can be reduced by subdividing the time sample more finely, estimating the standard deviation is not a problem. Dimson and his co-authors of *The Millenium Book*[2] estimate real annual rates of return on the market at 20.3% and 20.1% for the US and UK, respectively.

But sample error is a potential problem for estimates of the mean. Take the authors' 100 year sample: the standard deviation of the sample mean is

$$\frac{0.20}{\sqrt{100}} = \frac{0.20}{10} = 0.02.$$

The Central Limit Theorem applies to the dispersion of means of randomly drawn samples. There is roughly one chance in three that when a normally distributed sample mean is 0.06 (6%), the true universe mean is less than 0.04 or more than 0.08. Although they can benefit greatly from reflecting on Dimson's numbers, we think investors have to make their own judgment about the market premium. Accordingly, we include in Table 1 a range

Table 1 Terminal reward versus terminal risk.

Expected dollar gain over career for a lifetime risk equivalent to one "terminal dollar."

	Market premium per year			
Career length	0.04	0.05	0.06	0.07
16	0.64	0.80	0.96	1.12
25	1.00	1.25	1.50	1.75
36	1.44	1.80	2.16	2.52
49	1.96	2.45	2.94	3.43
64	2.56	3.20	3.84	4.48

Standard deviation of terminal wealth

Career length	0.04	0.05	0.06	0.07
16	0.80	0.80	0.80	0.80
25	1.00	1.00	1.00	1.00
36	1.20	1.20	1.20	1.20
49	1.40	1.40	1.40	1.40
64	1.60	1.60	1.60	1.60

Expected career gain/standard deviation of terminal risk

Career length	0.04	0.05	0.06	0.07
16	0.80	1.00	1.20	1.40
25	1.00	1.25	1.50	1.75
36	1.20	1.50	1.80	2.10
49	1.40	1.75	2.10	2.45
64	1.60	2.00	2.40	2.80

of market premiums, as well as a range of possible career lengths.

5 Terminal dollars

The terminal impact of the dollar gains and losses of particular years depends on the riskless interest rate. Unless investors' riskless rates are zero, a current dollar corresponds to a different number of terminal dollars, depending on their age. But if they are time diversifying, then they want their potential gains and losses at different ages to have the same terminal impact. So it is useful for them to measure their current risk in terms of what it represents for their terminal wealth—to measure their current risk in terminal dollars. Then, they can time diversify by maintaining a fixed number of "terminal dollars" worth of current risk. In Table 1, for example, the expected gains and associated risks are expressed in terms of one dollar of terminal risk.

The first two panels in Table 1 sum up market premium and market risk across investment careers varying from 16 to 64 years. Then, the third panel computes ratios of terminal reward to terminal risk. This is done for a range of assumptions about the hard-to-measure market premium.

The risk that investors will be worse off at the end of their career for having taken stock market risk depends on this ratio. If terminal risks are normally distributed, for example, that probability is 0.0036—three chances in 1000—for the most favorable case (a 64 year career length and a 7% risk premium).

Dimson estimates the real riskless rate at 1.2% per annum for the century 1900–2000. It is curious that this number is in the range of what many mutual funds charge shareholders. The effective rate for the time-diversifying investor should also allow for trading costs and taxes. But we defer further discussion until we get to inflation.

6 Constant dollar risk

Is there such a thing as a financial asset with constant dollar risk? Such an asset would permit the investor who owned it to achieve time diversification without trading.

All commercial risk measurement services focus on *relative* risk—surprise in an asset's value, divided by its beginning value. The only justification for such commercial measures is that the probability distribution of the ratio is stationary (see Figure 1). But, then, dispersion of the asset's dollar risk—surprise in its dollar value—fluctuates with fluctuations in the asset's value.

These comments apply to both individual common stocks and portfolios, including portfolios intended to proxy the value of the whole stock market. Let the stock market level—the value of the market portfolio—be x and the value of an asset with constant dollar risk be y, and let dx and dy represent dollar surprise in x and y, respectively. If both assets are completely diversified, then, the market level x determines the value of y. Let the relation between the two values be

$$y = f(x).$$

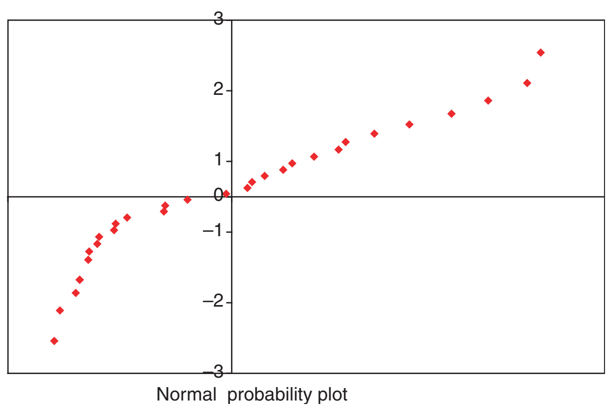

Figure 1 Rate of return on US market 1971–2000.

We ask: What functional dependence of y on x translates the constant relative risk of x into the desired constant dollar risk of y?

When the functional relation between y and x is such that, for all market levels, we have

$$\mathrm{d}y = \frac{\mathrm{d}x}{x}.$$

The right-hand side is of course the rate of return on the market. As noted, many finance scholars believe its risk is stationary. The left-hand side and the right-hand side being equal, they will necessarily have the same probability distribution. In particular, if the right-hand side—the relative return on the stock market—is stationary across time, the left-hand side will also be stationary. But, whereas the right-hand side is the *relative* change in x—dx divided by the level x—the left-hand side dy is the *dollar* change in y. So if, as the market level x fluctuates, its relative risk is truly stationary, then the dollar risk in y is also stationary.

If we take indefinite integrals of both sides, we have

$$y = \ln x + \ln k$$

where $\ln k$ is a constant of integration, or

$$y = \ln kx.$$

The asset with constant dollar risk is the asset whose value varies with the logarithm of the market level.

7 Inflation

We do not have the option of investing in the real market level. The values of the market and our log approximation are nominal values. But the risk we want to maintain constant over time—as the price level changes—is the *real* risk. If, as we have argued, the risk in nominal market return is stationary, then the risk of nominal dollar gains and losses in the log portfolio is also stationary. But this means that if, for example, the price level is rising, then the risk of real dollar gains and losses is falling.

Let x be the nominal market level and y be the nominal value of a portfolio that varies with the logarithm of the market level, and let the respective real values be x' and y', where the price level is p. We have

$$x' = \frac{x}{p}, \quad y' = \frac{y}{p}.$$

For investment surprises we have

$$\mathrm{d}x' = \frac{\mathrm{d}x}{p}, \quad \mathrm{d}y' = \frac{\mathrm{d}y}{p}.$$

The logarithmic portfolio is defined by a relation between nominals

$$\mathrm{d}y = \frac{\mathrm{d}x}{x}.$$

Substituting, we have

$$p\,\mathrm{d}y' = \frac{p\,\mathrm{d}x'}{p\,x'} = \frac{\mathrm{d}x'}{x'}.$$

We see that, if surprise in the rate of return on the real market level is stationary, surprise in the nominal rate of return will also be stationary.[3] But if surprise in the nominal value of the logarithmic portfolio is stationary, surprise in its real value

$$\mathrm{d}y' = \frac{\mathrm{d}y}{p}$$

will not be. This means that if, for example, the price level is rising over the investors' career, the real risk in their logarithmic portfolio is falling.

Consider, first, the case where the real riskless rate of interest is zero. To offset the effect of inflation, investment positions in recent years in the investor's career should be rescaled relative to early years, with the rescaling from year to year equaling that year's inflation rate.

Then, consider the case where inflation is not a problem but the riskless interest rate is positive rather than zero. Then, investment positions in recent years should be rescaled relative to early years, with the rescaling from year to year being equal to the riskless interest rate.

We see that inflation causes the late nominal gain/loss to have less impact than an early gain/loss and the same is true for the real riskless rate. On the other hand, management fees, trading costs and taxes cause an early gain/loss to have less impact on terminal wealth than a late gain/loss. So, their annual rate of attrition subtracts from the sum of the real rate and the inflation rate—i.e. from the nominal interest rate. If the gain from trading just offsets management fees and the portfolio is not subject to taxes, the terminal impact of a current dollar of nominal gain or less will appreciate at the nominal interest rate.

8 An approximation

The logarithmic asset is probably not available in today's security markets. But it can readily be approximated using assets that are. Consider the following Taylor series expansion of the logarithmic function, where a is greater than zero:

$$\ln\frac{x}{a} = \left(\frac{x-a}{a}\right) - \frac{1}{2}\left(\frac{x-a}{a}\right)^2 + \frac{1}{3}\left(\frac{x-a}{a}\right)^3 - \cdots.$$

Although the accuracy of the approximation increases with the number of terms retained in the series,[4] we retain only the first two. Expanding these terms we have

$$\ln\left(\frac{x}{a}\right) \approx 2\left(\frac{x}{a}\right) - \frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{3}{2}.$$

The investor who seeks time diversification is actually concerned with the corresponding risks. How well does the risk of the right-hand side approximate the risk of the left-hand side? The dollar risk on both sides depends on a product. One factor in the product is the rate of change with respect to the market level x. We have for the respective sides

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x}{a}\right) = \frac{1}{a}\left(\frac{1}{x/a}\right) = \frac{1}{x} \approx \frac{1}{a}\left(2 - \frac{x}{a}\right).$$

The other factor in both products is the dollar risk in x. But, if $\mathrm{d}x/x$ is stationary, then, the dollar risk in x is proportional to the (known, non-risky) value of x.

If we invest in the approximation portfolio when x equals a, then, the above rate of change is $1/a$ for both the logarithmic portfolio and the approximation. But the risk in the approximation drifts away from the log portfolio as the market level x moves away from a.

9 The role of beta

We have noted that beta is a measure of how much an asset's value changes when the general market level changes—that, specifically, it is the ratio of two rates of excess return. Define x as the market level, y as the (fully diversified) asset's value and level of relative risk by the Greek letter β. Then, we have

$$\frac{\mathrm{d}y/y}{\mathrm{d}x/x} = \beta,$$

$$\frac{\mathrm{d}y}{y} = \beta\frac{\mathrm{d}x}{x}.$$

Taking the indefinite integral, we have

$$\ln y = \beta \ln x + \ln k$$

where $\ln k$ is a constant of integration. Taking antilogs we have

$$y = k x^{\beta}.$$

We see that a diversified asset's value is linked to the market level by a power that equals its beta. Our truncated Taylor series approximation to the logarithmic function of the market level contains two powers of the market level x. Evidently, the terms containing these powers correspond to investment positions in diversified assets with betas of 1 and 2.

10 Accuracy of the approximation

How bad are the errors in the approximation portfolio? Let

$$\begin{aligned} a &= \text{beginning market level} \\ x &= \text{market level at the end of the year} \\ dx &= \text{change in market level} \\ \sigma_{dx} &= \text{standard deviation of change} \\ y &= \text{value of approximation portfolio} \\ dy &= \text{change in value of approximation} \\ \sigma_{dy} &= \text{standard deviation of change} \end{aligned}$$

As noted, its dollar risk is the product of its rate of change with respect to the market and the dollar risk in the market. The first column in Table 2 displays a range of possible ratios of the ending market level x to the beginning market level a. The second column shows the resulting new market levels. The third column shows the standard deviation of the market's dollar risk for the following year—assuming its relative risk, the standard deviation of its rate of return, is still 20%.

Table 2 Approximation errors.

x/a	x	σ_{dx}.	dx/dy	σ_{dy}	% Error
1.30	1.30a	0.26a	0.70/a	0.1820	9.00
1.25	1.25a	0.25a	0.75/a	0.1875	6.25
1.20	1.20a	0.24a	0.80/a	0.1920	4.00
1.15	1.15a	0.23a	0.85/a	0.1955	2.25
1.10	1.10a	0.22a	0.90/a	0.1980	1.00
1.05	1.05a	0.21a	0.95/a	0.1995	0.25
1.00	1.00a	0.20a	1.00/a	0.2000	0.00
0.95	0.95a	0.19a	1.05/a	0.1995	0.25
0.90	0.90a	0.18a	1.10/a	0.1980	1.00
0.85	0.85a	0.17a	1.15/a	0.1955	2.25
0.80	0.80a	0.16a	1.20/a	0.1920	4.00
0.75	0.74a	0.15a	1.25/a	0.1875	6.25
0.70	0.70a	0.14a	1.30/a	0.1820	9.00

The fourth column shows the rate of change of the approximation portfolio with respect to change in the stock market level. The fifth column is the product of the third and fourth columns. Because the third column measures dollar risk in the market level, and the fourth column measures its rate of change with respect to that level, the fifth column measures dollar risk in the approximation portfolio.

The dollar risk in the ideal, logarithmic portfolio is 20% of the initial market level a, no matter what the subsequent change in market level. But the approximation is imperfect. The fifth column shows how its dollar risk drifts progressively farther from the correct, constant value as the new market level x moves away from the beginning level a. (It may be worth noting, however, that the dollar risk of the approximation portfolio is always less than or equal to the correct value.) The sixth column expresses the errors as percentages of the correct dollar risk.

Table 2 shows that a 20% move up or down in the market level changes the dollar risk in the approximation portfolio by only 4%. To trade back to constant dollar risk every time their portfolio changed 4%, conventional investors would have to trade

$$\left(\frac{0.20}{0.04}\right)^2 = 5^2 = 25,$$

that is, 25 times as often. (If the dispersion of random fluctuations over a time interval varies with the square root of its length, the length of the time interval varies with the square of the stipulated dispersion.) Is this why conventional investors do not attempt to time diversify?

11 Rebalancing

We have seen that, when the market has moved up or down one standard deviation, or 20%, the new standard deviation for the approximation portfolio is no longer 20% of the original dollar investment, but merely 18.2%. (Roughly one year in three, the market moves more than 20%.) When the market level x moves away from the "beginning" level a, two things happen:

1. the approximation breaks down as the risky positions' 4 : 1 ratio breaks down;
2. the scale, or magnitude, of net risk moves away from beginning net risk.

There are many combinations of the two risky positions that will satisfy the 4 : 1 condition and, hence, restore the logarithmic character of the portfolio. Also, there are many combinations that will restore the original net risk. But one, and only one, combination of the two positions can satisfy both conditions. If the investor changes the "beginning" market level a in this ratio to the current market level x, the ratio reverts to its original value of 1. But when the values of the risky positions were based on a ratio value of 1, they

1. were in the accurate 4 : 1 ratio; and
2. had the desired level of net dollar risk that the investors wanted to maintain over their lifetime.

What the new value of a does not do is retain the same net investment in the two risky positions they had before we changed the ratio back to 1. This is where the third, constant, "riskless" term in the Taylor series formula comes in: when we are making the trades in the risky assets dictated by the change in the ratio, these trades free up or absorb cash, which then flows to or from the third, riskless, position. (Obviously, changes in the value of the riskless position do not change the portfolio's risk[5] so if, after these trades, the risky positions have the correct risk, so has the portfolio.)

In Table 3, the beginning market level is arbitrarily set at 1000. Then, the long position is

$$2(1000) = 2000,$$

and the short position is

$$\tfrac{1}{2}(1000) = 500.$$

So, the net value of the two risky positions (the "risky equity") is then

$$2000 - 500 = 1500.$$

Each rebalancing returns the risky equity to 1500. But offsetting transfers to or from the riskless asset preserve the investor's total equity.

Table 3 shows how the approximation portfolio would have functioned using actual US stock market data for end-of-year levels from 1977 to 2000. Although, given the limited data, rebalancings could not be triggered by daily market closes, there were 11 rebalancings during this period.

Table 3 devotes three stages of calculation (separated by semicolons in the third column) to each year (except 1978). For the current value of a, the first stage calculates the ratios x/a and $(x/a)^2$. The second stage applies the coefficients in the approximation formula to the respective ratios, and then multiplies all three terms in the formula by 1000. (For example, the initial value of the riskless term becomes -1500.) The third stage calculates the new risky equity, and the change since the last rebalancing.

Rebalancing makes the third stage of calculation more complicated. Since each rebalancing wipes out the difference between the current risky equity and the original investment (in this example, 1500), the third stage also calculates the new value of the riskless asset, reflecting the cash freed up or absorbed in returning the risky positions to their original values.

Table 3 Calculations for approximation portfolio 1977–2000 (see text).

Year	US mkt index	
1977	169	
1979	179	179/169 = 1.0592, 1.0592^2 = 1.1218; 2(1059) − 1/2(1122); 2118 − 561 = 1557; 1557 − 1500 = 57
1980	210	210/169 = 1.243, 1.243^2 = 1.544; 2(1243) − 1/2 (1544); 2486 − 772 = 1714, 1714 − 1500 = 214
1981	225	225/210 = 1.0714, 1.0714^2 = 1.1479; 2(1071) − 1/2(1148); 2142 − 574 = 1568, 1568 + 214 − 1500 = 282
1982	208	208/210 = 0.990, 0.990^2 = 0.9810; 2(990) − 1/2(981); 1980 − 491 = 1489, 1489 + 214 − 1500 = 203
1983	281	281/210 = 1.3381, 1.3381^2 = 1.7905; 2(1338) − 1/2(1790) = 2676 − 895 = 1781; 1781 + 214 − 1500 = 495
1984	283	283/281 = 1.007, 1.007^2 = 1.014; 2(1007) − 1/2 (1014); 2014 − 507 = 1507, 1507 + 495 − 1500 = 502
1985	324	324/281 = 1.1530, 1.1530^2 = 1.328; 2(1153) − 1/2 (1328); 2306 − 665 = 1641, 1641 + 495 − 1500 = 636
1986	409	409/281 = 1.456, 1.456^2 = 2.119; 2(1456) − 1/2 (2119); 2912 − 1059 = 1853, 1853 + 495 − 1500 = 848
1987	516	516/409 = 1.2616, 1.2616^2 = 1.5917; 2(1262) − 1/2 (1592); 2524 − 796 = 1727, 1727 + 848 − 1500 = 1075
1988	478	478/409 = 1.169, 1.169^2 = 1.366; 2(1169) − 1/2 (1366); 2338 − 683 = 1655, 1655 + 848 − 1500 = 1003
1989	577	577/409 = 1.411, 1.411^2 = 1.990; 2(1411) − 1/2 (1990); 2822 − 995 + 1827, 1827 + 848 − 1500 = 1175
1990	609	609/577 = 1.0554, 1.0554^2 = 1.114; 2(1055) − 1/2(1114); 2110 − 557 = 1553, 1553 + 1175 − 1500 = 1228
1991	695	695/609 = 1.141, 1.141^2 = 1.302; 2(1141) − 1/2(1302); 2282 − 651 = 1631, 1631 + 1228 − 1500 = 1359
1992	765	765/695 = 1.1007, 1.1007^2 = 1.2116; 2(1101) − 1/2(1212); 2202 − 606 = 1596, 1596 + 1359 − 1500 = 1455
1993	806	806/695 = 1.160, 1.160^2 = 1.345; 2(1160) − 1/2(1345); 2320 − 672 = 1648, 1648 + 1359 − 1500 = 1455
1994	841	841/806 = 1.0434, 1.0434^2 = 1.0887; 2(1043) = 1/2(1088); 2086 − 544 = 1542, 1542 + 1507 − 1500 = 1549
1995	1000	1000/841 = 1.189, 1.189^2 = 1.414; 2(1189) − 1/2(1414); 2378 − 707 = 1671, 1671 + 1549 − 1500 = 1720
1996	1235	1235/1000 = 1.2350, 1.2350^2 = 1.5252; 2(1235) − 1/2(1525); 2470 − 763 = 1707, 1707 + 1720 − 1500 = 1927
1997	1593	1593/1235 = 1.290, 1.290^2 = 1.664; 2(1290) − 1/2(1664); 2580 − 832 = 1748, 1748 + 1927 − 1500 = 2175
1998	1987	1987/1593 = 1.2473, 1.2473^2 = 1.5558; 2(1247) − 1/2(1556); 2494 − 778 = 1716, 1716 + 2175 − 1500 = 2391
1999	2513	2513/1987 = 1.2647, 1.2647^2 = 1.5995; 2(1265) − 1/2(1600); 2530 − 800 = 1730, 1730 + 2391 − 1500 = 2621
2000	2728	2728/2513 = 1.0856, 1.0856^2 = 1.1784; 2(1086) − 1/2(1178); 2172 − 589 = 1583 + 2621 − 1500 = 2704

The value of the approximation portfolio to the investor includes the net value of both his risky positions and the accumulating sum of these (algebraic) additions to the riskless asset. Thus, the three-stage entry for a rebalancing year reflects both the effect of rebalancing, which takes place at the beginning of that year, and the effect on the two risky positions of the subsequent change in market level, between the beginning and the end.[6]

12 The evidence

The last three decades of the century included several painful market collapses as well as a celebrated bull market. The nominal market level increased 16 times, the real level four. Surely this period is a worthy test of whether

1. the risk in the markets' rate of return is really stationary;
2. the dollar risk in the logarithmic portfolio is really stationary.

In order to test whether risks were stationary, we need to be able to measure *ex ante* risk *ex post*. Actuaries use a special kind of graph paper called "probability paper" to do this. Its vertical axis is conventional, with horizontal lines equally spaced. But its horizontal axis is variously compressed and stretched so that, when drawings from a normal sample are ranked from lowest to highest and then accorded equal probability increments (rather than equal distances) on that axis, they plot as a straight line. Depending on the chosen scale of the conventional vertical axis, the slope of that line reflects the sample's dispersion.

The point, of course, is that if the sample is drawn from a universe with different dispersions—if, across time, the risk is not stationary—then, the sample cannot plot as a straight line.

Were the two risks really stationary over the sample period? Figure 1 displays the data for the market's rate of return. Figure 2 displays the data for the year-to-year change in the dollar value of the logarithmic portfolio.

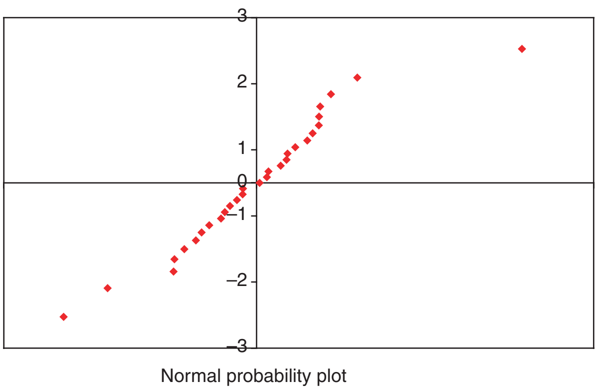

Figure 2 Year to year changes in the dollar value of a portfolio that varies with the logarithm of the US market (1972–2000).

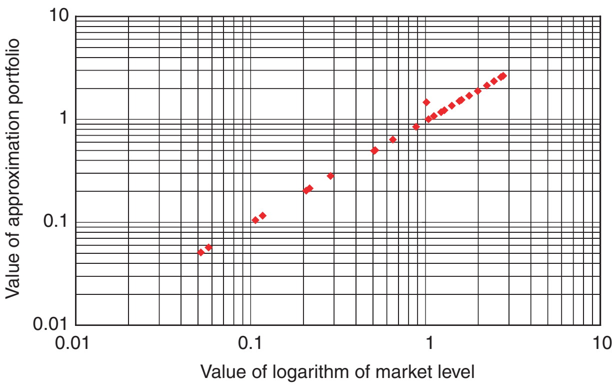

Figure 3 US experience 1980–2000.

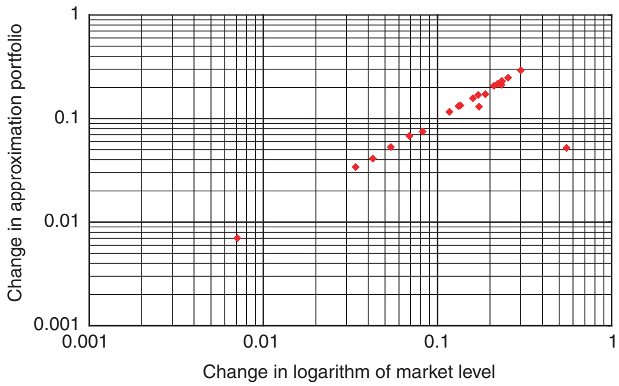

Figure 4 US experience 1972–2000.

Did the approximation portfolio really track the logarithmic portfolio? Figure 3 displays the data for the dollar values. Figure 4 displays the data for the year-to-year changes in dollar value of the two portfolios—i.e. for their risks.

13 Implementing the approximation portfolio

As the market level goes up, the value of the short position increases, even as the value of the long position increases. Rebalancing entails reducing the long and short positions after the stock market has gone up and increasing the long and short positions after the stock market has gone down.

Brokers who borrow the stock the investor sells short will demand "margin"—valuable assets to protect them in case the investor is unable to cover because the market has risen too much. If the investors deposit their long position with the broker, their margin does not start to shrink until the market level has doubled (five standard deviations). It does not run out until the market level has quadrupled ($3 \times 5 = 15$ standard deviations of annual stock market return). But, in the meantime, the investor has rebalanced to less risky positions, over and over.

On the other hand, when the market falls the investors lose margin. But they do not lose all of it until the market level reaches zero. The 4 : 1 target ratio assures that the long position will always provide more margin for the short position than even the most timid broker would require.

14 Should risk decline with age?

We have argued that, if their real riskless rate is zero—or just large enough to offset trading and other costs—investors who want to time diversify should take the same dollar risk in the last year of their investment career as they take in the first. Does not this prescription conflict with the intuition that an old investor should take less risk than a young investor?

We have seen that, if they have time diversified, investors approaching the end of their career are likely to be richer than when they began. But, then, the same dollar risk at the end of their career represents a smaller relative risk; and relative risk is the way most investors—especially conventional investors—think about risk.

Is time diversification (constant dollar risk) just an unfamiliar way of expressing a familiar intuition?

Notes

1 Obviously, the investor's savings at various points in his career also contribute to terminal wealth, appreciated forward at the effective riskless rate. Let his savings in year t be Δt. Then, their contribution to terminal wealth is

$$s_0(1+r)^T + s_1(1+r)^{T-1} + \cdots + s_T = \sum s_t(1+r)$$

2 Dimson, E., Marsh, P., and Staunton, M. (2000). *The Millenium Book*. ABN-AMRO and the London Business School.

3 Past inflation has the same effect on the units of measure for the numerator and denominator. Current inflation adds algebraically to both market gains and losses, but affects the mean of these numbers rather than the dispersion.

4 There are other power series approximations—even other Taylor series approximations—to the logarithmic function.

5 When we use year-end data for the market level, we restrict our opportunities for rebalancing back to an accurate approximation of the logarithmic asset. In practical applications, changes in the market level can be followed and responded to almost continuously.

When increasing approximation error forces us to rebalance back to our original investment positions, these positions should be scaled up from those of the previous rebalancing by a factor reflecting appreciation over the interval between rebalancings. (If the price level is inflating very rapidly, rescaling does not have to wait for the next rebalancing. Then, however, the investor incurs additional trading costs.)

6 Question: if rebalancing restores the original dollar risky positions at rebalancing, why is this not evident in JLT's 22 year example using actual US stock market data? Answer: Whereas rebalancing occurs at the beginning of the year, the worksheet numbers are based on market level at the end.

Keywords: Time diversification.