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FUND MANAGERS MAY CAUSE THEIR BENCHMARKS TO BE PRICED "RISKS"

*Michael Stutzer**

The presence of a positive intercept ("alpha") in a regression of an investment fund's excess returns on a broad market portfolio's excess return (as in the CAPM), and other "factor" portfolios' excess returns (e.g. the Fama and French factors) is frequently interpreted as evidence of superior fund performance. This paper theoretically and empirically supports the notion that the additional factors may proxy for benchmark portfolios that fund managers try to beat, rather than proxying for state variables of future risks that investors (in conventional theory) are supposed to care about.

1 Introduction

The CAPM is a linear, single excess return factor model, derivable by assuming that all investors are "rational," in the sense of choosing the tangency portfolio of risky assets on the mean–variance efficiency frontier. This portfolio is the single factor. But authors, too numerous to mention, have argued that additional factors are also present. For example, Fama and French (1992) documented the ability of the following linear, 3-factor model (their Eq. (1)) to predict anomalous expected excess returns earned by some well-known stock portfolio strategies:

$$E(R_i) - R_f = b_i[E(R_M) - R_f] + s_i E(\text{SMB}) + h_i E(\text{HML}) \qquad (1)$$

where M denotes the broad market portfolio, SMB denotes the return on a portfolio that sells relatively big cap stocks and buys relatively small cap stocks, and HML denotes the return on a portfolio that sells relatively low book-to-market stocks and buys relatively high book-to-market stocks. Because (1) arises by taking expectations of both sides of multiple linear regression specification without an intercept, it subsequently became quite common to regress the returns of managed investment funds on the factors in (1) with an intercept α, and then to interpret a statistically significantly positive α as indicative of superior fund performance (e.g. see Davis, 2002).

Fama and French (1992) could not find empirical support for the standard *theoretical* frameworks that permit derivation of factor models like this, concluding with the following summary:

Finally, there is an important hole in our work. Our tests to date do not cleanly identify the two consumption–investment

*Professor of Finance and Director, Burridge Center for Securities Analysis and Valuation, University of Colorado, 419 UCB, Boulder, CO 80309-0419, USA. Tel.: 303 492 4348; e-mail: michael.stutzer@colorado.edu

state variables of special hedging concern to investors that would provide a neat interpretation of the results in terms of Merton's (1973) ICAPM or Ross' (1976) APT.

Researchers subsequently struggled to do this, focusing on the possibility that the additional factors proxy for state variables of financial distress. Yet, a recent paper by Vassalou and Xing (2002) concludes that:

> Fama and French [Fama and French (1996)] argue that the SMB and HML factors of the Fama and French (FF) model proxy for financial distress. Our asset pricing model results show that, although SMB and HML contain default-related information, this is not the reason that the FF model can explain the cross section. SMB and HML appear to contain important price information, unrelated to the default risk.

Of course, this failure has not and will not stop others from compiling statistics favorable to claims that the factors proxy for predictors of other future risks that investors (in conventional theory) should care about, although this should be done with some care (as noted by Cochrane, 2001, p. 171). Given the easily observed relative scarcity of alternative quantitative theories presented in finance journals (relative to statistical studies loosely guided by old theories or no theory at all), perhaps it is time to propose plausible alternative *theoretical* explanations for the success of multifactor models. In a refreshing attempt to do so, Shefrin and Statman (1995) provide evidence that investors (likely wrongfully) presume that the stock of healthy, thriving companies will be unusually good investments, and conjecture that this might be the cause of the additional Fama and French factors. But they did not produce a quantitative financial theory that derives an equation like (1), nor more importantly, derives additional quantitative equations that could be tested.

In contrast, this paper does provide an alternative quantitative financial theory for the presence of non-market factor portfolios' expected excess returns in the following linear, excess return factor generalization of the CAPM:

$$E(R_i) - R_f = \beta_{im}[E(R_m) - R_f] + \sum_{b \in B} \beta_{ib}[E(R_b) - R_f]; \quad i = 1, \ldots, N \quad (2)$$

where B is a set of n benchmark portfolios that different classes of portfolio managers (and/or other investors) try to "beat." While Fama and French did not write the linear model (1) in the excess return form (2), many other studies do use the excess return form, e.g. Gruber (1996) or Elton *et al.* (1996). It is a very widely accepted assumption that portfolio managers are motivated to try to outperform specific benchmark portfolios, e.g. see Bailey (1992) and Bailey and Tierney (1993, 1995). A typical example of benchmarking, which is of more than just professional interest to academic readers of this paper, is contained in the following statement by the TIAA-CREF Trust Company:

> Different accounts have different benchmarks based on the client's overall objectives . . . Accounts for clients who have growth objectives with an emphasis on equities will be benchmarked heavily toward the appropriate equity index—typically the S&P 500 index—whereas an account for a client whose main objective is income and safety of principal will be measured against a more balanced weighting of the S&P 500 and the Lehman Corporate/Government Bond Index. [TIAA-CREF (2000, p. 3)]

Roll (1992, p. 13) argued that "This is a sensible approach because the sponsor's most direct alternative to an active manager is an index fund matching the benchmark." The ability of funds like TIAA-CREF to attract individual investors by the use of relative performance objectives implies the possibility that this may be the voluntary choice of (possibly bounded rational, or possibly differently motivated) investors, who think the funds are better able to beat their desired benchmarks and hence attain their desired objectives. In fact, it is certainly possible that *individual* investors also try to beat benchmarks.

Perhaps the best known quantitative model of benchmark beating behavior is the Tracking Error Variance (TEV) model of Roll (1992). TEV investors try to earn a higher expected return than a specific benchmark portfolio's expected return, while minimizing the variance of the difference of the two returns. This paper will show that the presence of benchmark portfolios as factors in (2) does not necessarily have to arise as a result of hedging of state variable risks by conventional investors, but instead could be due to the concurrent portfolio choice behavior of TEV investors attempting to beat the benchmark portfolios. That is, the very attempt to beat these benchmarks results in them occurring as priced factors in (2).

To eventually establish this, Section 2 starts by briefly contrasting conventional mean–variance investor behavior in the presence of a riskless asset with Roll's TEV behavior. Proposition 1 shows that in the presence of a riskless asset, the Information Ratio (Goodwin, 1998; Gupta *et al.*, 1999; Clarke *et al.*, 2002) produced by substituting a benchmark portfolio return for the riskless return in the conventional Sharpe Ratio (1994), plays a role analogous to that played by the Sharpe Ratio in conventional mean–variance theory. Despite the widespread use of both the TEV criterion and the Information Ratio, it does not appear that this result has been previously published. Proposition 2 provides the following plausible frequentist rationale for maximizing the Information Ratio: it is consistent with maximizing the probability of outperforming the benchmark on-average. Section 2.1 makes a brief descriptive and prescriptive case for this behavioral criterion.

Proposition 4 in Section 3 shows that the linear excess return factor model (2) is a capital market equilibrium relation resulting from the aggregate asset demands of both conventional investors (if any) and TEV investors. This proves that the theory is capable of delivering any linear multifactor model, after substituting a set of factor mimicking benchmark portfolios for the proposed factors. Because of this, empirical tests different from the mere goodness-of-fit of a particular specification of (2) must be used to differentiate this theory from others that imply (2). Fortunately, Proposition 3 shows that this theory implies an unusual sign restriction on the intercepts in regressions of each (i.e. the market and the benchmarks) portfolio's excess returns on the others' excess returns. This sign restriction—an inherently sharper test than the mere search for statistical connections between the factors and financial distress and/or other conjectured risk variables—is tested in Section 4, where we show that some of the explanatory power of multiple factors appearing in stock returns may be explained by the presence of separate classes of TEV investors who respectively try to beat growth and value benchmarks. Section 5 concludes, and suggests some topics for future research.

2 Conventional mean–variance versus TEV investing

The concise derivation of conventional mean–variance investing in Huang and Litzenberger (1988, pp. 76–78) is presented and then contrasted with its TEV generalization. Conventional investors choose an unrestricted weight vector q_p of the N risky assets, investing the rest of their investable funds in the riskless asset. They do so by minimizing the return variance, subject to a portfolio expected return constraint:

$$\min_q \tfrac{1}{2}q'Vq \quad \text{s. t. } q'[E(R) - \mathbf{1}R_f] = E(R_p) - R_f = E(R_p - R_f) \tag{3}$$

where V denotes the covariance matrix of the return vector R with expectation vector $E(R)$, and $\mathbf{1}$ denotes a vector of ones. Note that $E(R_p)$ denotes the expected return on a portfolio that could contain the riskless asset. The risky asset vector q_p satisfies the following Lagrangian first-order condition:

$$Vq_p = \lambda[E(R) - \mathbf{1}R_f] \tag{4}$$

Premultiplying both sides of (4) by V^{-1}, and then dividing each side of the resulting equation by the sum of its respective components (assuming that the sum of the risky asset weights is not equal to zero), we obtain the tangency portfolio w_T of the risky assets:

$$\frac{q_p}{\mathbf{1}'q_p} = \frac{V^{-1}[E(R) - \mathbf{1}R_f]}{\mathbf{1}'V^{-1}[E(R) - \mathbf{1}R_f]} \equiv w_T \tag{5}$$

Expression (5) is the familiar result that all conventional investors purchase risky assets in the same proportions as the tangency portfolio w_T. Letting $q^i_{pf} \equiv 1 - \mathbf{1}'q_p$ denote conventional investor i's weight on the riskless asset, (5) shows that the risky asset weight vector of the ith conventional investor is:

$$q^i_p = (1 - q^i_{pf})w_T \tag{6}$$

A few more routine calculations (see Huang and Litzenberger, 1988, pp. 76–77) show that the conventional Sharpe Ratio of the chosen portfolio is:

$$\frac{E(R_p - R_f)}{\sqrt{\mathrm{Var}(R_p - R_f)}} = \frac{E(R_p) - R_f}{\sqrt{\mathrm{Var}(R_p)}} = \sqrt{[E(R) - \mathbf{1}R_f]'V^{-1}[E(R) - \mathbf{1}R_f]} \equiv \sqrt{H} \tag{7}$$

where $\sqrt{H}$ is the *maximum* conventional Sharpe Ratio among mean–variance efficient risky asset portfolios, attained by the tangency portfolio (5).

Now, consider another class of investors, comprised of individuals and/or fund managers who use a portfolio q_b with return R_b as a benchmark against which performance is measured. According to the TEV hypothesis of Roll (1992), they would choose a risky asset weight vector q_p by solving

$$\min_q \tfrac{1}{2}[q - q_b]'V[q - q_b]$$
$$\text{s. t. } [q - q_b]'[E(R) - \mathbf{1}R_f] = E(R_p - R_b) \geq 0 \tag{8}$$

That is, a TEV-efficient investor minimizes the TEV $\mathrm{Var}(R_p - R_b)$ required to exceed the expected return of the benchmark by some chosen amount. The tradeoff between that chosen amount, and the minimum TEV required to achieve it, is dubbed the TEV frontier. Just as the conventional mean–variance frontier is simplified by the introduction of a riskless asset, I will now show that the TEV frontier is similarly simplified. Define $x \equiv q - q_b$ to be the unrestricted risky asset vector in excess of the benchmark's. Substituting x into (8), we have the equivalent problem:

$$\min_x \tfrac{1}{2}x'Vx \quad \text{s. t. } x'[E(R) - \mathbf{1}R_f] = E(R_p) - R_f - (E(R_b) - R_f) = E(R_p - R_b) \tag{9}$$

which is formally equivalent to (3). Assuming the solution $x_p \equiv q_p - q_b$ does not sum to zero, i.e. the weight placed on the riskless asset in the managed portfolio does not equal the weight placed on the riskless asset in the benchmark, the solution is found by substitution into (5), yielding

$$\frac{x_p}{\mathbf{1}'x_p} = \frac{V^{-1}[E(R) - \mathbf{1}R_f]}{\mathbf{1}'V^{-1}[E(R) - \mathbf{1}R_f]} \equiv w_T \tag{10}$$

and substituting $\mathbf{1}'x_p = \mathbf{1}'[q_p - q_b] = (1 - q_{pf}) - (1 - q_{bf}) = q_{bf} - q_{pf} \neq 0$ into (10) results in the following risky asset weight vector for the jth TEV investor, now denoted by q^j_{pb}:

$$q^j_{pb} = q_b + (q_{bf} - q^j_{pbf})w_T \tag{11}$$

Because (11) shows that $q^j_{pb} - q_b$ is proportional to the tangency portfolio whose Sharpe Ratio is the right-hand side of (7), the analogous finding for

TEV investors is

$$\frac{E(R_p - R_b)}{\sqrt{\mathrm{Var}(R_p - R_b)}} = \frac{E(R_p) - E(R_b)}{\sqrt{\mathrm{Var}(R_p - R_b)}} = \sqrt{[E(R) - \mathbf{1}R_f]'V^{-1}[E(R) - \mathbf{1}R_f]} \equiv \sqrt{H} \tag{12}$$

The left-hand side of (12) is the *Information Ratio* (Goodwin, 1998; Gupta *et al.*, 1999; Clarke *et al.*, 2002). A survey of the TEV literature failed to uncover the following proposition characterizing the symmetry between the conventional mean–variance and TEV optimal portfolios in the presence of a riskless asset:

Proposition 1: *When a riskless asset exists, conventional mean–variance investors choose risky asset weight vectors by maximizing the conventional Sharpe Ratio. Normalization of the vectors produces the Tangency Portfolio. Analogously, TEV investors choose risky asset weight vectors by maximizing the Information Ratio. Normalization of the difference between a risky asset weight vector and the benchmark's risky asset weight vector produces the same Tangency Portfolio.*[1]

Proposition 1 uses traditional theorizing to establish that in the presence of a riskless asset, the TEV hypothesis is a natural extension of the mean–variance hypothesis relative to a benchmark. But there is also a quite plausible frequentist foundation for TEV behavior. Anyone desiring to beat the benchmark return R_b will, at the very least, endeavor to beat it *on-average* over some span of time T that possibly differs across them. That is, anyone desiring to beat the benchmark would like $\sum_{t=1}^{T} R_{pt}/T > \sum_{t=1}^{T} R_{bt}/T$ for some T. If $T = \infty$, the law of large numbers dictates that he/she needs only choose a portfolio p with $E[R_p] > E[R_b]$ in order to ensure this. But over the finite time horizons T faced by real-world managers and/or investors, there is a nonzero probability that this might not happen, i.e. $\mathrm{Prob}\left[\sum_{t=1}^{T} (R_{pt} - R_{bt})/T \le 0\right]$. Assuming that $R_{pt} - R_{bt}$ is an IID normally distributed process, as commonly (albeit sometimes implicitly) assumed in textbook presentations of applied mean–variance analysis, $\sum_{t=1}^{T} (R_{pt} - R_{bt})/T \sim \mathcal{N}\left(E[R_p] - E[R_b], \sqrt{(\mathrm{Var}[R_p - R_b])/T}\right)$. So, by transforming this normally distributed variate to the standard normal variate Z, the underperformance probability is

$$\begin{aligned}\mathrm{Prob}\left[\sum_{t=1}^{T} (R_{pt} - R_{bt})/T \le 0\right] &= \mathrm{Prob}\left[Z \le \frac{-(E[R_p] - E[R_b])}{\sqrt{(\mathrm{Var}[R_p - R_b])/T}}\right] \\ &= \mathrm{Prob}\left[Z > \sqrt{T}\frac{E[R_p] - E[R_b]}{\sqrt{\mathrm{Var}[R_p - R_b]}}\right]\end{aligned} \tag{13}$$

Someone who wants to minimize the left-hand side of (13), i.e. the probability of failing to beat the benchmark on-average over a finite time horizon T, will choose the risky asset portfolio p that minimizes the right-hand side of (13). We immediately see that this is the same portfolio that maximizes the Information Ratio, *independent of the time horizon T*. Because the probability of *out*performing the benchmark on-average is one minus the left-hand side of (13), it is equally valid to state that this portfolio maximizes the probability of outperforming the benchmark on-average.

This frequentist interpretation of the Information Ratio is exactly true only when returns in excess of the benchmark are IID normal, no matter how many periods T are used to form the average return. But Central Limit Theorems (e.g. see Lehmann, 1999, Chap. 2) prove that this interpretation is still *approximately* true for suitably large T, in many cases when returns in excess of the benchmark are independently, non-normally distributed. That is, the average of T returns will be approximately normally distributed once T is large enough, making the above probability calculations accurate once T is suitably large.[2] These results are summarized in the following proposition.

Proposition 2: *Assuming the presence of independent, normally distributed returns measured in excess of a TEV investor's benchmark, maximization of the Information Ratio is equivalent to maximization (minimization) of the probability of outperforming (underperforming) the benchmark portfolio on-average over any number of evaluation periods T. Without the normality assumption, this interpretation is approximately valid for suitably large T.*[3]

In conjunction with Proposition 1, Proposition 2 shows that the criterion of maximizing (minimizing) the probability of outperforming (underperforming) the benchmark on-average is a natural generalization of the TEV hypothesis and its associated Information Ratio criterion function [see Browne (1999a,b) for analyses of portfolio choice based on more complex outperformance probability based criteria]. It provides a new, frequentist interpretation of existing studies that used the Information Ratio, e.g. Gupta *et al.* (1999) and Clarke *et al.* (2002).[4]

Finally, comparing (11) to (6), we see that a TEV investor's risky asset weight vector is no longer proportional to the tangency portfolio w_T, but instead is an affine transformation of it, displaced by the benchmark portfolio's risky asset weight vector q_b. Hence, TEV investors' portfolios will not be mean–variance efficient, i.e. a linear combination of two mean–variance efficient portfolios, unless the benchmark itself is.

2.1 Description versus prescription

The TEV hypothesis, and especially its aforementioned interpretation as maximizing (minimizing) the probability of outperforming (underperforming) a benchmark, is at least as plausible a *description* of fund manager behavior as the conventional mean–variance hypothesis is. Chan *et al.* (1999, p. 938) note that "Since managers are evaluated relative to some benchmark, it has become standard practice for them to optimize with respect to tracking error volatility." They provide further corroboration of the outperformance probability interpretation given in Proposition 2 above, by stating that (Chan *et al.*, 1999, p. 956) "Since professional managers are paid to outperform a benchmark, they are in general not concerned with the absolute variance of their portfolios, but are concerned with how their portfolio deviates from the benchmark. Underperforming the benchmark for several years typically results in the termination of a manager's appointment." Some direct evidence for this was provided by Olsen (1997), who conducted a series of surveys of portfolio managers and strategists, randomly selected from US-based Chartered Financial Analysts (CFAs). He asked these professionals to "list those things that first come into your mind when you think about *investment risk*," and found that 47% of them placed either "a large loss" or "return below target" first on their lists, which was more than twice as high as any other response. Finally, Goodwin (1998, p. 34) notes that "Most money managers routinely report their products' information ratios to investors, and some investors rely on information ratios to hire and fire money managers." The close connection between Information Ratio maximization and outperformance probability maximization, detailed in Proposition 2, shows that those money managers and their clients are at least implicitly concerned with the probability of outperforming their benchmark. Is it not reasonable to presume that the massive amount of professionally managed capital, invested in attempts to beat benchmarks, has had *some* influence on the returns of assets favored or disfavored by this criterion? Those proposing alternative explanations for multiple priced factors implicitly presume that fund management is irrelevant.

While it may be a better *description* of portfolio managers' behavior than the conventional mean–variance hypothesis, Roll (1992) worried that it may not be a good *prescription* for funds' investors. *If* a manager's benchmark portfolio is not on the mean–variance efficiency frontier, Roll (1992) foresaw a role for portfolio constraints that would induce fund managers to choose more mean–variance efficient portfolios. But, it is quite difficult for investors, fund managers, and/or regulators to ascertain whether or not a particular benchmark portfolio is on the mean–variance frontier. As Roll (1992, p. 19) notes:

> Estimation error is severe in portfolio analysis. No one knows where the global total return efficient frontier is really located. Its position depends, *inter alia*, on individual asset expected returns, which can be estimated only with substantial error because of the large component of noise in observed returns.

Furthermore, it is possible that measuring performance relative to a benchmark is a second-best, principal-agent mechanism desirably employed by investors (i.e. the principals), coping with an asymmetry in which portfolio managers (the agents) have better information about the efficiency frontier than they do. Measuring performance relative to a benchmark subtracts out a common shock faced by investors, which in the words of Brown *et al.* (1996, p. 87) would "allow the principal to separate some of the variation in outcome due to the state of nature from the agent's contribution." Moreover, Roll (1992, p. 20 and footnote 10) notes that doing so helps cope with estimation error, because when the correlation coefficient of the benchmark portfolio's return with a managed portfolio's return is in excess of half the ratio of their volatilities, "estimated *differences* between portfolio returns can be estimated more precisely."

Finally, the prescriptive case against measuring performance relative to a benchmark presumes that all investors *should* be worried about possible portfolio mean–variance inefficiency. *A priori*, it is equally plausible that *some* investors *should* be worried that their investments will not outperform a particular benchmark that provides a floor for their satisfaction, and accordingly either seek to employ a manager that will choose a portfolio with the highest probability of beating that benchmark on-average, or attempt to do it themselves.

3 Capital market equilibrium

Following Brennan (1993), capital market equilibrium is derived analogously to the standard CAPM: one imposes the equilibrium condition that the aggregate risky asset demand vector must equal the vector of market supplies. The demand arises from both conventional investors and the different classes of benchmark investors. The equilibrium condition is:

$$\sum_i W^i q_p^i + \sum_{b \in B} \sum_j W_b^j q_{pb}^j = W^m w_m \qquad (14)$$

where the W^i in the first term denotes the total wealth invested by the conventional mean–variance investor i, q_p^i denotes the risky asset weight vector chosen by that investor, W_b^j is the total wealth invested by the TEV investor j utilizing the benchmark b, and q_{pb}^j denotes the risky asset weight vector chosen by that TEV investor. The right-hand side of (14) multiplies the vector of market portfolio weights w_m by the total value of the market W^m to obtain the vector of risky assets' market supplies.

Now substitute (6) into the first term on the left-hand side of (14), and (11) into the rest of it, to produce the aggregate demand vector. The sums can be simplified by noting that the factor portfolios identified in the literature, e.g. the aforementioned papers of Fama and French, are comprised solely of risky assets, in which case we can let $q_{bf} = 0$ in (11). Letting W_r^c denote the aggregate value of wealth invested by conventional mean–variance investors in *risky* assets, W_f^b denote the aggregate wealth invested in the *riskless* asset by TEV investors with benchmark b, and W^b denote the aggregate wealth invested by TEV investors with benchmark b, we derive

$$w_T * \left(W_r^c - \sum_{b \in B} W_f^b \right) = W^m w_m - \sum_{b \in B} W^b q_b \tag{15}$$

Now, substitute (5) for w_T in (15), multiply both sides by the covariance matrix V of the risky assets' returns, and rearrange to obtain:

$$\begin{aligned} E(R) - \mathbf{1}R_f &= \frac{\mathbf{1}'V^{-1}[E(R) - \mathbf{1}R_f]}{W_r^c - \sum_{b \in B} W_f^b} \left[W^m V w_m - \sum_{b \in B} W^b V q_b \right] \\ &\equiv \frac{A}{W} \left[W^m V w_m - \sum_{b \in B} W^b V q_b \right] \\ &\equiv \frac{A}{W} \left[W^m \operatorname{Cov}(R1, R_m) - \sum_{b \in B} W^b \operatorname{Cov}(R_1, R_b) \right] \\ &\vdots \\ &\frac{A}{W} \left[W^m \operatorname{Cov}(R_N, R_m) - \sum_{b \in B} W^b \operatorname{Cov}(R_N, R_b) \right] \\ &\equiv \mathrm{COV} \frac{A}{W} [W^m, -W^{b_1}, \ldots, -W^{b_n}]' \end{aligned} \tag{16}$$

where COV denotes the matrix of the N risky assets' covariances with the market and the n benchmark portfolios' returns, $A = \mathbf{1}'V^{-1}[E(R) - \mathbf{1}R_f]$ is one of the four numbers that Huang and Litzenberger (1988, p. 64) used to characterize the mean–variance efficient set, and $W = W_r^c - \sum_{b \in B} W_f^b$. The covariance terms arise because the product of V and any portfolio weight vector is the vector of the risky assets' return covariances with that portfolio's return. This will soon be transformed into the exact linear factor model (2). But first, let us examine an important implication of it.

Premultiplying both sides of (16) by any portfolio's risky asset weight vector produces a linear relationship between that portfolio's expected excess return and its covariances with the market and benchmarks' returns. Successively doing so for market and the n benchmark portfolios yields the following system of linear equations:

$$\begin{aligned} E(R_m) - R_f &= \frac{A}{W} \Big[W^m \operatorname{Cov}(R_m, R_m) - \sum_{b \in B} W^b \operatorname{Cov}(R_m, R_b) \Big] \\ E(R_{b_1}) - R_f &= \frac{A}{W} \Big[W^m \operatorname{Cov}(R_{b_1}, R_m) - \sum_{b \in B} W^b \operatorname{Cov}(R_{b_1}, R_b) \Big] \\ \vdots &= \vdots \\ E(R_{b_n}) - R_f &= \frac{A}{W} \Big[W^m \operatorname{Cov}(R_{b_n}, R_m) - \sum_{b \in B} W^b \operatorname{Cov}(R_{b_n}, R_b) \Big] \end{aligned} \tag{17}$$

It is convenient to write (17) in matrix form as:

$$E(\mathcal{R}) - \mathbf{1}R_f = \Sigma \frac{A}{W} [W^m, -W^{b_1}, \ldots, -W^{b_n}]' \tag{18}$$

where $E(\mathcal{R})$ denotes the vector of market and benchmark portfolios' expected returns on the left-hand side of (17) and Σ is the square covariance matrix of the market and benchmark portfolios' returns. Premultiply both sides of (18) by Σ^{-1} to produce $\Sigma^{-1}[E(\mathcal{R}) - \mathbf{1}R_f]$, and then use the seminal result in Stevens (1998, Eq. (9), p. 1826) to rewrite it as follows:

$$\frac{A}{W}[W^m, -W^{b_1}, \ldots, -W^{b_n}]' = \Sigma^{-1}[E(\mathcal{R}) - \mathbf{1}R_f]$$

$$= \left[\frac{\alpha_m}{\mathrm{Var}(\varepsilon_m)}, \frac{\alpha_{b_1}}{\mathrm{Var}(\varepsilon_{b_1})}, \ldots, \frac{\alpha_{b_n}}{\mathrm{Var}(\varepsilon_{b_n})} \right]' \tag{19}$$

where α_m is the intercept and $\mathrm{Var}(\varepsilon_m)$ is the variance of the error term in the following linear excess return factor model for the market portfolio:

$$R_m - R_f = \alpha_m + \sum_{j=1}^{n} \beta_{mb_j}(R_{b_j} - R_f) + \varepsilon_m \tag{20}$$

and αb_i and $\mathrm{Var}[\varepsilon_{b_i}]$ are the counterparts in the following linear excess return factor model for the ith benchmark portfolio:

$$R_{b_i} - R_f = \alpha_{b_i} + \beta_{b_i m}(R_m - R_f) + \sum_{j \neq i} \beta_{b_i b_j}(R_{b_j} - R_f) + \varepsilon_{b_i} \tag{21}$$

Because the theory does not imply a particular sign for A/W,[5] (19) only implies that α_m in the factor model (20) for the market portfolio has a sign opposite to the predicted common sign of each α_{b_i} in the factor model (21) for the benchmark portfolio i. This implication is summarized as the following proposition:

Proposition 3: *The intercept (i.e. alpha) in a linear factor regression model of the market portfolio's excess return on the benchmark portfolios' excess returns should be opposite in sign to the intercept in a linear factor regression model of any benchmark portfolio's excess return on the excess returns of the market portfolio and the other benchmark portfolios.*

While a test of the sign restriction in Proposition 3 is conducted in Section 4, there is another implication of the theory that has already received vast testing and commentary. This is (2), the general linear, multi-excess return factor generalization of the CAPM. Let us derive (2) as an equilibrium relationship; (19) permits the substitution of $\Sigma^{-1}[E(\mathcal{R}) - \mathbf{1}R_f]$ for $(A/W)[W^m, -W^{b_1}, \ldots, -W^{b_n}]'$ in (16), resulting in the vector of equations

$$E(R) - \mathbf{1}R_f = \mathrm{COV}\,\Sigma^{-1}[E(\mathcal{R}) - \mathbf{1}R_f] \tag{22}$$

whose ith component is the factor model (2), i.e. $\mathrm{COV}\ \Sigma^{-1}$ is the $N \times n+1$ matrix of the N risky asset returns' betas on the market, and the n benchmark portfolios that separate classes that TEV investors want to beat.[6] This is summarized in the following proposition:

Proposition 4: *The linear, excess return factor model (2) of expected asset returns arises as a market equilibrium in the presence of a riskless asset, with both mean–variance investors and classes of TEV investors, who respectively attempt to beat one of the benchmarks in (2).*

4 Some empirical evidence

Let us test the theory's implied sign restriction given in Proposition 3, using the portfolios that were essential in the Fama and French equity factor model tests. Fama and French (1996, Table IX, pp. 70–71) documented that "equivalent descriptions of returns" are provided when omitting an explicit size factor from their three factor model (1), using just $R_M - R_f$, $R_L - R_f$, and $R_H - R_f$ as excess return factors in a three factor model, where R_M is the return on the CRSP Value Weighted portfolio, R_L is the return on their portfolio of low book-to-market ratio (i.e. growth) stocks, and R_H is the return on their portfolio of high book-to-market ratio (i.e. value) stocks. In fact, even their earlier study (Fama and French, 1992, pp. 447–448) concluded that "Unlike the size effect, the relation between book-to-market equity and average return is so strong that it shows up reliably in both the 1963–1976 and the 1977–1990 subperiods . . . The subperiod results thus support the conclusion that, among the variables considered here, book-to-market equity is consistently the most powerful for explaining the cross-section of average stock returns." Corroborating this emphasis on book-to-market equity, Knez and Ready (1997) employed an outlier-robust regression technique to re-examine the Fama and French evidence, concluding that (Knez and Ready, 1997, p. 1380) "the negative relation between firm size and average returns is driven by a few extreme positive returns in each month," that (Knez and Ready, 1997, p. 1356) "most small firms actually do worse than larger firms," but (Knez and Ready, 1997, p. 1357) "that the risk premium on book-to-market is not affected by extreme observations once you control for size."

In light of this evidence for the efficacy of a linear model (2) of stock returns with three excess return factors $R_m - R_f$, $R_L - R_f$, and $R_H - R_f$, let us follow Fama and French in using the CRSP Value Weighted equity portfolio as the "market" portfolio, and test Proposition 3 assuming that there are two classes of TEV investors: one class tries to beat a growth stock benchmark, proxied by the Fama and French growth stock portfolio L, while the other class tries to beat a value stock benchmark that is proxied by the Fama and French value stock portfolio H. Using the same July 1963 to December 1993 data period adopted by Fama and French, the OLS-estimated (with t-statistic in parentheses) $\alpha_m = -0.059\%$ per month (−2.42) in (20). The L benchmark portfolio's $\alpha_L = 0.059\%$ per month (1.64) in (21). The H benchmark portfolio's $\alpha_H = 0.277\%$ per month (4.21). The significantly negative sign of the market alpha is opposite to the significantly positive signs of the L and H alphas, consistent with Proposition 3. Given the aforementioned Fama and French evidence on the efficacy of the L and H benchmark factors in explaining stock returns, it is not surprising that the market regression (20) and the two benchmark factor regressions (21) all had values for their adjusted R^2 in excess of 90%.[7]

Additional corroborating evidence for this theory was found in the two factor model of Gomez and Zapatero (in press). They used the MSCI US equity index as the proxy for the market portfolio, and the S&P 500 as a benchmark portfolio that all TEV investors try to beat. They estimated an orthogonalized version of the resulting linear two-factor model (2) on a large number of stocks, and concluded that the addition of the S&P 500 factor did indeed help explain the expected returns of those stocks. Because of Fama and French's (1992, p. 446) finding that "large stocks are more likely to be firms with . . . lower book-to-market equity," the findings of Gomez and Zapatero overlap with Fama and French's findings about the explanatory ability of their low book-to-market "L" portfolio. Gomez and Zapatero (in press) did not derive an analog of the sign restriction in Proposition 3, and hence did not subject their proposed factor model to the additional test conducted above.

5 Conclusion

A linear excess return factor model was derived as a consequence of equilibrium asset demands from a class of conventional mean–variance investors and different classes of other investors, each of whom tries to beat a different benchmark portfolio in accord with the TEV hypothesis of Roll (1992). That is, each TEV investor chooses a portfolio to minimize the variance of its return about a benchmark portfolio's return, while trying to exceed the benchmark's expected return by some amount. In the presence of a riskless asset, this is equivalent to choosing a risky asset weight vector portfolio that maximizes the well-known Information Ratio, calculated by using the benchmark portfolio's return in place of the riskless asset return in the conventional Sharpe Ratio. It was shown that maximization of the Information Ratio is also consistent with maximization (minimization) of the probability of outperforming (or underperforming) the benchmark used to compute it, particularly so over longer time horizons. This frequentist criterion seems plausible, and there does not appear to be any direct evidence that either institutional investors (e.g. managed mutual funds, banks, insurers, and other financial intermediaries) or individual investors behave more in accord with conventional mean–variance theory than TEV theory.

The theory implies that an asset's expected excess return is linearly related to the excess return of the market portfolio's excess return, as in the CAPM. But the excess returns of each of the benchmark portfolios in wide use will also be factors priced in this way, whether or not they are mean–variance efficient portfolios. The theory also implies that if one builds a separate excess return factor model for the market portfolio with the benchmarks as factors, and builds separate excess return factor models for each benchmark portfolio with the market and the *other* benchmark portfolios as factors, the market model's alpha must have a sign opposite to that of the usual one for all the benchmark portfolios' alphas. An empirical test of this implication indicated that the seemingly anomalous empirical multifactor findings of Fama and French (1992, 1996) may be explained by the presence of two classes of TEV investors, trying to beat growth stock and value stock benchmarks, respectively.

An ambitious future theoretical topic is to use the generalized theory of benchmark investing developed in Stutzer (2003) and Foster and Stutzer (2002) to extend the theoretical predictions beyond the normally distributed TEV paradigm. This should yield analogous nonlinear multifactor models. A good empirical topic would be to incorporate a widely adopted bond benchmark and a value-weighted blended market portfolio, in order to jointly test the theory on both stocks and bonds.

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Notes

1 Here is the proof. A risky asset vector that maximizes the Information Ratio also maximizes the squared ratio. Again, using the notation $q - q_b \equiv x$ as in (9), the squared Information Ratio is $\left(\sum_i x_i[E(R_i) - R_f]\right)^2 / \sum_i \sum_j V_{ij}x_ix_j$. Because q_b is fixed, maximizing over q can be achieved by maximizing over x. Because neither risky asset vectors q nor q_b need sum to one, the x_k are unrestricted variables, so the first derivatives of the Information Ratio with respect to each must be zero at a maximum. The kth first derivative condition can be written as $\sum_j V_{kj}\lambda x_j = E(R_k) - R_f$, where $\lambda \equiv \left(\sum_i x_i[E(R_i) - R_f]\right) / 2\sum_i \sum_j V_{ij}x_ix_j$. Substituting variables defined by $y_j \equiv \lambda x_j$ transforms these into an exactly determined system of linear equations, with solution vector $y = V^{-1}[E(R) - \mathbf{1}'R_f] \equiv \lambda x$. Dividing both sides of this expression by $\sum_i y_i = \mathbf{1}'y = \lambda \mathbf{1}'x$ we obtain $y/\mathbf{1}'y = x/\mathbf{1}'x = w_T$. This is the same as Eq. (10), which is used to characterize the TEV portfolio.

2 Restrictions on the process $R_{pt} - R_{bt}$ that are required for a CLT approximation of the average return's distribution are given in many texts, e.g. Lehmann (1999). Such process restrictions are often implicit in commonly applied time series analyses of financial returns.

3 The Gärtner–Ellis Large Deviations Theorem (Bucklew, 1990, Chap. 2) provides an alternative to the CLT approximation when T is large. In the special case of IID, but not necessarily normally distributed $R_p - R_b$, Stutzer (2000) used it to show that the Information Ratio should be replaced by the alternative performance criterion $\max_\theta E\left[-e^{-\theta(R_p - R_b)}\right]$, i.e. the expected CARA utility of $R_p - R_b$, when evaluated at the coefficient of absolute risk aversion θ that maximizes the expected CARA utility of $R_p - R_b$. A straightforward calculation shows that this is half the squared Information Ratio when $R_p - R_b$ is normally distributed, and is thus equivalent to using that ratio, consistent with Proposition 2.

4 It is also possible to formulate the probability of cumulatively outperforming the benchmark by a specific *calendar* time T, rather than on-average over T periods, by substituting $\log R_p - \log R_b$ for $R_p - R_b$ before forming the average. Stutzer (2003) and Foster and Stutzer (2002) studied and applied this formulation of the outperformance probability hypothesis. But the on-average criterion is needed here to derive the exact linear factor model (2) of non-log returns that is commonly used in practice.

5 Remember that $A/W \equiv (\mathbf{1}'V^{-1}[E(R) - \mathbf{1}R_f])/\left(W_r^c - \sum_{b \in B} W_f^b\right)$. The case analysis and figures in Huang and

Litzenberger (1988, pp. 77–78) assume that $A/\mathbf{1}'\Sigma^{-1}\mathbf{1} > 0$. Because Σ^{-1} is positive definite, this is tantamount to assuming that $A > 0$. But the sign of W depends on whether or not the aggregate wealth invested by conventional mean–variance investors in risky assets exceeds the aggregate wealth invested by all TEV investors in the riskless asset. Because there appears to be no direct evidence that conventional mean–variance investing in risky assets is any more common than TEV investing (among individuals as well as institutional investors) in riskless assets, the theory does not imply that $W > 0$. Hence, the sign of A/W is indeterminate.

6 The derivation simplifies the development in Brennan (1993). Brennan only allowed conventional investors the right to invest in the riskless asset. In his model, the TEV investors are not allowed to use the riskless asset. As a result, Brennan derives a more complicated relationship than (2), which necessitates replacing R_f in (2) by a complicated function of it, the return on the minimum variance portfolio, and investors' wealths and absolute coefficients of risk aversion (see Brennan (1993) for details). By symmetrically allowing the TEV investors the same right to invest a fraction of the funds in the riskless asset that conventional investors have, the simpler relationship (2) results, which has a form commonly used in empirical studies (e.g. in Fama and French, 1996, Table IX, pp. 70–71; Gruber, 1996; Elton *et al.*, 1996).

7 A referee wrote that a small cap benchmark should be included in these regressions, because small cap benchmarks "are in common usage among investors." Adding the excess returns from a benchmark portfolio of small cap stocks (specifically, a portfolio representing the smallest 20% of CRSP stocks' market capitalization, as reported on Kenneth French's website) resulted in a sign pattern analogous to that reported above. That is, the estimated α_m in (20) is statistically significantly negative, while the intercepts in two of the three possible regressions (21) of a benchmark portfolio excess return on the market's and the other two benchmarks' excess returns were statistically significantly positive. While the estimated intercept in the other regression (the excess return of the small cap benchmark on the excess return of the market and the excess returns of the L and H benchmarks) was negative, it was statistically insignificant (its t-statistic was only 1.10). Hence, the addition of a small cap benchmark does not change the empirical support for the theory's Proposition 3.

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