Journal of Investment Management, Vol. 1, No. 1, (2003), pp. 45–54

JOIM
www.joim.com

GREAT MOMENTS IN FINANCIAL ECONOMICS: I. PRESENT VALUE

*Mark Rubinstein**

This is the first in a series of articles to appear in this journal on the history of significant ideas in financial economics. Perhaps the most basic of these is the idea of present value. Early contributors include Johan de Witt (1671), the famous mathematician Abraham de Moivre (1725), and the famous scientist Edmund Halley (1761). But it was Irving Fisher who in 1930 laid the theoretical foundations behind the concept as a by-product of the standard inter-temporal model of rational consumption choice. In 1938, John Burr Williams applied the model to the discounting of dividends and derived what later became known as the Gordon growth formula.

Ideas are seldom born fully clothed, but are rather dressed by a slow and arduous process of accretion. In the study of many fields, to achieve deep knowledge of the current state-of-the-art, it is necessary to appreciate how its ideas have evolved—What are their origins? By what paths of thought are they elaborated? How does one idea lead to others? Why was there once confusion about ideas that now seem obvious?

Such an understanding has a special significance in the social sciences. In the humanities, there is little sense of chronological progress. For example, who would argue that, in the last three centuries, English poetry or drama has been written that surpasses the works of Shakespeare? In the natural sciences, knowledge accumulates by largely uncovering pre-existing and permanent natural processes; whereas knowledge in the social sciences can affect the social evolution that follows its discovery, which, in turn, largely determines the succeeding social theory.

In this spirit, this article and its successors that are to appear in subsequent issues of this journal provide a history of aspects of the theory of financial economics, emphasizing the earliest foundation-setting contributions. It is not, however, a history of the practice of finance, and only occasionally refers to the large real world outside of theoretical finance. Nonetheless, the "history of financial economics" is construed quite broadly to include the historical development of methodological and theoretical tools used to create this theory, including economics and mathematics.

*Professor University of California at Berkeley.
E-mail: mr@in-the-money.com

Of necessity, I have selected only a small, but I hope the most important, portion of the full body of research that is available on each subject. Some papers are significant because they plant a seed, ask what turns out to be the right question, or develop important economic intuitions; others are important because they formalize earlier concepts, making all assumptions clear, and prove results with mathematical rigor. Mathematical statements or proofs are also provided, usually set off by small print, of important and condensable results primarily to compensate for the ambiguity of words. However, the proofs are seldom necessary for an intuitive understanding.

Perhaps this field is like others, but I am nonetheless dismayed to see how little care is taken by many scholars to attribute ideas to their original sources. Academic articles and books, even many of those that purport to be historical surveys, occasionally of necessity but often out of ignorance, oversimplify the sequence of contributors to a finally fully developed theory, attributing too much originality to too few scholars. No doubt that has inadvertently occurred in these articles as well, but hopefully to a much lesser extent than earlier attempts. Even worse, an important work can lie buried in the forgotten past; occasionally, this work is even superior in some way to the later papers that are typically referenced.

For example, the Modigliani–Miller theorem received possibly its most elegant exposition, at its inception, in a single paragraph contained in a rarely referenced but amazing book by John Burr Williams published 20-years before Modigliani and Miller in 1938 (Williams, 1938). Had his initial insight been well known and carefully considered, decades of confusion might have been spared. Sometimes models and formulae are mistakenly named, implicitly attributing them to the wrong source; an example of this is the "Gordon growth formula". Unfortunately, once this type of error takes hold, it is very difficult to shake lose. Indeed, the error becomes so ingrained that even prominent publicity is unlikely to change old habits.

Also, researchers occasionally do not realize that an important fundamental aspect of a theory was discovered many years earlier. To take a prominent example, although the Black–Scholes model developed in the early 1970s is surely one of the great discoveries of financial economics, fundamentally it derives its force from the idea that it may be possible to make up for missing securities in the market by the ability to revise a portfolio of the few securities that do exist over time. Kenneth Arrow, 20-years earlier in 1953, was the first to give form to a very similar idea. In turn, shades of Arrow's idea can be found in the famous correspondence between Blaise Pascal and Pierre de Fermat three centuries earlier. One of the delightful by-products of historical research is the connections that one often uncovers between apparently disparate and unrelated work—connections that may not have been consciously at work, but no doubt through undocumented byways, must surely have exercised an influence.

Of course, financial economics is not alone in its tendency to oversimplify its origins. For example, consider the calculus, well known to have been invented by Issac Newton and Gottfried Willhelm Leibnitz. Yet, the invention of calculus can be traced back to the classical Greeks, in particular, Antiphon, Euxodus, and Archimedes, who anticipated the concept of limits and of integration in their use of the "method of exhaustion" to determine the areas and volumes of geometric objects (e.g., to estimate the area of a circle, inscribe a regular polygon in the circle; as the number of sides of the polygon goes to infinity, the polygon provides an increasingly more accurate approximation of the area of the circle). Galileo Galilei's work on motion implies that velocity is the first derivative of distance with respect to time and acceleration is the second derivative of distance with respect to time. Fermat devised the method of tangents that in substance we use today to determine maxima and minima of functions. Isaac Barrow used the notion of differential to find the tangent to a curve and described theorems for the differentiation of the product and quotient of two functions, the differentiation of powers of x, the change of variable in a definite integral, and the differentiation of implicit functions.

Despite this view that ideas accumulate by a slow process usually involving contributions from several individuals, occasionally it will seem, even after all available evidence is examined, that a particular individual has made a significant intuitive leap that surmounts a seemingly impassible barrier. When, in my opinion that has happened, these articles will celebrate that event by calling it a "Great Moment" in the history of financial economics.

One caveat: I have tried my best, for each paper or book cited, to clarify its marginal contribution to the field. Despite my best efforts, I am certain that I have omitted MANY important discoveries or even attributed ideas to the wrong sources, unaware of even earlier work. I hope the reader will forgive me. Even better, I ask the reader to take the constructive step of letting me know the error of my ways.

1 Johan de Witt, Abraham de Moivre, and Edmund Halley

No doubt the idea of present value has had a long undocumented history. For example, we know that the classical Greeks applied their mathematical acumen to the inverse problems of calculating simple and compound interest. In modern times, our first **Great Moment** in financial economics came with the publication in 1671 of *Value of Life Annuities in Proportion to Redeemable Annuities* by Johan de Witt (1625–1672).

A life annuity is a contract that pays the annuitant a given constant amount every year until the death of a given individual, the "nominee" (usually the same as the annuitant), with no repayment of principal. A generalization of a life annuity is a tontine (named after a government funding proposal recommended to the French Cardinal Mazarin in 1652 by Lorenzo Tonti). In a typical arrangement, a designated group of annuitants equally divide among themselves a given constant total every year. As the annuitants drop out because of their death, those remaining divide the same total, leaving a greater payment to each. After only one annuitant remains, he receives the entire amount. Once he dies, all payments cease.

Bequests in ancient Rome often took the form of a life annuity to children who were not the first born; and, beginning in the seventeenth century, life annuities were used by governments to raise funds. Although the Romans may have used a crude adjustment for the expected life of the nominee (a controversial issue among ancient historians), little attempt was made to make this adjustment in the seventeenth century before de Witt. In what may be regarded as the first formal analysis of an option style derivative, de Witt proposed a way to calculate the value of life annuities that takes account of the age of the nominee. His method was crude by modern standards, but he did make use of one of the first mortality tables. de Witt assumed nominees would die according to the list given below: Out of every 768 nominees:

6 will die every six months for the first 50 years
4 will die every six months for the next 10 years
3 will die every six months for the next 10 years
2 will die every six months for the next 7 years

Assuming a compound interest rate of 4%, for each of the 768 times to death, he calculated the present value of the corresponding annuity and then took their arithmetic average to be the price of the annuity.

Further progress with the valuation of annuities followed quickly, occupying the attentions of illustrious scientists and mathematicians. Abraham de Moivre (1657–1754), perhaps best known for his proof of the normal approximation to the binomial, a precursor of the central limit theorem, in *Treatise of Annuities on Lives* (1725), also worked on the life annuity problem, deriving a closed-form analytical approximation assuming that the probability of remaining alive decreased with age in an arithmetic progression. According to Geoffrey Poitras (2000), in *The Early History of Financial Economics: 1478–1776: From Commercial Arith-metic to Life Annuities and Joint Stocks*, de Moivre also provided both exact and quick-calculation approximation results for the tontine valuation problem, as well as for annuities written on successive lives.[1]

In the forgotten age before computers, once it was desired to determine the effects of interest rates on contracts, much work was devoted to developing fast means of computation. These include the use of logarithms, pre-calculated tables, and closed-form algebraic solutions to present value problems. Edmund Halley (1656–1742) in a paper titled "Of Compound Interest", published posthumously in 1761, better known, of course, for the comet that bears his name and many other far more important achievements, derives (probably not for the first time) the formula for the present value of an annual annuity beginning at the end of year 1 with a final payment at the end of year T: $[X/(r-1)][1-(1/r^T)]$, where r is one plus the annual discrete interest rate of the annuity and X is the annual cash receipt from the annuity. Another relatively early derivation can be found in Irving Fisher (1867–1947), *The Nature of Income and Capital* (1906).

Although valuation by present value had appeared much earlier, Irving Fisher, in *The Rate of Interest: Its Nature, Determination and Relation to Economic Phenomena* (1907), may have been the first to propose that <u>any</u> capital project should be evaluated in terms of its present value. Using an arbitrage argument, he compared the stream of cash flows from the project to the cash flows from a portfolio of securities constructed to match the project. Despite this, according to Faulhaber and Baumol (1988), neither the *Harvard Business Review* from its founding in 1922 to World War II, nor widely used textbooks in corporate finance as late as 1948, made any reference to present value in capital budgeting. It was not until 1951 that Joel Dean in his book *Capital Budgeting* popularized the use of present value.

2 Irving Fisher

Another *Great Moment* in financial economics surely occurred in 1930 with the publication of Irving Fisher's *The Theory of Interest: As Determined by Impatience to Spend Income and Opportunity to Invest It*. Here is the seminal work for most of the financial theory of investments during the twentieth century. Fisher's 1930 book refines and restates many earlier results that had appeared in his *Appreciation and Interest* (1896), *The Nature of Capital and Income* (1906), and *The Rate of Interest* (1907), and, as Fisher states, were foreshadowed by John Rae (1796–1872: 1834), to whom Fisher dedicates his book. Fisher develops the first formal equilibrium model of an economy with both intertemporal exchange and production. In so doing, at one swoop, he not only derives present value calculations as a natural economic outcome in calculating wealth, he also justifies the maximization of present value as the goal of production and derives determinants of the interest rates that are used to calculate present value.

He assumes each agent is both the consumer and producer of a single aggregate consumption good under certainty. This single good simplification allows him to abstract from the unnecessary complications of the multi-commodity Walrasian paradigm, and has ever since been at the heart of theoretical research in finance. At each date, exchange is effected by means of a short-term riskless bond maturing at the end of the period. In this context, among its many contributions to economic thought are (I) an analysis of the determinants of the real rate of interest and the equilibrium intertemporal path of aggregate consumption; (II) the "Fisher effect" relating the nominal interest rate to the real interest rate and the rate of inflation; and (III) the "Fisher separation theorem" justifying the delegation of production decisions to firms that maximize present value, without any direct dependence on shareholder preferences, and justifying the separation of firm financing and production decisions. Most subsequent work in the financial theory of investments can be viewed as further elaboration, particularly to considerations of uncertainty and to more complex financial instruments, for the allocation of consumption across time and across states of the world.

Fisher reconciles the two previous explanations of the rate of interest, one based on productivity ("opportunity") and the other based on consumer psychology, or time-preference ("impatience", a termed coined by Fisher in *The Rate of Interest*), showing that they are jointly needed for a comprehensive theory: "So the rate of interest is the mouthpiece at once of impatience to spend income without delay and of opportunity to increase income by delay" (p. 495).

Fisher describes his economy in three ways: in words, with graphs, and with equations. It is interesting that, even at this time in the development of economic thought, Fisher finds it necessary to justify the usefulness of algebraic formulations, pointing out that by this method one could be sure that the number of unknowns and number of independent equations are the same. In addition, he writes:

"The contention often met with that the mathematical formulation of economic problems gives a picture of theoretical exactitude untrue to actual life is absolutely correct. But, to my mind, this is not an objection but a very definite advantage, for it brings out the principles in such sharp relief that it enables us to put our finger definitely on the points where the picture is untrue to real life." (p. 315)

Fisher develops a simple example with just two time periods and three consumers for the case where only consumer time-preference determines interest rates. Let

r be the equilibrium riskless return
$\underline{C}_0^i$, $\underline{C}_1^i$ be the endowed consumption of consumer i at dates 0 and 1
x_0^i, x_1^i be the amount of borrowing or lending of consumer i at dates 0 and 1 which each consumer can choose subject to his budget constraint: $\boldsymbol{x}_0^i + x_1^i/r = 0$
$C_0^i \equiv \underline{C}_0^i + x_0^i$, $C_1^i \equiv \underline{C}_1^i + \boldsymbol{x}_1^i$ be the optimal amounts of consumption that consumer i chooses at dates 0 and 1

He then assumes that a consumer's rate of time-preference will depend on his chosen consumption stream:

$f_i = F_i(C_0^i, C_1^i)$ is the rate of time-preference of consumer i

In the appendix to Chapter 12, Fisher relates the rate of time-preference to the utility of consumption, $U_i(C_0^i, C_1^i)$ such that: $f_i = [U_i'(C_0^i)/U_i'(C_1^i)] - 1$.

He argues that in equilibrium the rate of time-preference of each consumer must equal the riskless return, so that

$$f_1 = f_2 = f_3 = r$$

For the market to clear, he requires that net borrowing and lending at each date across all consumers be 0: $x_0^1 + x_0^2 + x_0^3 = 0$ and $x_1^1 + x_1^2 + x_1^3 = 0$. The seven unknowns, C_0^1, C_0^2, C_0^3, C_1^1, C_1^2, C_1^3, and r are matched by seven independent equations.

A modernized representative agent proof including production would be given as below. Let

$U(C_0)$, $U(C_1)$ be the utility of consumption at dates 0 and 1

ρ be the rate of patience

Ω_0 be the initial endowment of the consumption good

X_0 be the amount of Ω_0 used up in production so that $C_0 = \Omega_0 - X_0$

$f(X_0)$ be the output from production of date 1 consumption so that $C_1 = f(X_0)$

W_0 be the current wealth of the consumer so that $W_0 = C_0 + C_1/r$

Assume that $U'(C) > 0$ (non-satiation), $U''(C) < 0$ (diminishing marginal utility), $0 < \rho < 1$ (tendency to prefer current over future consumption), $f'(X_0) > 0$ (more input yields more output), $f''(X_0) < 0$ (diminishing returns to scale).

The production problem for the consumer is:

$$\max_{C_0,C_1} U(C_0) + \rho U(C_1)$$
$$\text{subject to } C_0 = \Omega_0 - X_0 \text{ and } C_1 = f(X_0)$$

Substituting in the constraints, differentiating the utility function, and setting the derivative equal to zero to characterize the maximum, it follows that:

$$U'(C_0)/[\rho U''(C_1)] = f'(X_0)$$

The exchange problem for the consumer is

$$\max_{C_0,C_1} U(C_0) + \rho U(C_1) \quad \text{subject to } W_0 = C_0 + C_1/r$$

Again, substituting in the constraint, differentiating the utility function, and setting the derivative equal to zero, it follows that

$$U'(C_0)/[\rho U'(C_1)] = r$$

Gathering these two results together:

$$U'(C_0)/[\rho U'(C_1)] = r = f'(X_0) \qquad [1]$$

Thus, we have Fisher's two-sided determinants of the interest rate: the equilibrium riskless return equals what we would call today the marginal rate of substitution (what Fisher called "the rate of time-preference") and it equals the marginal productivity of capital.

For a more concrete example, suppose $U(C_t) = \log C_t$ and $f(X_0) = \alpha X_0^{\beta}$ with $0 < \beta < 1$ and $\alpha > 0$. These satisfy the derivative conditions on utility and the production function required above. α can be interpreted as a pure measure of productivity since the greater α the more output from any given input. Substituting into Eq. [1]:

$$\rho^{-1}(C_1/C_0) = r = \alpha\beta X_0 \beta^{-1}$$

Solving this for the unknowns C_0 and r:

$$C_0 = (1 + \rho\beta)^{-1}\Omega_0 \text{ and}$$
$$r = \alpha\beta[(\rho\beta/(1 + \rho\beta))\Omega_0]^{\beta-1}$$

Differentiating the solution for the riskless return:

$$\mathrm{d}r/\mathrm{d}\alpha = \beta[(\rho\beta/(1 + \rho\beta))\Omega_0]^{\beta-1} > 0 \text{ (productivity)}$$
$$\mathrm{d}r/\mathrm{d}\rho = \alpha(\beta - 1)\Omega_0^{\beta-1}\rho^{-2}\,(\rho\beta/(1 + \rho\beta))^{\beta} < 0 \text{ (time-preference)}$$

So, we see a pure isolation of the effects of Fisher's impatience (ρ) and opportunity (α) on the interest rate.

Fisher also claims that separate rates of interest for different time periods are a natural outcome of economic forces, and not something that can be arbitraged away in a perfect market.

"The other corollary is that such a formulation reveals the necessity of positing a theoretically separate rate of interest for each separate period of time, or to put the same thing in more practical terms, to recognize the divergence between the rate for short terms and long terms. This divergence is not merely due to an imperfect market and therefore subject to annihilation, as Böhm-Bawerk, for instance, seemed to think. They are definitely and normally distinct due to the endless variety in the conformations of income streams. No amount of mere price arbitrage could erase these differences." (p. 313)

More generally, Fisher argued that the rate of interest was determined by: (i) the relative distribution of endowed resources across time; (ii) time-preferences of consumer/investors; (iii) production opportunities that provide a way of transforming aggregate current endowments into aggregate future consumption; (iv) the general size of endowed resources; (v) risk aversion and the time-structure of risk; and (vi) the anticipated rate of inflation. With a noticeably behavioral orientation, Fisher attributed factor (ii) to lack of foresight, lack of self-control, habit formation, expected lifetime, and a bequest motive. He showed how all six factors would affect the decisions made by economic agents and how these decisions would aggregate up to determine the equilibrium rate of interest.

Fisher then considered a number of potential objections to his theory. An objection still popular is that tying the determinants of interest to aspects of intertemporal consumption choice may be elegant, but it is too narrow. In fact, interest is largely determined by the "supply and demand for loanable funds". Fisher replies that this supply and demand is the intermediate effect of the fundamental underlying needs of producers to maximize present value and of consumers to optimally balance their consumption over their lifetimes. But he also admits that there may be a myriad of institutional influences on interest rates that he has not considered, but that these factors will be secondary.

Fisher worded his separation result as follows:

"But we see that, in such a fluid world of options as we are here assuming, the capitalist reaches the final income through the cooperation of two kinds of choice of incomes which, under our assumptions, may be considered and treated as entirely separate. To repeat, these two kinds of choice are: first, the choice from among many possible income streams of that particular income stream with the highest present value, and secondly, the choice among different possible modifications of this income stream by borrowing and lending or buying and selling. The first is a selection from among income streams of differing market values, and the second, a selection from among income streams of the same market value." (p. 141)

This "separation" must be carefully interpreted to mean that the second choice is not independent of the first choice. In order to know what second choice to make, the implications of the first choice must be known. However, the first choice can be made before making the second. Fisher also made it quite clear that his separation result depends on a competitive market where the capitalist is "unconscious" of any impact he might have on interest rates, and he made it clear that his result requires the equivalency of borrowing and lending rates (perfect markets).

To derive the separation theorem, continuing with our earlier example, suppose the production decision were delegated to a competitive present value maximizing firm. Such a firm would then choose X_0 to:

$$\max_{X_0} \; -X_0 + f(X_0)/r$$

where it disregards any influence it may have over r (that is, it chooses X_0 as if $\mathrm{d}X_0/\mathrm{d}r = 0$). Differentiating the present value and setting the derivative equal to zero, it follows that: $r = f'(X_0)$, precisely the decision the representative consumer would have made on his own.

This suggests that, provided firms act as competitive present value maximizers, firms can make the same production decisions its shareholders would make on their own without knowledge of their time-preferences or their endowments. If true, this dramatically simplifies the problem of resource allocation in a competitive economy.

Fisher may also have been the first economist to emphasize the role of what are now called "real options" in increasing the flexibility of production opportunities, which now play a key role in modern treatments of present value for corporate investments:

"This brings us to another large and important class of options; namely the options, of effecting, renewals and repairs, and the options of effecting them in any one of many different degrees. . . . But the owner has many other options than that of thus maintaining a constant stock of goods. He may choose to enlarge his business as fast as he makes money from it. . . . A third option is gradually to go out of business. . . . Another case of optional income streams is found in the choice between different *methods* of production, especially between different degrees of so-called capitalist production. . . . The alternatives constantly presented to most business men are between policies which may be distinguished as temporary and permanent. The temporary policy involves use of easily constructed instruments which soon wear out, and the permanent policy involves the construction at great cost of instruments of great durability. . . . In all cases, the 'best' results are secured when the particular series of renewals, repairs, or betterments is chosen which renders the present value of the perspective income stream the maximum." (pp. 194–199)

He also discussed dynamic properties of interest rate changes, whereby, for example, increasing interest rates leads to a change in the utilization of production opportunities, which in turn, tends to stabilize interest rates, creating the mean reversion we typically observe.

While Fisher provided a qualitative discussion of the first-order effects of uncertainty, he expressed considerable pessimism about prospects for formal generalization of his theory:

"To attempt to formulate mathematically in any useful, complete manner the laws determining the rate of interest under the sway of chance would be like attempting to express completely the laws which determine the path of a projectile when affected by random gusts of wind. Such formulas would need to be either too general or too empirical to be of much value." (p. 316)

So Fisher left it for others to explain a wide variety of economic phenomena such as insurance, the use of both debt and equity, the demand for liquidity, the use of diversified portfolios and the extreme diversity of types of securities with differing returns, all of which largely rely on uncertainty for their existence.

3 John Burr Williams

John Burr Williams (1899–1989), the author of the insufficiently appreciated classic *The Theory of Investment Value* (1938), was one of the first economists to interpret stock prices as determined by "intrinsic value" (i.e., discounted dividends). Harry Markowitz (1991) writes in his Nobel Prize autobiography: "The basic concepts of portfolio theory came to me one afternoon in the library while reading John Burr Williams' *The Theory of Investment Value*."

While, as we have seen, Williams did not originate the idea of present value, he, nonetheless, develops many implications of the idea that the value of a stock under conditions of certainty is the present value of all its future dividends. His general present value formula is

$$P_0 = \sum_{t=1}^{\infty} D_t / r(t)^t$$

where D_t is the dividend paid at date t, $r(t)$ the current (date $t = 0$) annualized riskless discount return for dollars received at date t, and P_0 is the current (date $t = 0$) value of the stock.

A nice way to build up to this is to start with the recursive relation $P_t = (D_{t+1} + P_{t+1})/r(t+1)$. Successive substitutions for P_t through date T, leads to

$$P_0 = \sum_{t=1}^{T} D_t / r(t)^t + P_T / r(T)^T.$$

The result then follows for $T = \infty$.

He argues against discounting earnings instead of dividends and quotes the advice an old farmer gave his son (p. 58):

A cow for her milk,
A hen for her eggs,
And a stock, by heck,
For her dividends.

His book contains the derivation of the simple formula for the present value of a perpetually and constantly growing stream of income, $P_0 = D_1/(r - g)$, where r is the constant annualized riskless discount rate and g the constant annualized growth rate in dividends.

Here is a proof. Define $a = D_1/r$ and $x = g/r$. Then, $P_0 = a(1 + x + 1 + x^2 + \dots)$. Multiplying both sides by x, we have $P_0 x = a(x + x^2 + x^3 + \dots)$. Subtracting this from the previous expression for P_0, $P_0(1 - x) = a$. Substituting back for a and x, $P_0(1 - (g/r)) = D_1/r$. Therefore, $P_0 = D_1/(r - g)$.

Williams actually writes this formula in the form $P_0 = D_0x/(1-x)$ where $x \equiv g/r$ (p. 88, Eq. 17a) and notes that finite stock prices require $g < r$. This is commonly and mistakenly called the "Gordon growth formula" after its restatement by Myron J. Gordon (1920–) and Eli Shapiro in 1956. Gordon and Shapiro popularized the formula by rewriting it as $k = (D_1/P_0) + g$, where k equals r under certainty, but under uncertainty could loosely be interpreted as the expected return to stock. Breaking apart this expected return into two components, the dividend yield and growth, translated William's formula into a language that popularized it amongst investment professionals. For example, in the early 1960s, although the dividend yield of U.S. Steel was higher than IBM's, IBM could have a higher k and P/E ratio because its prospects for growth were so spectacular.

Following in the footsteps of de Moivre (1725) and Halley (1761), Williams also develops a very extensive analysis of a variety of generalizations; for example, for a constant growth rate over n years, followed by dividends that exponentially level off toward a limiting amount that is twice the dividend in the nth year (p. 94, Eq. 27a):

$$P_0 = (D_1/r^n)\{[(g^n - r^n)/(g-r)] + [(2gr - r - 1)/(r-1)(gr-1)]\}$$

4 Conclusion

We have seen that the concept of present value has had a long and illustrious history with contributions by famous mathematicians, scientists, and economists. In the course of this development, we have highlighted two Great Moments: Johan de Witt's publication of *Value of Life Annuities in Proportion to Redeemable Annuities* in 1671 and Irving Fisher's publication in 1930 of *The Theory of Interest*. By 1938, the refinement of the concept of present value was far from over. Other Great Moments were to follow. Work on present value in the 1950s focused on the use of substitute criteria such as the internal return of an investment. Then, in the 1960s and thereafter, present value calculations under uncertain cash flows were generalized to include the effects of economy-wide risk aversion. In the 1970s, serious work began on extending present value to cash flows from derivatives. The 1980s saw extensions to the effect of real options on present value and the effects of uncertainty on the term structure of interest rates. Tracing these developments, which are no doubt better known than those we have discussed here, will have to await another opportunity.

Notes

[1] This recent and innovative book has been of invaluable assistance in my construction of the early history of present value, and I wish to thank Nils Hakansson for bringing it to my attention.

References

de Witt, J. (1671). *Value of Life Annuities in Proportion to Redeemable Annuities*, published in Dutch. Translated from Dutch into English in Hendricks, F. (1852). "Contributions of the History of Insurance and the Theory of Life Contingencies." *Assurance Magazine* 2, 232–249.

de Moivre, A. (1725). *Treatise of Annuities on Lives*. Reprinted as an appendix to de Moivre's *The Doctrine of Chances*, 1967.

Halley, E. (1761). "Of Compound Interest," in Henry Sherwin, *Sherwin's Mathematical Tables*, published posthumously after Halley's death in 1742. London: W. and J. Mount, T. Page and Son.

Rae, J. (1834). *Statement of Some New Principles on the Subject of Political Economy, Exposing the Fallacies of the System of Free Trade, and Some Other Doctrines Maintained in "The Wealth of Nations."* Boston: Hilliard Gray & Co., 1834.

Fisher, I. (1906). *The Nature of Income and Capital* (Macmillan 1906, reprinted by New York: Augustus M. Kelley, 1965).

Fisher, I. (1907). *The Rate of Interest: Its Nature, Determination and Relation to Economic Phenomena*. New York: Macmillan, 1907.

Fisher, I. (1930). *The Theory of Interest: As Determined by Impatience to Spend Income and Opportunity to Invest It*. New York: Macmillan. Reprinted, New York: Augustus M. Kelley, 1955.

Williams, J. B. (1938). *The Theory of Investment Value*. Cambridge, MA: Harvard University Press, 1938. Reprinted, Burlington, VT: Fraser Publishing, 1997.

Dean, J. (1951). *Capital Budgeting*. New York: Columbia University Press.

Gordon, M. J. and Shapiro, E. (1956). "Capital Equipment Analysis: The Required Rate of Profit." *Management Science* 3, 102–110.

Faulhaber, G. R. and Baumol, W. J. (1988). "Economists as Innovators: Practical Products of Theoretical Research." *Journal of Economic Literature 26*, 577–600.

Markowitz, H. (1991). "Foundations of Portfolio Theory." *Les Prix Nobel 1990*. Nobel Foundation, p. 292.

Poitras, G. (2000). *The Early History of Financial Economics: 1478–1776: From Commercial Arithmetic to Life Annuities and Joint Stocks*. Cheltenham, UK: Edward Elgar, 2000.