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LIFECYCLE CONSUMPTION-INVESTMENT POLICIES AND PENSION PLANS: A DYNAMIC ANALYSIS*

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This paper explores the optimal design of personal pensions based on the economic theory of the life cycle. It assumes that individuals derive utility from consumption of goods and leisure and that at some date they retire and stop earning income from labor. The existence of this retirement phase of the life cycle has a profound impact on optimal consumption and portfolio policy. We describe the properties of the optimal pension contract and derive the dynamic trading strategy that hedges the contract. In view of the popularity of age-based strategies—like target date funds—as default options in 401k and other defined contribution retirement plans, some of our results are particularly noteworthy. All target date funds start with a high proportion in equities at a young age and reduce it as a person ages. We identify conditions where the fraction of wealth optimally invested in equities increases or decreases over time as an individual ages. We also analyze the dynamics of pension plans, wealth and optimal policies. Distributional properties of endogenous variables are examined and the robustness of patterns to variations in parameters such as risk aversion and mortality risk is examined.

1 Introduction

How should individuals save and invest over the course of life? How should they invest for retirement? What is the optimal design of a pension plan? These questions have been hotly debated for decades. Their resolution is central to the welfare of individuals in society. This paper examines the behavior of optimal household decisions over the life cycle. It seeks to shed further light on complex issues confronting

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households, financial service producers, and policy makers.

For the purpose of simplicity, the paper considers stylized models of the life cycle in a perfect financial market in which all risks other than mortality risk can be hedged. Yet, these models are already rich enough to deliver interesting insights about the structure and behavior of optimal decisions. Several useful and somewhat surprising properties are identified. It is shown, for instance, that the *fraction of total wealth* optimally invested in equities, can increase over time as an individual ages. Likewise, the *fraction of financial wealth* (the portfolio value) optimally invested in equities can increase over time. But it can also decrease, even if wages and stock returns are perfectly correlated. Explicit conditions for these behaviors are provided when financial wealth and the wage are kept fixed. Numerical experiments show that some of these behaviors can also occur when stochastic fluctuations in endogenous financial wealth and wage are taken into account. These results shed additional light on standard wisdom and common expert advice recommending a reduction in the share of equities in a portfolio as retirement approaches.[1] Perspective is also provided on the incentives to grow financial wealth to secure for retirement. It is clear that the prospect of future retirement induces individuals to save and build up resources to sustain retirement consumption. What is perhaps less evident is the magnitude of this effect and the structure and behavior of portfolio policies needed to achieve the desired buildup. Even less obvious is the distribution of possible outcomes over the life cycle. Finally, the paper also sheds light on pension plan design. It derives and values the optimal pension plan, then examines the behavior of the pension plan value over the household's life cycle.

The paper is part of a broad literature revolving around life cycle finance. Broadly speaking, life cycle finance studies the behavior of households in regards to their consumption, labor/leisure and investment decisions over the course of life. It seeks to address a variety of questions revolving around these issues, such as the questions raised above. Life cycle finance finds its origin in seminal writings by Merton (1969, 1971) and Samuelson (1969). Countless papers have been written on the issues raised therein and their ramifications. Critical extensions of the base models were carried out in Bodie *et al.* (1992) with the addition of flexible labor and a labor/leisure decision. The subsequent literature is vast and has explored various aspects of the life cycle model (see Bodie *et al.* (2009) for a review and an extensive list of references). Recent contributions discussing aspects of models with flexible labor income include Bodie *et al.* (2004), Cvitanic *et al.* (2007), Gomes *et al.* (2008) and Chai *et al.* (2010). The present paper refines the analysis in certain dimensions. In particular, it sheds light on dynamic properties of decision variables such as consumption and labor and induced quantities such as financial wealth and pension plans. Properties are derived based on the analysis of specific sample paths or the analysis of distributions and their evolutions. Confidence bands provide further information about the variety of possible scenarios that may occur. The insights provided deepen our understanding of optimal structures and designs and can help to shape the production of optimal financial services.

Section 2 describes the ingredients of the model. Sections 3 and 4 examine the properties of consumption, labor/leisure, investments and wealth over the life cycle. Section 5 analyzes the optimal pension plan. Densities and statistics of endogenous variables are discussed in Section 6. Mortality risk is examined in Section 7. A discussion of other important considerations for life cycle policies appears in Section 8. Conclusions follow.

Formulas underlying the analysis are presented in the appendix.

2 The life cycle model

This section describes the structure of the financial market, the characteristics of the household, the evolution and constraints on financial wealth, and the household decision problem.

2.1 *The financial market*

The objective of this paper is to highlight fundamental aspects of optimal life cycle decisions. For this reason, a very stylized and standard financial market is considered. Households/individuals can invest in equities (the market portfolio of risky assets) and in a riskless asset (a money market account). The return on equities, dR_t, is given by

$$dR_t = \mu dt + \sigma dW_t, \tag{1}$$

where μ represents the expected return and σ the return volatility. Both μ and σ are assumed to be constants. The quantity dW_t is the increment of a Brownian motion process, which captures shocks to the equity market. The return on funds invested in the money market account is r, a constant. Investments in equities and the money market account are assumed to be unrestricted (frictionless capital markets).

2.2 *Households*

Households have a finite life $[0, T]$, which can be divided into an accumulation (or active) phase $[0, T_r)$ and a retirement phase $[T_r, T]$.

During the accumulation phase, households work and consume. In return for work, they earn a wage w which evolves stochastically according to

$$dw_t = w_t(\mu^w dt + \sigma^w dW_t). \tag{2}$$

The parameter μ^w is the expected growth rate of wages, σ^w is the volatility of the growth rate of wages. Both are assumed to be constant. The correlation between the wage growth rate and equity returns is perfect. This assumption is strong. It best captures the situation of a household whose income is tied to the performance of the stock market. Money managers, hedge fund managers, and executives of financial corporations are examples of professions that fall in this category. Our model is especially relevant for this class of individuals.

Households have a work capacity equal to $\bar{h}$, that represents the maximum number of work hours that are feasible. The maximum labor income (obtained by working full capacity) is therefore $w_t\bar{h}$. This amount is the human capital at time t. Human capital can be spent on consumption and leisure. Leisure, in particular, reduces the number of available hours spent on work and therefore has an opportunity cost equal to wages. Leisure effectively represents a commodity that can be bought at a price w_t.

During the retirement phase, households consume. Consumption is financed by the capital accumulated during the first phase of the life cycle.

Household preferences are represented by the standard von Neumann–Morgenstern (expected utility) criterion

$$U(c, l) \equiv E\left[\int_0^{T_r} a_s u^a(c_s, l_s) ds + \phi \int_{T_r}^{T} a_s u^r(c_s, \bar{h}) ds\right], \tag{3}$$

where $u^a(c_s, l_s)$ (resp. $u^r(c_s, \bar{h})$) is the instantaneous utility derived from consumption and leisure during the active (resp. retirement) phase. The term $a_s \equiv \exp(-\beta s)$ is a subjective discount factor with constant discount rate β. The constant ϕ is a weight that captures the relative importance of the retirement period in the household's welfare. Utility functions have the Cobb–Douglas structure

$$u^a(c,l) = \frac{(c^\eta l^{1-\eta})^{1-R}}{\eta(1-R)}, \quad u^r(c,\overline{h}) = \frac{c^{1-R}}{1-R}, \tag{4}$$

where $R > 0$ is the relative risk aversion coefficient and $\eta \in (0,1)$ is a weight parameter which quantifies the relative importance of consumption and leisure in the basket of goods consumed. When η increases, instantaneous utility becomes more sensitive to consumption.

2.3 *Investments and financial wealth*

Households invest in equities and in the money market account. Let X be the value of the portfolio account (financial wealth). If π is the amount invested in equities, c the amount consumed and l the amount of leisure, financial wealth evolves according to

$$dX_t = (rX_t + w_t\overline{h}1_{\{t<T_r\}} - c_t - w_t l_t 1_{\{t<T_r\}})dt + \pi_t\sigma(\theta dt + dW_t), \tag{5}$$

where $1_{\{t<T_r\}}$ is the indicator of the active phase $\{t < T_r\}$. The initial value $X_0 \equiv x$ is the amount of initial capital. Human capital generates a potential inflow at time $t < T_r$, that is credited to the portfolio account. Consumption and leisure are withdrawals that reduce the financial wealth. The difference $w_t(\overline{h} - l_t)1_{\{t<T_r\}}$ is the salary collected for the effective amount of time worked.

Human capital is the present value of the maximal labor income that can be generated by the individual. Total wealth is the sum of financial wealth and human capital. If H_t denotes human capital and N_t total wealth, then $N_t = X_t + H_t$. Given perfect capital markets, in which all risks can be synthesized, it is natural to require that total wealth be nonnegative at all times

$$N_t \geq 0, \quad \text{for } t \in [0,T]. \tag{6}$$

This constraint allows the household to borrow against future labor income, but it also ensures that there are always enough resources to cover potential liabilities.

2.4 *The household decision problem*

Households maximize welfare (3) with respect to consumption, leisure and investment, subject to the evolution of financial wealth (5) and the nonnegativity constraint (6).

It can be shown that this problem is equivalent to the maximization of (3) with respect to consumption and leisure, subject to the static budget constraint

$$x + H_0 = E\left[\int_0^{T_r} \xi_t(c_t + l_t w_t)dt\right] + E\left[\int_{T_r}^{T} \xi_t c_t dt\right], \tag{7}$$

where

$$H_0 = \overline{h}E\left[\int_0^{T_r} \xi_t w_t dt\right] \tag{8}$$

and $\xi_t = \exp(-rt - \frac{1}{2}\theta^2 t - \theta W_t)$ with $\theta = \sigma^{-1}(\mu - r)$. In this formulation H_0 does represent human capital at date 0, i.e., the present value of the maximum salary that can be generated given the work capacity of the household and labor market conditions. The quantity $N_0 \equiv x + H_0$ is the total wealth of the household at the outset. The budget constraint (7) states that the present value of future expenditures on consumption and leisure must equal initial resources (total wealth). In (7) and (8) consumption and labor income are discounted using the stochastic discount factor ξ. This quantity is a risk-adjusted discount factor that converts risky future cash flows into values at the initial date. The risk adjustment is embedded in the market price of risk θ, equal to the risk premium per unit risk. This market price of risk is also called the Sharpe ratio of the stock market.

3 Optimal policies: general remarks

This section gives basic properties of optimal consumption, labor/leisure, and investment decisions. The focus is on the structure of these demand functions.

3.1 The single-phase model

To build intuition, consider first the setting without retirement phase (single-phase model). In this context, optimal total wealth

$$N_t^* = X_t^* + H_t = E_t\left[\int_t^T \xi_{t,v}(c_v^* + l_v^* w_v)dv\right], \qquad (9)$$

where $\xi_{t,v} = \xi_v/\xi_t$ is the stochastic discount factor for the valuation at t of cash flows received at a subsequent date v, finances expenditures on future consumption and leisure. Optimal expenditures are

$$c_t^* + l_t^* w_t = m(t,T)N_t^*, \qquad (10)$$

where $m(t,T)$ is a function of time. Expenditures are thus proportional to total wealth. The function $m(t,T)$ represents the marginal propensity to spend out of total wealth. Optimal consumption and leisure are given by

$$c_t^* = \eta m(t,T)N_t^*, \quad l_t^* = (1-\eta)m(t,T)\frac{N_t^*}{w_t}. \qquad (11)$$

Inspection of these formulas shows that consumption and leisure are fractions of expenditures where the fractions are the respective weights of consumption and leisure in the utility derived from the basket of goods.

The optimal portfolio has the decomposition

$$\pi_t^* = \pi_t^m + \pi_t^w - \pi_t^H \qquad (12)$$

with

$$\frac{\pi_t^m}{N_t^*} = \frac{1}{R}\sigma^{-1}\theta, \qquad \frac{\pi_t^w}{N_t^*} = (1-\eta)\rho\sigma^{-2}\sigma\sigma^w, \qquad (13)$$

$$\frac{\pi_t^H}{N_t^*} \equiv \frac{H_t}{N_t^*}\sigma^{-2}\sigma\sigma^w, \qquad (14)$$

where $\rho = 1 - 1/R$. The optimal portfolio demand has three parts.

(1) The first one, π_t^m, is a standard mean–variance component. This term is proportional to total wealth, with proportionality factor determined by relative risk aversion of the household and by the Sharpe ratio per unit risk of the stock market (which is also the risk premium divided by the variance of the equity return). The proportionality factor is inversely related to relative risk aversion. When risk aversion goes to infinity (extremely risk averse household) the mean–variance demand vanishes.

(2) The second component, π_t^w, is a wage hedge. This term is motivated by the desire to protect against stochastic fluctuations in the wage rate, which affects the demand for leisure. The wage hedge has the nature of a static hedge: it insures against instantaneous fluctuation in the cost of leisure. It is proportional to total wealth and increasing in relative risk aversion. As risk aversion goes to infinity, the wage hedge converges to $(1-\eta)\sigma^{-2}\sigma\sigma^w N_t^*$. When risk aversion is one, which corresponds to logarithmic utility $\eta \log c + (1-\eta)\log l$, the wage hedge vanishes. It also vanishes when the consumption weight in the utility function goes to one ($\eta \to 1$). In this case, the household ceases to care about leisure and the need to hedge against the cost of leisure disappears.

(3) The last component, π_t^H, is a human capital hedge. The human capital hedge is also

motivated by instantaneous fluctuations in the wage rate. Variations in wages affect the full capacity salary, hence human capital, prompting the household to hedge. The human capital hedge is again static in nature. It is proportional to human capital and unrelated to risk aversion. It does not vanish when risk aversion equals one. Even logarithmic utility is exposed to human capital fluctuations and therefore needs to hedge. The human capital hedge is time dependent.

Some of the dynamic properties of optimal policies can already be inferred from the formulas above. Optimal wealth in the single-phase model converges to zero as the terminal date approaches (see Corollary 1 in the Appendix). The marginal propensity to spend out of wealth goes to infinity, so as to ensure that all resources are exhausted by the terminal date. Consumption and leisure converge to positive and finite limits. The mean–variance component and the wage hedge vanish, because wealth vanishes. The human capital hedge also vanishes, because human capital is exhausted in the limit. The demand for equities becomes null in the limit.

More specific results concerning the age-dependence of the portfolio can also be derived (see Corollary 2 in the Appendix for details). Suppose that the wage and financial wealth are held fixed. If t_a denotes the age of the household, then

$$\frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} = -\frac{\partial(H_t/N_t^*)}{\partial t}\sigma^{-2}\sigma\sigma^w, \quad (15)$$

where

$$\frac{\partial(H_t/N_t^*)}{\partial t} = -\frac{H_t}{N_t^*}\left(1-\frac{H_t}{N_t^*}\right) \times \frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)}. \quad (16)$$

When the wage growth rate is positively correlated with equity returns,

$$\sigma\sigma^w > 0 \quad (17)$$

the fraction of *total wealth* in equities increases (decreases) with age as long as financial wealth is positive (negative). In this case the human capital hedge is positively (negatively) related to age. This effect is prompted by the reduction (increase) in human capital with age. The behavior of the fraction of *financial wealth* in equities is more intricate. In this case,

$$\frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} = \frac{\partial(H_t/X_t^*)}{\partial t}\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta) \times \rho\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right), \quad (18)$$

where

$$\frac{\partial(H_t/X_t^*)}{\partial t} = -\frac{H_t}{X_t^*}\frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)}. \quad (19)$$

The condition

$$\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\rho\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w > 0 \quad (20)$$

determines the behavior. When (20) is satisfied, the fraction of financial wealth in equities decreases (increases) with age as long as financial wealth is positive (negative). The expression on the left-hand side of (20) is the sum of the aging effects on the mean–variance term ($(1/R)\sigma^{-1}\theta$), on the wage hedge ($(1-\eta)\rho\sigma^{-1}\sigma^w$) and on the human capital hedge ($-\sigma^{-2}\sigma\sigma^w$).

The results concerning the impact of age on the optimal portfolio are quite interesting in light of debates in academic circles and among practitioners. The results obtained show that an aging household whose wage growth rate is sufficiently volatile relative to equity returns, to such an extent that the left-hand side of (20) is negative, will optimally increase the fraction of financial wealth in equities. The possibility of a positive association between age and the fraction invested in equities was noted by Jagannathan and Kocherlakota (1996). The converse of condition (20) identifies the precise circumstances under which this behavior is optimal in the single-phase model described above. Condition (20) also shows that the reverse behavior can be optimal, even though the wage is perfectly correlated with equity returns.

3.2 *The two-phase model*

In the presence of a retirement period (two-phase model), optimal behavior depends on the phase of the life cycle.

Consider first the active phase. Total wealth, during that period of the life cycle, serves to finance expenditures on consumption and leisure during the remaining part of the active phase, as well as expenditures on consumption during retirement. Optimal wealth can therefore be split into two parts,

$$N_t^* = N_{at}^* + N_{rt}^*, \tag{21}$$

where N_a^* finances accumulation period expenditures and N_r^* finances retirement consumption

$$\begin{aligned} N_{at}^* &= E_t\left[\int_t^{T_r} \xi_{t,v}(c_v^* + l_v^* w_v)dv\right] \\ N_{rt}^* &= E_t\left[\int_{T_r}^{T} \xi_{t,v} c_v^* dv\right]. \end{aligned} \tag{22}$$

In the remainder of this paper, N_{at}^* is called *accumulation wealth* and N_{rt}^* is *retirement wealth* (as will become clear later, N_{rt}^* also represents the value of the optimal pension plan).[2] The fraction of resources devoted to the financing of expenditures during the active phase is

$$\Psi(w_t, t) = \frac{N_{at}^*}{N_{at}^* + N_{rt}^*} \tag{23}$$

The expression can be written entirely in terms of the wage w_t (see (A.3) in the Appendix). Optimal expenditures now take the form:

$$c_t^* + l_t^* w_t = \Psi(w_t, t) m(t, T_r) N_t^*, \tag{24}$$

where $m(t, T_r)$ is the marginal propensity to spend out of accumulation wealth. Expenditures are thus proportional to accumulation wealth N_{at}^*. The marginal propensity to spend out of accumulation wealth is a (nonstochastic) function of time. In contrast, expenditures are not a deterministic proportion of total wealth N_t^*. Optimal consumption and leisure are given by

$$\begin{aligned} c_t^* &= \eta \Psi(w_t, t) m(t, T_r) N_t^*, \\ l_t^* &= (1-\eta)\Psi(w_t, t) m(t, T_r) \frac{N_t^*}{w_t}. \end{aligned} \tag{25}$$

As before consumption and leisure are fractions of expenditures where the fractions are the respective utility weights with respect to the two goods. Consumption and leisure are also deterministic proportions of accumulation wealth, but not of total wealth.

The optimal portfolio has the decomposition (12) with π^m, π^H as in (13) and (14) but

$$\frac{\pi_t^w}{N_t^*} = (1-\eta)\rho \Psi(w_t, t)\sigma^{-2}\sigma\sigma^w. \tag{26}$$

The wage hedge is now proportional to accumulation wealth, but not total wealth ($\pi_t^w / N_{at}^* = (1-\eta)\rho\sigma^{-2}\sigma\sigma^w$). This is intuitive, as the exposure to wage fluctuations is limited to the active phase of the life cycle.

The same formulas apply during the retirement phase, with the proviso that total wealth is then entirely composed of financial wealth (human capital is null during retirement) and that expenditures consist solely of consumption.

Optimal consumption is again proportional to wealth. The optimal portfolio reduces to the mean–variance part, which is also proportional to wealth.

The presence of a retirement date has a sharp impact on the dynamic profile of optimal policies. Total wealth now converges to retirement wealth at the retirement date. Human capital is exhausted at that point, but the household accumulates resources to finance retirement consumption. The marginal propensity to spend out of wealth remains finite in the limit. Consumption and leisure have finite limits. Labor of course goes to zero at retirement. The need to hedge also vanishes in the limit: the wage hedge because accumulation wealth goes to zero, the human capital hedge because human capital is exhausted. The demand for equities thus converges to the mean–variance demand component, which is positive under typical market conditions.

The presence of a retirement phase also has a deep effect on the relation between the optimal portfolio and age. In this instance,

$$\frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} = \frac{\partial(\pi_t^w/N_t^*)}{\partial t} - \frac{\partial(\pi_t^H/N_t^*)}{\partial t} \equiv \left((1-\eta)\rho\frac{\partial\Psi(w_t,t)}{\partial t} - \frac{\partial(H_t/N_t^*)}{\partial t}\right) \times \sigma^{-2}\sigma\sigma^w, \tag{27}$$

where

$$\frac{\partial\Psi(w_t,t)}{\partial t} = -\Psi(w_t,t)(1-\Psi(w_t,t)) \times \left(\frac{\exp(-g(\eta,R)(T_r-t))}{G(t,T_r)} + g(1,R)\right) \tag{28}$$

$\partial(H_t/N_t^*)/\partial t$ is given in (16), and $G(t,T_r)$, $g(\eta,R)$ are defined in (A.4), (A.7) in the appendix ($g(1,R) \equiv (1/R)\beta + \rho(r+\frac{1}{2}(1/R)\theta^2)$). When the wage growth rate is positively correlated with equity, as in (17), the fraction of *total wealth* in equities increases (decreases) with age as long as $(1-\eta)\rho\partial\Psi(w_t,t)/\partial t - \partial(H_t/N_t^*)/\partial t$ is positive (negative). Note that the second term is due to the behavior of the human capital hedge recorded in the single-phase model. The presence of a retirement period adds a new effect, captured by the first term, which reflects the behavior of the wage hedge $\partial(\pi_t^w/N_t^*)/\partial t$. It can be positive or negative depending on the sign and magnitude of $\rho = 1-1/R$. When relative risk aversion exceeds 1, this component is unambiguously negative and reduces the effect due to the behavior of the human capital hedge. When relative risk aversion falls below 1, the component is positive as long as $g(1,R)$ does not become too negative.

Likewise, aging has two effects on the fraction of financial wealth invested in equities. The impact due to the change in human capital, which is the sole effect in the single-phase model, is now augmented by an additional term capturing the change in $\Psi(w_t,t)$ in the wage hedge. Specifically,

$$\frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} = \frac{\partial(H_t/X_t^*)}{\partial t}\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta) \times \rho\Psi(w_t,t)\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right) + \frac{N_t^*}{X_t^*}(1-\eta)\rho\frac{\partial\Psi(w_t,t)}{\partial t}\sigma^{-1}\sigma^w,$$

where $\partial(H_t/X_t^*)/\partial t$ is given by (19) and $\partial\Psi(w_t,t)/\partial t$ in (28). When the condition

$$\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\rho\Psi(w_t,t)\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w > 0$$

(its converse) holds, the first component is negative (positive), as long as financial wealth is positive. The sign of the second component depends on relative risk aversion. If risk aversion exceeds 1 this term is negative. Otherwise it can be positive. Various types of behaviors can therefore be optimal depending on the parameters of the model and the wage level. This complements the results documented in the previous section for the single-period model.

The comparative static analysis above shows that deterministic age-based portfolio rules like those implemented in target date funds (TDF) are suboptimal in many circumstances. This is further documented in Section 6, in a dynamic comparison of the distributions associated with a TDF and with the optimal retirement plan. It should also be noted that the aging effects described above are entirely due to the passage of time, keeping variables such as financial wealth and wage frozen. But these quantities also change, due to shocks affecting the economy. The numerical analysis carried out in the next section takes account of these effects and sheds additional light on the dynamic behaviors of the optimal portfolio and other endogenous variables.

4 Dynamic policy behavior: a numerical study

The static behavior in the two settings (with and without retirement phase) is described in Bodie *et al.* (2009). This section carries out a numerical analysis to gauge the importance and uncover the dynamic behavior of the various components that determine consumption and investment over the life cycle. Section 4.1 describes the behavior in a single-phase model. Section 4.2 extends the analysis to a setting with a retirement phase. Illustrations are provided for the parameter values in Table 1.

The retirement date is $T_r = 45$. This corresponds to age 65 if the active phase starts at age 20. The investment horizon $T = 60$ indicates that the individual dies after 15 years of retirement (age 80 if work starts at age 20). In the single-phase model $T = T_r = 45$. These and other parameter values are standard. The utility weight ϕ is selected so that $\phi^{1/R}$ is of the same order of magnitude as $w_0^{(1-\eta)\rho}(1/\eta - 1)^{-(1-\eta)\rho}$ (so that $\eta\Psi(w_0)/G(T_r, T_r) \approx \phi^{1/R}G_0(T_r, T)$). This serves to control the disruption in consumption pattern as the household enters the retirement phase (see (A.1) and (A.3)).

4.1 The single-phase model

Figure 1 shows the stock market index and the wage for a particular trajectory of the underlying Brownian motion W. Realizations are sampled every 6 months for 45 years, giving 90 data points. In this scenario the stock market experiences a sharp increase during the last 5 years of the sample. Wages follow the same pattern.

Figure 2 shows the corresponding behavior of optimal policies. Consumption varies between 70, 000 and 100, 000 most of the period, except during the last 5 years when it rises sharply,

Table 1 Parameter values.

Subjective discount rate	$\beta = 0$	Market price of risk	$\theta = 0.3$
Relative risk aversion	$R = 4$	Interest rate	$r = 0.02$
Relative weight	$\eta = 2/3$	Market return volatility	$\sigma = 0.2$
Utility weight	$\phi = 20^4$	Expected wage growth	$\mu^w = 0.01$
Work capacity	$\bar{h} = 1$	Volatility wage growth	$\sigma^w = 0.03$
Investment horizon	$T = 60$		
Retirement date	$T_r = 45$		

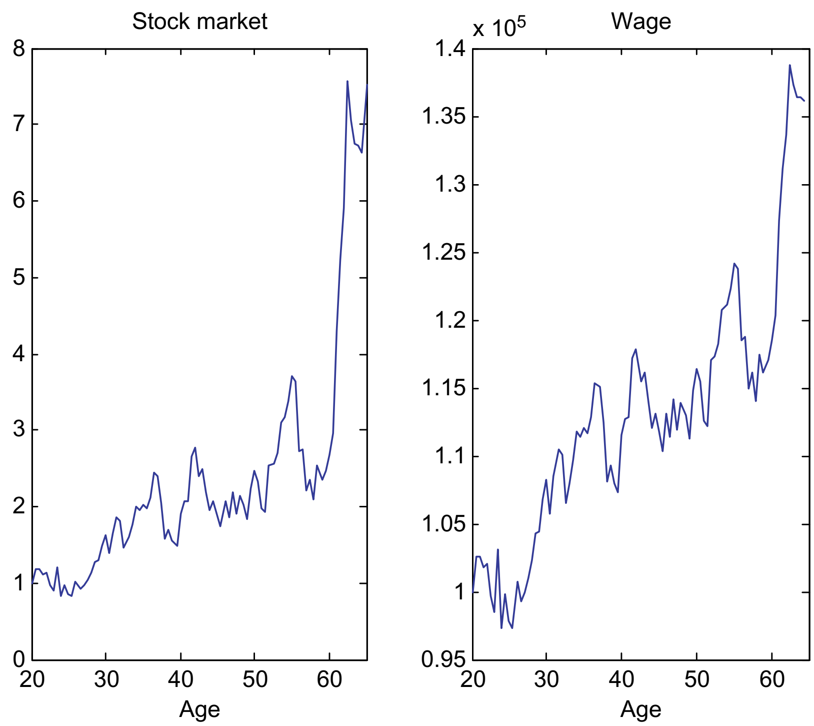

Figure 1 This figure shows a trajectory of the stock market (left panel) and the wage (right panel).

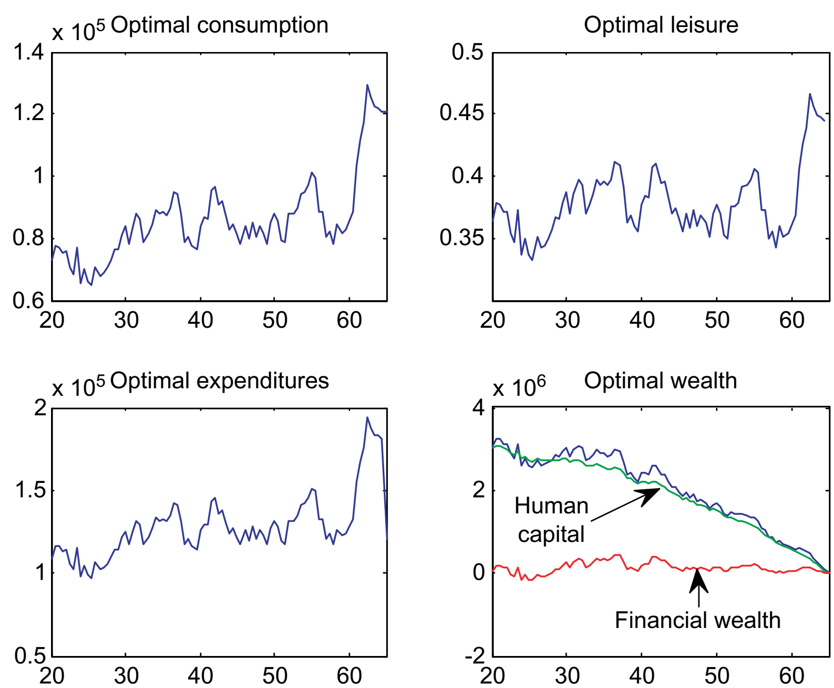

Figure 2 This figure shows optimal consumption and leisure (top panels) and optimal expenditures and wealth (bottom panels). The panel for optimal wealth shows the human capital (green path) and financial wealth (red path) components.

reflecting the behavior of the stock market and wages. Leisure follows a similar pattern. The behavior of the wealth process and its components is displayed in the lower right panel. Human capital is large at the outset, outweighing financial wealth by a factor of about 60. It then experiences moderate fluctuations and decreases on average, to eventually converge to zero at the terminal date. Financial wealth, in contrast, experiences more semester-to-semester variations, but remains low throughout, also converging to zero as the terminal date approaches.

Figure 3 shows the optimal portfolio behavior. The overall fraction invested in equities increases on average during the period. It eventually converges to the mean–variance demand of 37.5%. The fundamental reason for this profile is the behavior of the human capital hedge. This term is initially negative, then increases on average during the period to eventually vanish at the terminal date. Human capital hedging considerations are behind the increasing share of equities in the portfolio during the life cycle. The amount invested in the stock market is significant and reflects the size of human capital. This amount is initially financed by borrowing 764, 708 at the riskfree rate, an equally sizeable amount.[3]

4.2 The two-phase model

Figure 4 is the continuation of Figure 1. It extends the path of the stock market over the household's life $[0, T]$ (the path over the active period remains the same). In this particular scenario the stock market experiences a downturn during the first part of the retirement period, followed by a boom and another downturn just prior to the terminal date.

Figure 5 shows the response of optimal consumption, leisure, and wealth to these events. As indicated before, consumption rises sharply during the boom just prior to retirement. In the

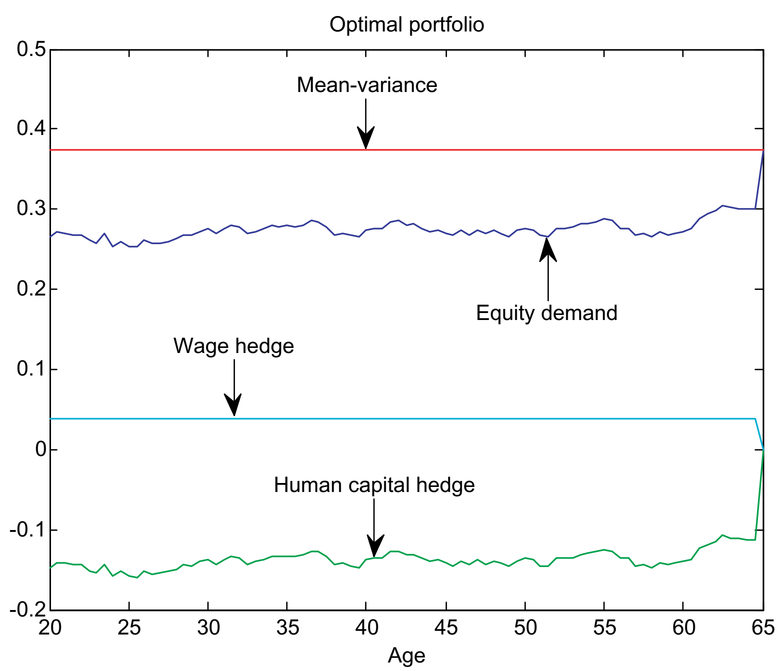

Figure 3 This figure shows the optimal portfolio and its components (mean–variance term, wage hedge and human capital hedge) as fractions of total wealth.

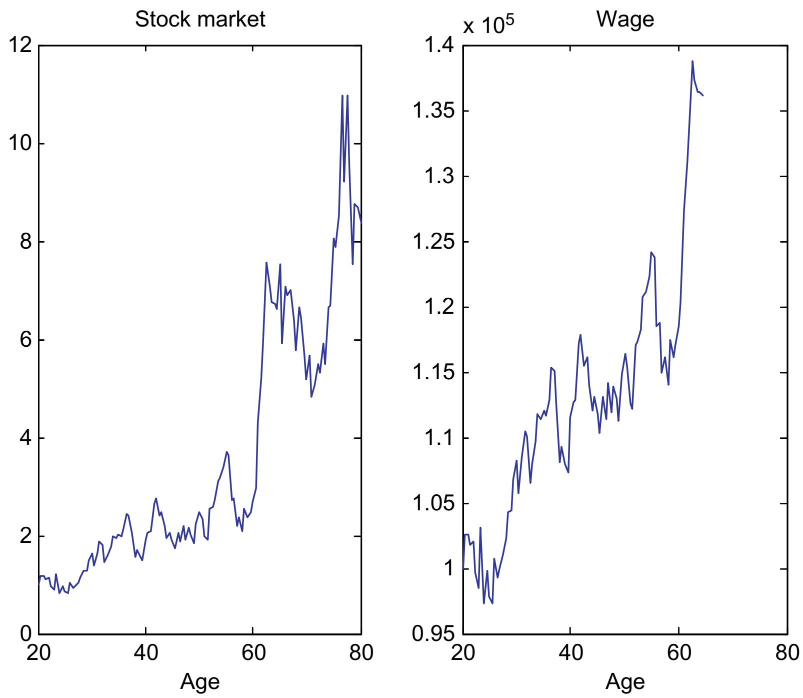

Figure 4 This figure shows a trajectory of the stock market over $[0, T]$ (left panel) and of the wage over $[0, T_r]$ (right panel).

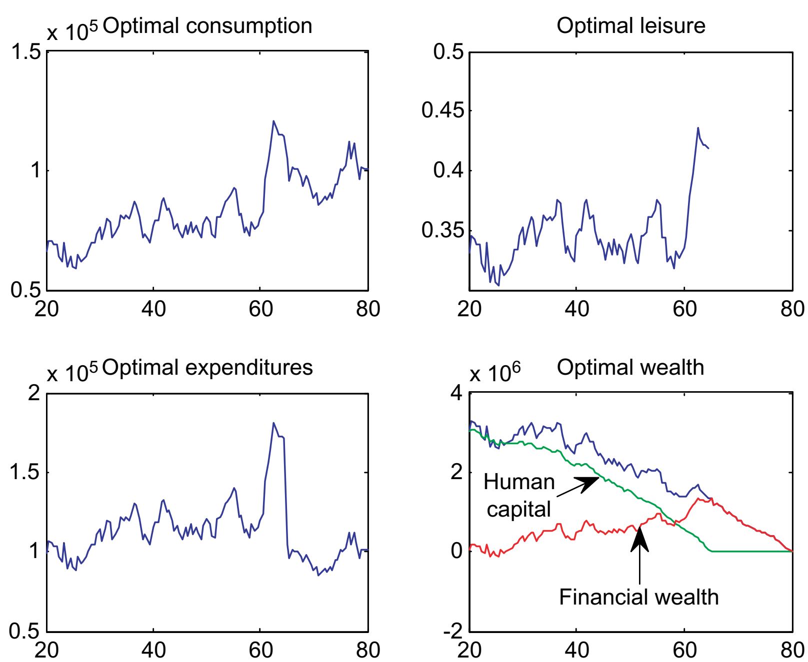

Figure 5 This figure shows optimal consumption and leisure (top panels) and optimal expenditures and wealth (bottom panels). The panel for optimal wealth shows the human capital (green path) and financial wealth (red path) components.

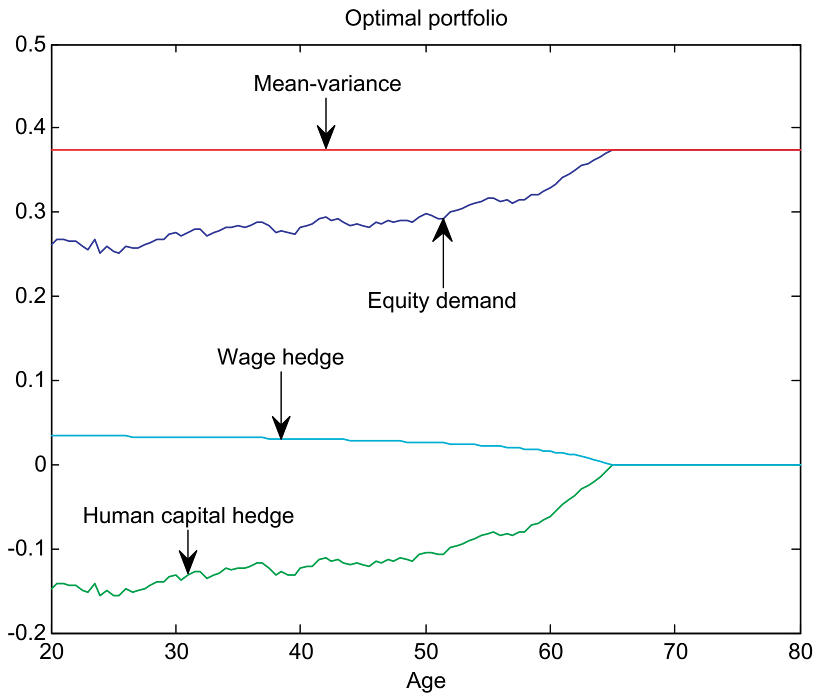

Figure 6 This figure shows the optimal portfolio and its components (mean–variance term, wage hedge and human capital hedge) as fractions of total wealth.

recession that follows it declines, but somewhat less sharply. This reflects a combination of two effects. The first one is the decline in the stock market. The second is the structural change in preferences and decision making as the household enters the retirement period. The reaction to subsequent booms and busts during retirement is more modest and simply reflects variations in the stock market valuation. Optimal wealth and its components (bottom right panel of the figure) show striking differences in pattern. With the prospect of a retirement period the household is seeking to accumulate wealth. Financial wealth is seen to increase on average throughout the period. This accumulation is achieved by reducing consumption and working more at the outset. The difference is significant. At retirement financial wealth amounts to 1, 325, 509. These resources serve to finance consumption during retirement.

The optimal portfolio behavior, displayed in Figure 6, is also striking. The human capital hedge, taken as fraction of total wealth, converges much more gradually to zero, reached at the retirement date. Similar gradual patterns characterize the wage hedge and the overall investment in equities. Financial wealth accumulation is the source of these patterns.

5 Optimal pension plans

The optimal pension plan is the plan that finances consumption during retirement. The optimal plan is therefore a contingent claim delivering the stream

$$c_t^* = m(t, T)N_t^*$$

during retirement, where $m(t, T) = 1/G_0(t, T)$ and $G_0(t, T)$ is defined in (A.5).[4] The cash flow paid out is an age-dependent fraction of the value of the pension plan. This value is

$$N_{rt}^* = \begin{cases} E_t\left[\int_{T_r}^{T} \xi_{t,v} c_v^* dv\right] & \text{for } t \le T_r \\ E_t\left[\int_{t}^{T} \xi_{t,v} c_v^* dv\right] & \text{for } t > T_r \end{cases},$$

which corresponds to the notion of retirement wealth introduced in (22). The portfolio financing the pension plan is the pure mean–variance portfolio

$$\frac{\pi_{rt}^m}{N_t^*} = \frac{1}{R}\frac{N_{rt}^*}{N_t^*}\sigma^{-1}\theta_t$$

(equivalently, $\pi_{rt}^m = (1/R)N_{rt}^*\sigma^{-1}\theta_t$). This portfolio is proportional to the value N_{rt}^* of the pension plan and is therefore easy to implement in practice. The fraction of the pension plan value invested decreases in risk aversion, converging to zero as risk aversion goes to infinity. Extremely risk averse households seek a pension plan delivering a constant consumption flow during retirement, i.e., a constant annuity. This annuity is financed by investing at the riskless rate. Households with positive tolerance for risk seek a plan delivering a stochastic consumption flow during retirement. This consumption stream is manufactured by paying dividends out of the plan at an age-dependent rate $m(t, T)$.

Figure 7 shows the evolution of the value of the pension plan for the trajectories of the stock market and wage displayed in Figure 1. The pension plan value increases fairly steadily over the course of the active period. It reaches a peak at the retirement date then decreases as it is depleted to finance retirement consumption. A notable feature is its lower sensitivity to short-term fluctuations in the stock market.

Figure 8 provides additional perspective on the relation between the pension plan and other components in the decomposition of optimal wealth. As seen from the discussions above total wealth has two decompositions. The first one involves

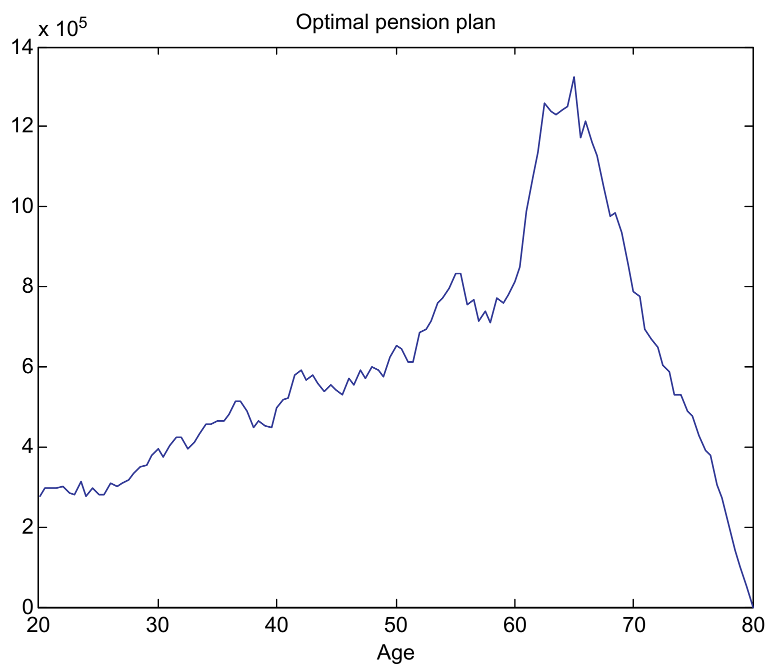

Figure 7 The value of the optimal pension plan over the life cycle.

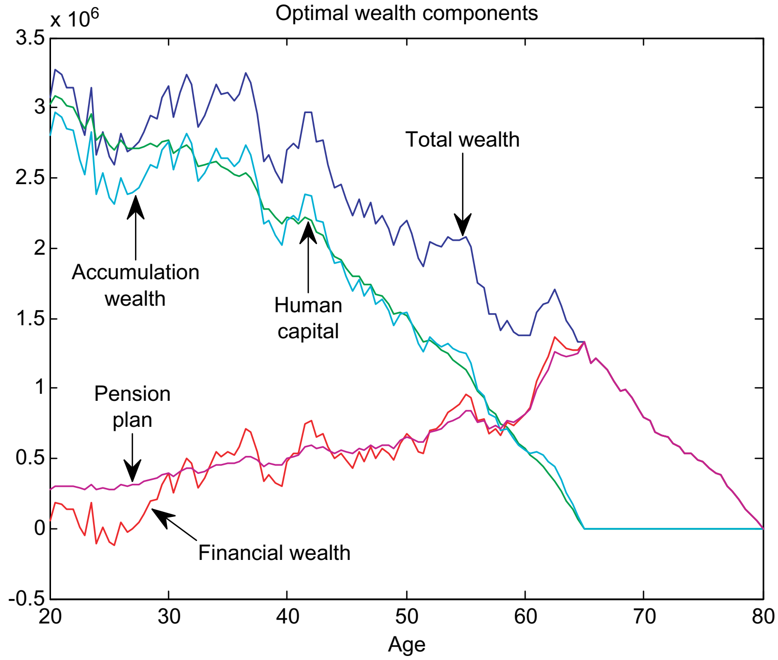

Figure 8 The components of wealth: accumulation wealth, retirement wealth (pension plan), Financial wealth, and Human capital.

financial wealth and human capital, the second accumulation wealth and retirement wealth (the pension plan). The figure shows the behavior of the various components for the trajectories in Figure 1. The pension plan value clearly shadows financial wealth. But it is generally more stable. At the retirement date the two notions coincide. The reduction in variability is compensated by the increased variability of accumulation wealth (its complement) relative to human capital.

6 Distributional analysis

In order to provide more comprehensive information about the behavior of optimal policies and other endogenous variables, two additional sets of results are reported. Confidence bands for all endogenous variables in the problem are provided in Section 6.1. The evolution of the endogenous density of retirement wealth is discussed in Section 6.2.

6.1 *Confidence bands*

Confidence bands at the $\alpha = 95\%$ level are graphed in Figure 9. The figure also shows the means of the variables considered as well as the specific trajectory discussed in prior sections. Several properties emerge from the plots. First, it is interesting to note that the trajectory investigated earlier lies below the average trajectory. It therefore gives a conservative view of the evolution of endogenous variables. For instance, it appears that average consumption, leisure and expenditures increase more during the life cycle than along this specific path.

Second, the behavior of averages reveals interesting patterns. Average optimal wealth increases mildly during the initial stages of the life cycle, then decreases. Average financial wealth increases up to retirement then decreases as the household draws down resources to sustain retirement consumption. Average human capital decreases up to the retirement date, time at which

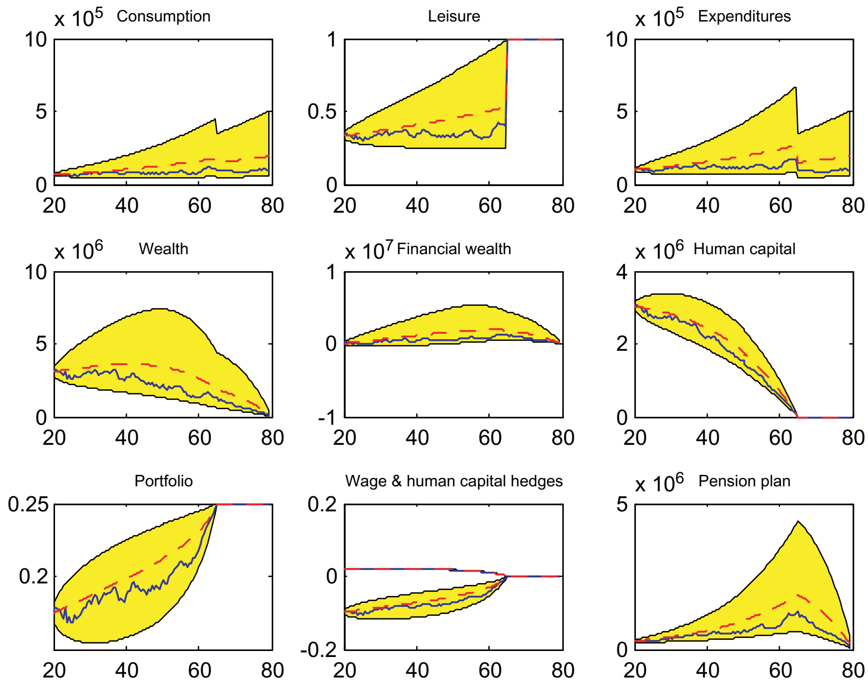

Figure 9 Confidence bands and averages during the life cycle. This figure shows 95% confidence bands (yellow regions), averages (red dotted curves), and the realized trajectories analyzed in earlier sections (solid trajectories) for endogenous variables in the model.

it is exhausted. The optimal investment in equities increases on average up to the retirement date. This reflects the strong increase in the average human capital hedge during the active phase. Finally, the average value of the pension plan is seen to increase sharply up to the retirement date and to decrease thereafter.

Finally, it is also interesting to note the dynamic behavior of the 95% confidence bands. In several cases the upper and lower boundaries reflect the behavior of the averages. What is remarkable is the behavior of the width of the bands. For instance, the widths of the bands for total wealth, financial wealth, and human capital experience an increase during the first part of the life cycle, before decreasing. The same behavior characterizes the widths of the bands for the portfolio share and its components. Most widths peak before retirement time. An exception is the width of the band for the pension plan value that peaks at the retirement date. Confidence interval widths can take large values. For instance, in the case of the pension plan, the lower bound (resp. upper bound) at retirement is about $613,000$ (resp. $4,410,000$) leading to a width of about $3,797,000$. This shows the wide range of pension plan values that can materialize in 95% of the possible scenarios. The reason for this property is that the standard market model and the wage process allow for a large set of possible trajectories and the pension plan value is sensitive to uncertainty. In contrast, the confidence band for the wage hedge is very small, reflecting its limited variability.

Figure 10 provides insights about the sensitivity of the confidence bands with respect to relative risk aversion. As Corollary 4 in the Appendix shows, the behavior of the upper and lower bounds and the widths of the confidence intervals is the result of multiple effects. No global systematic pattern exists as risk aversion increases. Figure 10

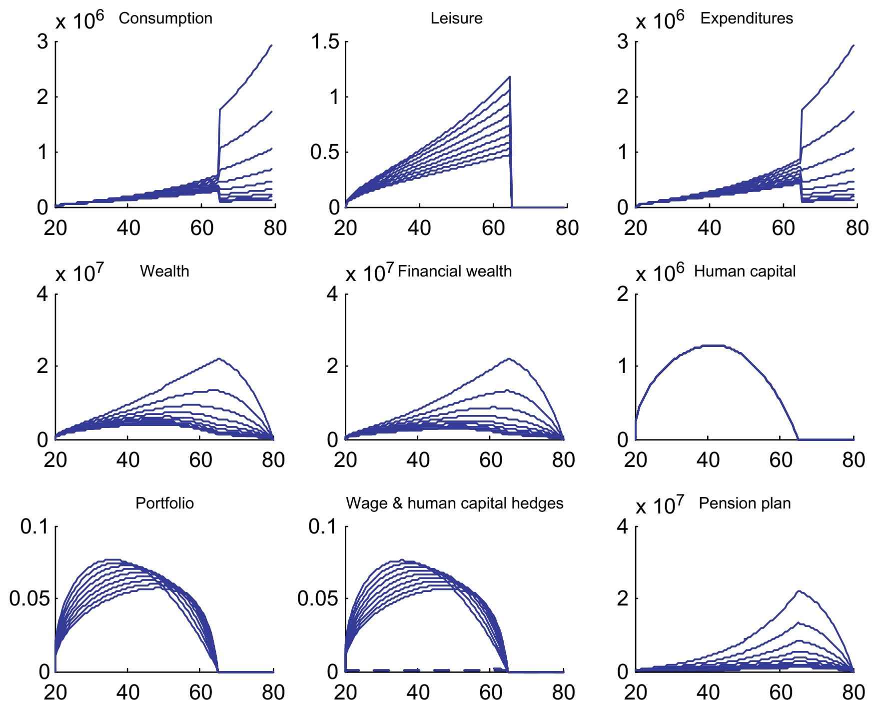

Figure 10 Confidence band widths during the life cycle. This figure shows the widths of 95% confidence bands as risk aversion increases from $R = 3$ to $R = 5$ in increments of 0.25.

shows a configuration of parameters and a range of risk aversion values for which the widths of the confidence bands for consumption, leisure, expenditures, total wealth, financial wealth, and the pension plan decrease as relative risk aversion increases. Those for the portfolio and its human capital hedge component exhibit non-monotone behavior: they decrease (increase) during the first (second) part of the active phase. It is striking to note that the confidence bands for certain endogenous variables, such as wealth, seem to remain quite wide when risk aversion increases. In fact, Corollary 4 shows that the width of the confidence band for wealth converges to a positive limit as risk aversion goes to infinity. This counterintuitive result is explained by the fact that even an extremely risk averse household is exposed to human capital uncertainty (i.e., wage uncertainty). The confidence band for human capital, of course, is not affected by risk aversion.

6.2 *Pension plan distribution dynamics*

Complementary information is provided by the evolution of the densities of the endogenous variables throughout the life cycle. Figure 11 focuses on the density of the pension plan value, shown at ages 25, 30, 40, 50, 60, 65, 70, 75, and 79. The density is seen to be single peaked and to drift to the right (resp. left) up to (resp. after) the retirement date. The standard deviation follows the same pattern, resulting in outcomes that are more spread out around the retirement date. This behavior corresponds to that of the confidence band discussed above. The density exhibits right-skewness, which becomes apparent after an initial phase. Skewness increases during the active phase and is pronounced at the retirement date.

The important lesson here is that the optimal pension plan can take a wide range of possible values. This is especially clear at the retirement date. In most instances, the retirement value of the plan

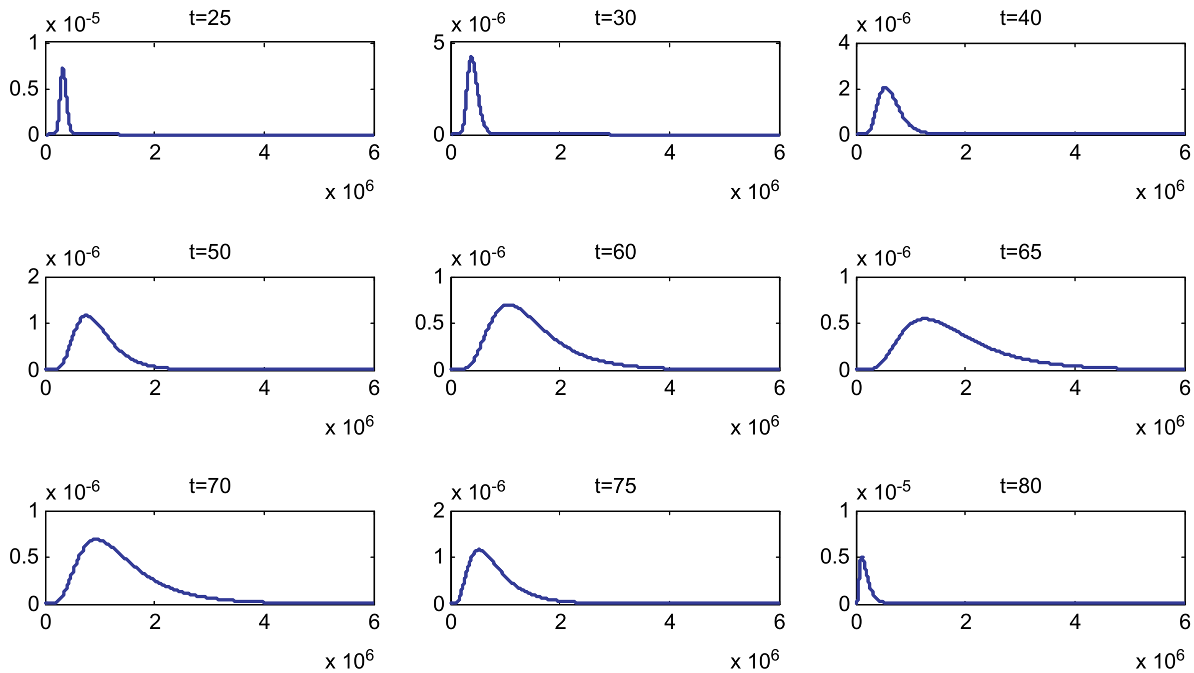

Figure 11 The density of the pension plan value at ages 25, 30, 40, 50, 60, 65, 70, 75, and 79.

exceeds $0.4M$. With low probability it exceeds $4M$. Even after retirement, the spread in outcomes is striking. Five years into retirement (age 70) potential values lie between $0.3M$ and $4M$. The household with constant relative risk aversion of $R = 4$ optimally chooses plans with large risk exposure. There is also a clear preference for right-skewness.

The behavior of the optimal pension plan can be contrasted with the behavior of a TDF. The standard TDF is structured so as to reduce the exposure to equities as the plan-holder ages. This typically reduces the range of possible outcomes as the target date approaches. Consider, for instance, the TDF based on the investment rule

$$\frac{\pi_t^{tdf}}{X_t^{tdf}} = a + b(T_r - t),$$

where $a = 0.04$ and $b = 0.01$ [see Bodie and Treussard (2007)]. This policy mandates a fraction in equities that is affine in time-to-retirement. It reduces the proportion in equities at the rate $-b$ as the fund holder ages. Figure 12 compares the densities and distributions generated by the TDF policy and the optimal pension plan, at ages 35, 50, and 65. The difference is striking. The density of the pension plan value lies systematically to the right of that of the TDF. The distribution plot shows that the optimal pension plan has higher mean and higher first, second (median), and third quartiles.

7 Mortality risk

Assume that mortality is determined by an inhomogeneous Poisson process with a jump arrival intensity $\lambda(v)$ that depends on time.[5] The random time of death corresponds to the time of the first jump in the Poisson process. As the probability of survival past a threshold is null, it is important to restrict the arrival intensity. Let T be the physical maximum of life. To ensure that death does not occur past the maximal lifetime, it suffices

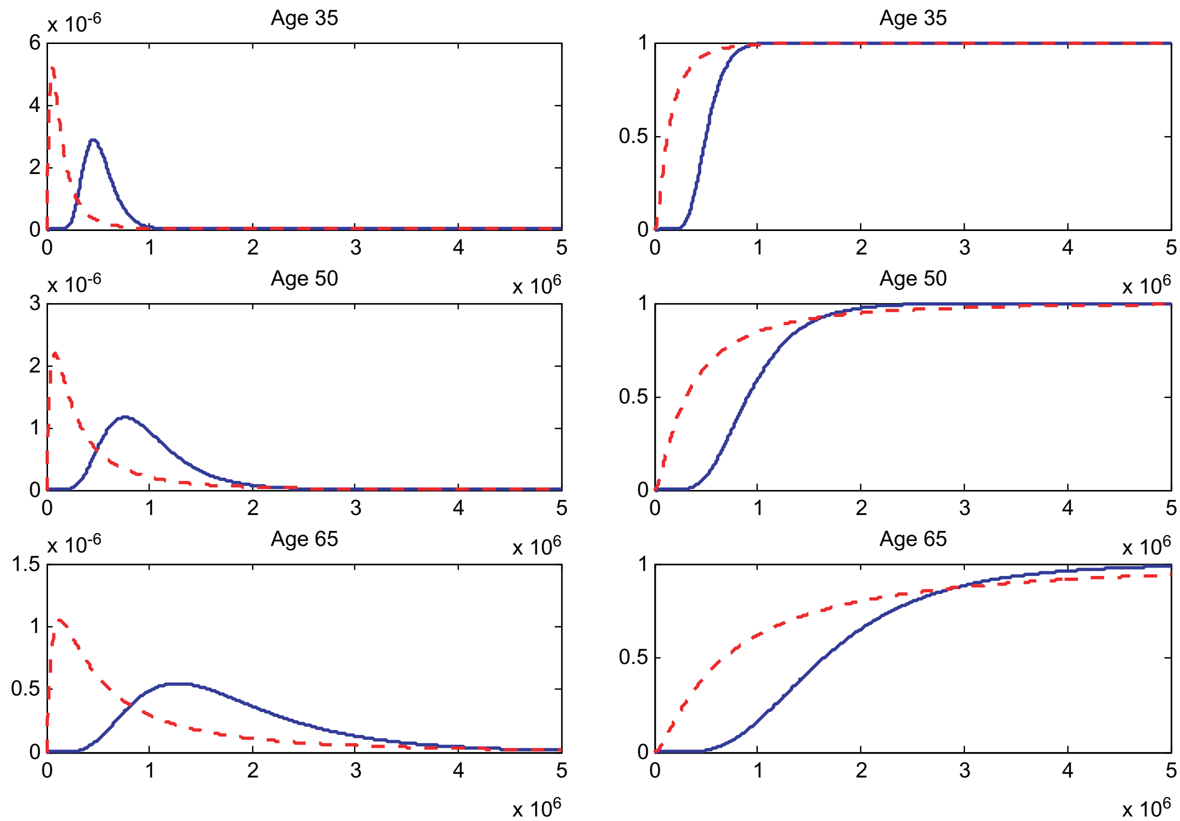

Figure 12 Densities (left panels) and cumulative distribution functions (right panels) of the optimal pension plan (solid curves) and the TDF policy $a + b(T_r - t)$ with $a = 0.04$ and $b = 0.01$ (dotted curve), at ages 35, 50, and 65.

to consider intensities such that $\lambda(v) = \infty$ for $v \in [T, T + \epsilon)$, $\epsilon > 0$. The probability $P(0, s)$ of surviving past s is

$$P(0, s) = \exp\left(-\int_0^s \lambda(v)dv\right),$$

which converges to zero as soon as s exceeds the maximal threshold T (the probability of surviving to $T + \epsilon$ is null for any $\epsilon > 0$). The conditional survival probability at time t is $P(t, s) = \exp(-\int_t^s \lambda(v)dv)$.

Optimal policies in this extended setting are described in Proposition 4 in the Appendix. This proposition shows that the model with mortality risk is the same as a model without mortality risk, but adjusted subjective discount rate and interest rate given by

$$\begin{aligned} \widehat{\beta}(t) &\equiv \beta + \lambda(t) \quad \text{and} \\ \widehat{r}(t) &\equiv r + \lambda(t). \end{aligned} \tag{29}$$

Implicitly, mortality risk raises both quantities by a common amount equal to the intensity of occurrence of death. Aside from this modification, the remaining structure of optimal policies is the same as before. The effects on policies are nevertheless complex. On one hand mortality risk reduces human capital. On the other hand, it also reduces the present value of a given consumption and leisure expenditure plan (due to the possibility of early death). These effects work in opposite directions. The first one implies a tighter budget constraint leading to a reduction in consumption and leisure. The second one relaxes the constraint and leads to an increase in consumption and leisure. The same trade-offs arise for optimal wealth. Ultimately, the behavior of endogenous variables depends on the relative strengths of these conflicting effects.

Figures 13 and 14 show the trajectories and confidence bands over the life cycle, for the model with random lifetime. To ease the comparison with previous sections, maximal lifetime is kept at $T = 60$ (maximum age is 80) in Figure 13. The intensity of death arrival is $\lambda = 0.12$, a typical estimate of an average population mortality. In this example the budget multiplier decreases with random mortality. This leads to an increase in consumption and leisure in every state in which the individual is alive. The impact on human capital is significant. The initial human capital decreases from $3,024,825$ to $718,043$. Subsequent values tend to be much lower as well. Financial wealth also tends to be lower along the trajectory graphed, although it occasionally exceeds the values obtained for $\lambda = 0$. This is the consequence of differences in optimal portfolios. The lower sensitivity to shocks results in smaller confidence bands. The most apparent difference is in the behavior of the pension plan. When λ increases from 0 to 0.12, the initial value of the pension plan decreases from $275,796$ to an insignificant amount, 670. At the retirement date, the respective values stand at $1,274,352$ versus $685,716$ if alive. The widths of the confidence intervals reflect these differences. The pension plan confidence band for $\lambda = 0.12$ is very small at the outset. It starts to widen much later during the life cycle and is generally much smaller.

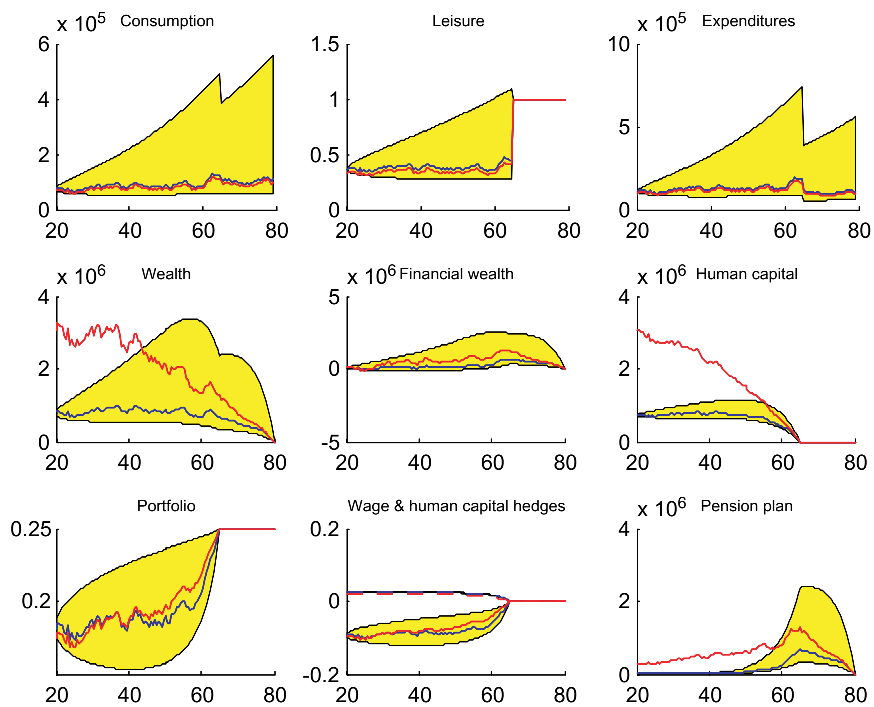

Figure 13 Confidence bands and trajectories during the life cycle, with random life. This figure shows 95% confidence bands for $\lambda = 0.12$ (yellow regions) and realized trajectories for $\lambda = 0.12$ (blue) and $\lambda = 0$ (red), for endogenous variables in the model with random lifetime. Realized trajectories are constructed from the wage and stock market paths described in earlier sections. Maximum lifetime is set at $T = 60$ (age 80).

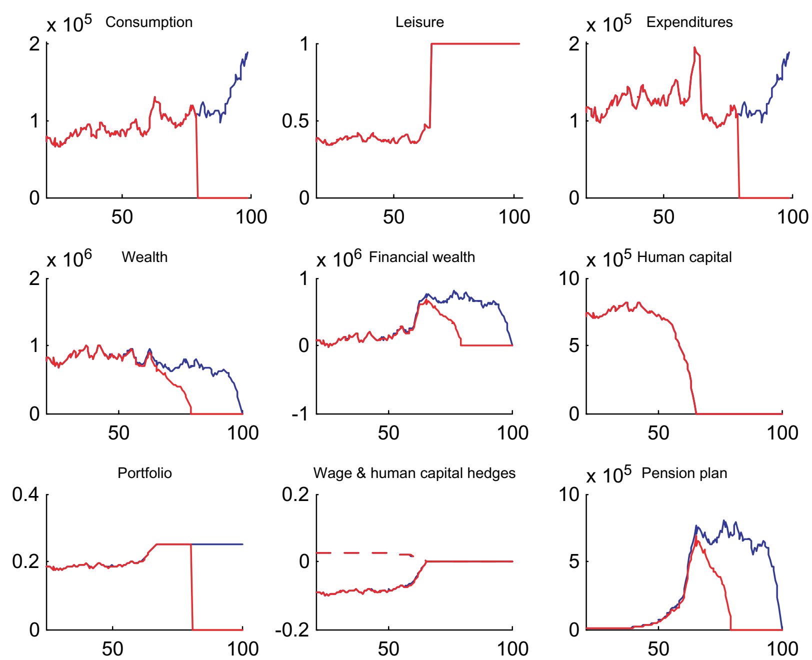

Figure 14 Trajectories during the life cycle, with random life and different maximal lifetimes. This figure shows realized trajectories for $T = 80$ (blue) and $T = 60$ (red), for endogenous variables in the model with random lifetime. Realized trajectories are constructed from the wage and stock market paths described in earlier sections. The intensity of death is $\lambda = 0.12$.

Table 2 Consumption and leisure for $T = 60$ and 80 (maximal age 80 and 100).

	Maximum age ($T + 20$)			
	$T = 60$ (age 80)		$T = 80$ (age 100)	
Age	c	l	c	l
30	70, 292	0.3521	70, 285	0.3521
40	81, 332	0.3811	81, 323	0.3810
50	88, 124	0.3955	88, 115	0.3955
60	76, 557	0.3566	76, 549	0.3566
65	97, 213	0.4123	97, 202	0.4123
70	84, 966	0.3759	84, 957	0.3758
75	85, 617	0.3751	85, 608	0.3750
80	85, 226	0.3714	85, 217	0.3714

As shown in Table 2, a subsequent increase in the terminal date from $T = 60$ to 80 (maximal age 100), keeping the mortality rate at $\lambda = 0.12$, has a small impact on consumption, labor, and expenditures during the common part of the life cycle (ages 20 to 80). Consumption decreases slightly. Leisure is virtually unchanged. The reason underlying these modest changes is the heavy discounting associated with contingencies far in the future. The cumulative impact of consumption decreases nevertheless eventually starts impacting wealth. A material effect is observed during the retirement period. Financial wealth and total wealth become larger with the longer horizon. Resources are preserved to finance consumption during the additional part of the retirement period. Figure 14 illustrates these phenomena.

8 Discussion

The previous analysis illustrates the dynamic properties of optimal choices in a simple model of the life cycle. While the analysis captures essential element of the decision problem confronting households it also abstracts from a number of important factors. Important aspects that remain to be incorporated in future research are as follows:

(1) Unhedgeable labor income components. Although compensation in some professions is closely tied to the performance of certain stock index benchmarks, in many cases correlation between individual labor income and the stock market is low. Evidence at the aggregate level seems to support a correlation close to zero [see, for instance, Heaton and Lucas (1997)].[6] In contrast, recent empirical evidence based on portfolio rebalancing decisions by individuals switching industry reveals significant portfolio effects motivated by human capital hedging concerns [Bettermier *et al.* (2010)]. The importance of the effects documented suggests correlations that differ from zero. From a theoretical perspective, inspection of the portfolio policies reveals that imperfect correlation has multiple effects on the human capital hedge. A direct effect reduces the size of this hedge, thereby lessening the incentive to increase equity holdings with age. This direct effect is complemented by an indirect effect, due to the structure of the endogenous shadow price of the incompleteness. Indeed, the shadow price depends on both hedgeable and unhedgeable risks, leading to the addition of a new hedging components in the portfolio demand. The size, sign, and dynamic behavior of these (shadow price) hedges remain to be identified. Some insights are provided by recent theoretical contributions. The numerical analysis in Benzoni *et al.* (2007), for instance, suggests that idiosyncratic shocks dampen the desire to hold equities. Bick *et al.* (2009) show that imperfect hedgeability of labor income has a limited impact on the welfare of an individual. Both papers base their conclusions on either approximation methods or numerical schemes with conjectured boundary conditions, applied in the context of specific parametric models. Definitive answers to the questions concerning the behavior of the portfolio and its hedging components hinge on the development of accurate methods to deal with imperfect correlation.

(2) Health risks. Health shocks constitute another type of imperfectly hedgeable risks that can have a significant impact on life cycle policies. Even individuals enrolled in a disability plan stand to lose an important fraction of their labor income in case of an event resulting in long-term disability. The occurrence of such an event reduces human capital. In addition, the riskiness of future income vanishes, eliminating the need to hedge. As a result, the fraction of total wealth invested in equities experiences a positive jump in the event of a negative health shock. Imperfect hedgeability also introduces an endogenous shadow price for health risk. As for unhedgeable labor income, this shadow price depends on all sources of risk, prompting an associated hedging demand. An additional complication relates to the impact of health on preferences. The health status of an individual might affect the marginal utility of consumption. This introduces an additional channel through which health shocks affect optimal policies. Edwards (2008) sheds some light on this last effect. In a stylized model without labor flexibility, he shows that health risk is akin to increasing risk aversion, prompting individuals to reduce risk taking as they age. The combination of the three effects, unfortunately, is nontrivial and remains to be analyzed. Other studies that have touched on health risk include Yogo (2009) and Hugonnier *et al.* (2009). Yogo considers optimal asset allocation during the retirement phase (but not the

accumulation phase) in the presence of health costs. Hugonnier *et al.* consider a similar setting, but assume nonseparable preferences in order to smooth discontinuities in preferences when death occurs.

(3) Liquidity constraints. Brokers typically require that the value of a portfolio account, i.e., financial wealth, be non-negative. This type of liquidity constraint is often slack, but can become binding when labor income becomes sufficiently large. The main impact of the constraint is to restrict consumption smoothing. Concretely, resources cannot be shifted out of states in which the constraint binds. The result is that constrained consumption will typically be lower in the early stages of the life cycle to eventually become larger if the constraint binds sufficiently often. Similarly, purchases of leisure are initially lower in the presence of a constraint. The precise magnitude of these effects is difficult to determine. The methods in El Karoui and Jeanblanc–Picque (1998) and the analysis in Detemple and Serrat (2003) can in principle be used to examine the question and shed light on the dynamics of optimal constrained policies.

(4) Borrowing and collateral constraints. Restrictions on the amount that can be borrowed are another relevant issue confronting households (see Campbell (2006) and references therein). As shown in Section 4.1, unconstrained policies borrow large amounts at the outset. A borrowing constraint is therefore likely to bind for many households in the early stages of the life cycle. The immediate consequence is a reduction in risk-taking. More importantly, the likelihood of a binding constraint in the future modifies optimal policies, even if the constraint does not bind in the present. The optimal portfolio, in particular, includes new hedging terms motivated by the desire to mitigate the impact of the constraint in the future. The precise extent of this effect is difficult to pinpoint, especially in the model with a retirement period.

(5) Mean reversion and incomplete information. Mean-reversion is an important feature of stock return dynamics, which has relevance for life cycle policies. Campbell and Viceira (1999), Campbell *et al.* (2004) and Detemple *et al.* (2003) examine the issue in various settings without labor income. In the context of the present paper, mean-reversion in stock returns and in the growth rate of wages may reduce the width of the confidence intervals derived in Section 6. Parameter uncertainty, as studied by Barberis (2000) and Wachter and Warusawitharana (2009), will further impact the width of these confidence bands. Resulting confidence bands can be larger or smaller depending on the nature of co-movements between estimation risk and market risk.

9 Conclusions

The analysis carried out in this paper brought to light some of the properties, in particular dynamic properties, of household decision variables and related quantities. In particular it identified the optimal pension plan and provided perspective on its relation with other components of household wealth, such as financial wealth.

The model underlying this analysis was very stylized by design. The driving goal was to produce basic insights in the simplest structure possible, yet one that retains essential features of real life situations. The presence of a retirement period is clearly a major structural component of the household decision-making framework. This element was shown to have major implications for the behavior of optimally invested wealth and pension plans.

It is equally clear that other important elements have been omitted. The aspects already discussed in the previous section deserve further critical thinking. Additional elements to be considered include the nature of household preferences, the multiplicity of risks confronting individuals and the variety of financial instruments available for transferring resources over time. These elements and their implications for optimal decision-making and structures will be examined in future research.

Appendix

This appendix gives the formulas underlying the analysis in the text.

It is useful to note that the value of human capital at date $t \leq T_r$ is

$$\begin{aligned} H_t &= \bar{h} E\left[\int_t^{T_r} \xi_{t,v} w_v dv\right] \\ &= w_t \bar{h} A(t, T_r; r - \mu^w + \sigma^w \theta), \end{aligned}$$

where $\xi_{t,v} = \xi_v/\xi_t = \exp(-r(v-t) - \frac{1}{2}\theta^2(v-t) - \theta(W_v - W_t))$ is the conditional state price density at t (the stochastic discount factor for valuation at t of cash flows received at v) and

$$A(t, T_r; x) = \frac{1 - \exp(-x(T_r - t))}{x}$$

is the usual annuity factor. Likewise, total wealth at t is

$$N_t = \begin{cases} X_t + H_t = E\left[\int_t^{T_r} \xi_{t,v}(c_v + l_v w_v) dv\right] + E\left[\int_{T_r}^{T} \xi_{t,v} c_v dv\right] & \text{for } t \leq T_r \\ X_t = E\left[\int_t^{T} \xi_{t,v} c_v dv\right] & \text{for } t > T_r \end{cases},$$

where the right-hand side is the present value of expenditures over the life cycle. Total wealth is the sum of financial wealth and human capital.

With this notation, optimal policies are as follows:

Proposition 1. *Optimal consumption and leisure are given by*

$$c_t^* = \begin{cases} \eta \Psi(w_t, t) \dfrac{1}{G(t, T_r)} N_t^* & \text{for } t \in [t, T_r) \\ \dfrac{1}{G_0(t, T)} N_t^* & \text{for } t \in [T_r, T] \end{cases} \tag{A.1}$$

$$l_t^* = \begin{cases} (1 - \eta)\Psi(w_t, t) \dfrac{1}{G(t, T_r)} \dfrac{N_t^*}{w_t} & \text{for } t \in [t, T_r) \\ \bar{h} & \text{for } t \in [T_r, T] \end{cases}, \tag{A.2}$$

where

$$\Psi(w_t, t) = \frac{w_t^{(1-\eta)\rho} f(\eta) G(t, T_r)}{w_t^{(1-\eta)\rho} f(\eta) G(t, T_r) + \phi^{1/R} \exp(-g(1, R)(T_r - t)) G_0(T_r, T)} \tag{A.3}$$

$$G(t, T_r) \equiv A(t, T_r; g(\eta, R)) = \frac{1 - \exp(-g(\eta, R)(T_r - t))}{g(\eta, R)} \quad \text{(A.4)}$$

$$G_0(t, T) \equiv A(t, T; g(1, R)) = \frac{1 - \exp(-g(1, R)(T - t))}{g(1, R)} \quad \text{(A.5)}$$

with

$$f(\eta) = \left(\frac{1-\eta}{\eta}\right)^{-(1-\eta)\rho} \frac{1}{\eta}, \quad \rho = 1 - \frac{1}{R} \quad \text{(A.6)}$$

$$g(\eta, R) \equiv \frac{1}{R}\beta + \rho\left(r + \frac{1}{2}\theta^2\right) - \rho(1-\eta)\left(\mu^w - \frac{1}{2}(\sigma^w)^2\right) - \frac{1}{2}\rho^2(\theta - (1-\eta)\sigma^w)^2. \quad \text{(A.7)}$$

Total wealth, the sum of financial wealth and the value of human capital, is

$$N_t^* = \begin{cases} N_{at}^* + N_{rt}^* & \text{for } t \in [t, T_r) \\ N_{rt}^* & \text{for } t \in [T_r, T] \end{cases} \quad \text{(A.8)}$$

with

$$N_{at}^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1/R} w_t^{(1-\eta)\rho} G(T_r \wedge t, T_r) f(\eta) \quad \text{(A.9)}$$

$$N_{rt}^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1/R} \phi^{1/R} \exp(-g(1, R) \times (T_r - T_r \wedge t)) G_0(T_r \vee t, T), \quad \text{(A.10)}$$

where $T_r \wedge t \equiv \min(T_r, t)$ and $T_r \vee t \equiv \max(T_r, t)$. The optimal portfolio has the decomposition

$$\frac{\pi_t^*}{N_t^*} = \frac{\pi_t^m}{N_t^*} + \frac{\pi_t^w}{N_t^*} - \frac{\pi_t^H}{N_t^*} \quad \text{(A.11)}$$

$$\frac{\pi_t^m}{N_t^*} = \frac{1}{R}\sigma^{-1}\theta,$$

$$\frac{\pi_t^w}{N_t^*} = (1-\eta)\rho\Psi(w_t, t)\sigma^{-1}\sigma^w, \quad \text{(A.12)}$$

$$\frac{\pi_t^H}{N_t^*} \equiv \frac{H_t}{N_t^*}\sigma^{-2}\sigma\sigma^w$$

during the accumulation period and

$$\frac{\pi_{rt}^*}{N_t^*} = \frac{\pi_{rt}^m}{N_t^*} = \frac{1}{R}\sigma^{-1}\theta \quad \text{(A.13)}$$

during retirement.

Proof of Proposition 1. Corollary 2 in Bodie *et al.* (2009) gives the solution for $\phi = 1$. When $\phi \neq 1$, the wealth ratio $\Psi(w_t, t)$ and retirement wealth N_{rt}^* are modified as described in (A.3) and (A.10). □

The marginal propensity to consume out of accumulation wealth, in (24) and (25) in the text, is $m(t, T_r) \equiv 1/G(t, T_r)$.

When T decreases to T_r the retirement period contracts and vanishes in the limit. The limiting case corresponds to the model without retirement (one-phase model).

Corollary 1. *In the absence of a retirement phase ($T_r = T$) optimal consumption, leisure and total wealth are given by*

$$c_t^* = \eta\frac{1}{G(t, T_r)}N_t^*, \qquad l_t^* = (1-\eta)\frac{1}{G(t, T_r)}\frac{N_t^*}{w_t} \quad \text{(A.14)}$$

$$N_t^* = N_{at}^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1/R} w_t^{(1-\eta)\rho} G(t, T_r) f(\eta) \quad \text{(A.15)}$$

for $t \in [t, T_r)$. The optimal portfolio has the decomposition (A.11) *with*

$$\frac{\pi_t^m}{N_t^*} = \frac{1}{R}\sigma^{-1}\theta, \qquad \frac{\pi_t^w}{N_t^*} = (1-\eta)\rho\sigma^{-1}\sigma^w \tag{A.16}$$

and π_t^H is as in Proposition 1 *above.*

The next corollary discusses properties of the portfolio in the one phase setting.

Corollary 2. *Consider the model without retirement phase* ($T_r = T$) *and suppose that the wage w_t and financial wealth X_t^* are given. Age is $t_a = t - t_0$, where t_0 is the birth date. The optimal portfolio is age-dependent. Age-dependence enters through human capital*:

$$\frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} = \frac{\partial(\pi_t^*/N_t^*)}{\partial t} = -\frac{\partial(\pi_t^H/N_t^*)}{\partial t} = -\frac{\partial(H_t/N_t^*)}{\partial t}\sigma^{-2}\sigma\sigma^w \tag{A.17}$$

$$\begin{aligned}\frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} &= \frac{\partial(\pi_t^*/X_t^*)}{\partial t} \\ &= \frac{\partial(\pi_t^m/X_t^*)}{\partial t} + \frac{\partial(\pi_t^w/X_t^*)}{\partial t} - \frac{\partial(\pi_t^H/X_t^*)}{\partial t} \\ &= \frac{\partial(H_t/X_t^*)}{\partial t}\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\rho\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right),\end{aligned} \tag{A.18}$$

where

$$\frac{\partial(H_t/X_t^*)}{\partial t} = -\frac{H_t}{X_t^*}\frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t, T_r; r-\mu^w+\sigma^w\theta)}. \tag{A.19}$$

(i) *The fraction of total wealth in equities increases* (*resp. decreases*) *with age if financial wealth X_t^* and the covariance between returns and wages have the same* (*resp. opposite*) *signs*:

$$\frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} \gtreqless 0 \Leftrightarrow X_t^*\sigma\sigma^w \gtreqless 0.$$

(ii) *The fraction of financial wealth in equities increases* (*resp. decreases*) *with age if financial wealth X_t^* and the expression $(1/R)\sigma^{-1}\theta + (1-\eta)\rho\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w$ have opposite* (*resp. the same*) *signs*:

$$\frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} \gtreqless 0 \Leftrightarrow X_t^*\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\rho\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right) \lesseqgtr 0. \tag{A.20}$$

Proof of Corollary 2. The relations (A.17) and (A.18) follow from the decomposition of the portfolio. To obtain (A.20) note that

$$\begin{aligned}\frac{\partial(H_t/N_t^*)}{\partial t} &= \frac{w_t\bar{h}X_t^*}{(X_t^* + w_t\bar{h}A(t, T_r; r-\mu^w+\sigma^w\theta))^2} \times \frac{\partial A(t, T_r; r-\mu^w+\sigma^w\theta)}{\partial t} \\ &= \frac{H_t}{N_t^*}\left(1-\frac{H_t}{N_t^*}\right)\frac{\partial A(t, T_r; r-\mu^w+\sigma^w\theta)/\partial t}{A(t, T_r; r-\mu^w+\sigma^w\theta)}\end{aligned}$$

$$= -\frac{H_t}{N_t^*}\left(1 - \frac{H_t}{N_t^*}\right)$$

$$\times \frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)} \gtreqless 0$$

$$\Leftrightarrow 1 - \frac{H_t}{N_t^*} \lesseqgtr 0$$

$$\Leftrightarrow X_t^* \lesseqgtr 0.$$

The relation (A.20) now follows immediately from (A.18) and the last inequality. □

The age-dependence of the portfolio becomes more complex when there is a retirement phase. The next result describes the effects that arise.

Corollary 3. *Consider the model with a retirement phase ($T_r > T$) and suppose that the wage w_t and financial wealth X_t^* are given. Also, define*

$$\begin{aligned} F(H_t, N_t, w_t, t) &\equiv (1-\eta)\rho\frac{\partial\Psi(w_t,t)}{\partial t} - \frac{\partial(H_t/N_t^*)}{\partial t} \\ &\equiv F_1(w_t,t) + F_2(H_t,N_t,w_t,t) \end{aligned} \quad \text{(A.21)}$$

and note that

$$\begin{aligned} F_1(w_t,t) &= -(1-\eta)\rho\Psi(w_t,t)(1-\Psi(w_t,t)) \\ &\quad \times \left(\frac{\exp(-g(\eta,R)(T_r-t))}{G(t,T_r)} + g(1,R)\right) \end{aligned} \quad \text{(A.22)}$$

$$\begin{aligned} F_2(H_t,N_t,w_t,t) &= \frac{H_t}{N_t^*}\left(1-\frac{H_t}{N_t^*}\right) \\ &\quad \times \frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)}. \end{aligned} \quad \text{(A.23)}$$

The optimal portfolio is age-dependent. Age-dependence enters through human capital and through the fraction of accumulation wealth to total wealth:

$$\begin{aligned} \frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} &= \frac{\partial(\pi_t^*/N_t^*)}{\partial t} \\ &= \frac{\partial(\pi_t^w/N_t^*)}{\partial t} - \frac{\partial(\pi_t^H/N_t^*)}{\partial t} \\ &= F(H_t,N_t,w_t,t)\sigma^{-2}\sigma\sigma^w \end{aligned} \quad \text{(A.24)}$$

$$\begin{aligned} \frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} &= \frac{\partial(\pi_t^*/X_t^*)}{\partial t} \\ &= \frac{\partial(\pi_t^m/X_t^*)}{\partial t} + \frac{\partial(\pi_t^w/X_t^*)}{\partial t} \\ &\quad - \frac{\partial(\pi_t^H/X_t^*)}{\partial t} \\ &= \frac{\partial(H_t/X_t^*)}{\partial t}\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\right. \\ &\quad \left.\times \rho\Psi(w_t,t)\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right) \\ &\quad + \frac{N_t^*}{X_t^*}(1-\eta)\rho\frac{\partial\Psi(w_t,t)}{\partial t}\sigma^{-1}\sigma^w, \end{aligned} \quad \text{(A.25)}$$

where the derivative $\partial(H_t/X_t^*)/\partial t$ is as given in (A.19).

(i) *The fraction of total wealth in equities increases* (*resp. decreases*) *with age if $F(H_t,N_t,w_t,t)$ and the covariance between returns and wages have the same* (*resp. opposite*) *signs*:

$$\frac{\partial(\pi_t^*/N_t^*)}{\partial t_a} \gtreqless 0$$

$$\Leftrightarrow F(H_t,N_t,w_t,t)\sigma^{-2}\sigma\sigma^w \gtreqless 0. \quad \text{(A.26)}$$

(ii) *The relation between the fraction of financial wealth in equities and age is determined by*

$$\frac{\partial(\pi_t^*/X_t^*)}{\partial t_a} \gtreqless 0$$

$$\Leftrightarrow \frac{\partial(H_t/X_t^*)}{\partial t}\left(\frac{1}{R}\sigma^{-1}\theta + (1-\eta)\rho\Psi(w_t,t)\sigma^{-1}\sigma^w - \sigma^{-2}\sigma\sigma^w\right) + \frac{N_t^*}{X_t^*}F_1(w_t,t)\sigma^{-1}\sigma^w \gtreqless 0$$

$$\Leftrightarrow \frac{N_t^*}{X_t^*}F(H_t,N_t,w_t,t)\sigma^{-2}\sigma\sigma^w - \frac{\pi_t^*}{N_t^*}\frac{H_t}{X_t^*}\frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)} \gtreqless 0. \qquad \text{(A.27)}$$

Proof of Corollary 3. To obtain the expressions for $F(H_t, N_t, w_t, t)$ and its components, note that

$$\frac{\partial G(t,T_r)}{\partial t} = -\exp(-g(\eta,R)(T_r-t))$$

$$\frac{\partial \Psi(w_t,t)}{\partial t} = \frac{w_t^{(1-\eta)\rho}f(\eta)G(t,T_r)\phi^{1/R}\exp(-g(1,R)(T_r-t))G_0(T_r,T)}{(w_t^{(1-\eta)\rho}f(\eta)G(t,T_r)+\phi^{1/R}\exp(-g(1,R)(T_r-t))G_0(T_r,T))^2}$$
$$\times\ (\partial \log G(t,T_r) - g(1,R))$$
$$= -\Psi(w_t,t)(1-\Psi(w_t,t))\left(\frac{\exp(-g(\eta,R)(T_r-t))}{G(t,T_r)} + g(1,R)\right) < 0$$

and

$$F_1(w_t,t) = (1-\eta)\rho\frac{\partial\Psi(w_t,t)}{\partial t}$$
$$= -(1-\eta)\rho\Psi(w_t,t)(1-\Psi(w_t,t))$$
$$\times\left(\frac{\exp(-g(\eta,R)(T_r-t))}{G(t,T_r)} + g(1,R)\right)$$

$$F_2(H_t,N_t,w_t,t)$$
$$= -\frac{\partial(H_t/N_t^*)}{\partial t}$$
$$= \frac{H_t}{N_t^*}\left(1-\frac{H_t}{N_t^*}\right)$$
$$\times\ \frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)}.$$

The relations (A.24) and (A.25) follow from the decomposition of the portfolio in the model with two phases and the definition of $F(H_t, N_t, w_t, t)$. The condition (A.26) is obtained from (A.24). The first line of (A.20) uses (A.25) and the definition of $F_1(w_t, t)$. The second line is based on (A.26) and

$$\frac{\partial(H_t/X_t^*)}{\partial t}$$
$$= -\frac{H_t}{X_t^*}\frac{\exp(-(r-\mu^w+\sigma^w\theta)(T_r-t))}{A(t,T_r;r-\mu^w+\sigma^w\theta)}.$$

This completes the proof. □

The following proposition identifies the optimal pension plan.

Proposition 2. *The optimal pension plan is a contingent claim producing the cash flow $c_t^* = N_t^*/G_0(t,T)$ during retirement. The value of the plan prior to retirement is*

$$N_{rt}^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1/R}\phi^{1/R}\exp(-g(1,R)(T_r-t))$$
$$\times\ G_0(T_r,T). \qquad \text{(A.28)}$$

The plan is engineered by investing $N_{r0}^* = (y^*/a_0)^{-1/R}\phi^{1/R}\exp(-g(1,R)T_r)G_0(T_r,T)$ at the initial date in the mean–variance portfolio $\pi_{rt}^*/N_{rt}^* = \pi_{rt}^m/N_{rt}^* = (1/R)(\sigma_t')^{-1}\theta_t$.

The final result provides confidence bands for the wealth components.

Proposition 3. *Assume $(1-\eta)\rho\sigma^w + \frac{1}{R}\theta > 0$ and define*

$$Z_t^{\pm}(\alpha;\eta) \equiv \left((1-\eta)\rho\left(\mu^w - \frac{1}{2}(\sigma^w)^2\right) + \frac{1}{R}\left(r + \frac{1}{2}\theta^2\right)\right)t \pm N^{-1}\left(1 - \frac{1}{2}\alpha\right) \times \left((1-\eta)\rho\sigma^w + \frac{1}{R}\theta\right)\sqrt{t},$$

where $N(\cdot)$ is the cumulative standard normal distribution. The α-confidence intervals for N_{at}^* and N_{rt}^* are $[z_l^a(\alpha), z_u^a(\alpha)]$ and $[z_l^r(\alpha), z_u^r(\alpha)]$ with

$$z_u^a(\alpha) = \left(\frac{y^*}{a_t}\right)^{-1/R} G(T_r \wedge t, T_r) f(\eta) \times \exp(Z_t^+(\alpha;\eta))$$

$$z_l^a(\alpha) = \left(\frac{y^*}{a_t}\right)^{-1/R} G(T_r \wedge t, T_r) f(\eta) \times \exp(Z_t^-(\alpha;\eta))$$

$$z_u^r(\alpha) = \left(\frac{y^*}{a_t}\right)^{-1/R} \phi^{1/R} \times \exp(-g(1,R)(T_r - T_r \wedge t)) \times G_0(T_r \vee t, T)\exp(Z_t^+(\alpha;1))$$

$$z_l^r(\alpha) = \left(\frac{y^*}{a_t}\right)^{-1/R} \phi^{1/R} \times \exp(-g(1,R)(T_r - T_r \wedge t)) \times G_0(T_r \vee t, T)\exp(Z_t^-(\alpha;1)).$$

Confidence intervals for other endogenous variables can be derived using the same procedure as for the wealth components. Endogenous variables are transformations of a Brownian motion and confidence intervals can be calculated from those for Gaussian variates.

Proof of Proposition 3. Let $g_1 \equiv (1-\eta)\rho(\mu^w - \frac{1}{2}(\sigma^w)^2) + (1/R)(r + \frac{1}{2}\theta^2)$, $g_2 \equiv (1-\eta)\rho\sigma^w + \theta/R$ and $g_3 \equiv (1/R)(r + \frac{1}{2}\theta^2)$. Using

$$\log(\xi_t^{-1/R} w_t^{(1-\eta)\rho}) = g_1 t + g_2 W_t,$$

$$\log(\xi_t^{-1/R}) = g_3 t + \frac{1}{R}\theta W_t$$

$$E[\log(\xi_t^{-1/R})] = g_3 t,$$

$$E[\log(\xi_t^{-1/R} w_t^{(1-\eta)\rho})] = g_1 t$$

gives

$$E[\log(N_{at}^*)] = \log\left(\left(\frac{y^*}{a_t}\right)^{-1/R} G(T_r \wedge t, T_r) f(\eta)\right) + g_1 t$$

$$E[\log(N_{rt}^*)] = \log\left(\left(\frac{y^*}{a_t}\right)^{-1/R} \phi^{1/R} \exp(-g(1,R) \times (T_r - T_r \wedge t)) G_0(T_r \vee t, T)\right) + g_3 t$$

$$\mathrm{std}(\log(N_{at}^*)) = g_2\sqrt{t},$$

$$\mathrm{std}(\log(N_{rt}^*)) = \frac{1}{R}\theta\sqrt{t}$$

and

$$\frac{\log(N_{at}^*) - E[\log(N_{at}^*)]}{\text{std}(\log(N_{at}^*))} = \frac{W_t}{\sqrt{t}},$$

$$\frac{\log(N_{rt}^*) - E[\log(N_{rt}^*)]}{\text{std}(\log(N_{rt}^*))} = \frac{W_t}{\sqrt{t}}.$$

To find confidence intervals, note that

$$\begin{aligned}
\frac{\alpha}{2} &= P(N_{at}^* \geq z_u^a(\alpha)) \\
&= P(\log(N_{at}^*) \geq \log(z_u^a(\alpha))) \\
&= P\left(\frac{\log(N_{at}^*) - E[\log(N_{at}^*)]}{\text{std}(\log(N_{at}^*))} \geq z(\alpha)\right) \\
&= P\left(\frac{W_t}{\sqrt{t}} \geq z(\alpha)\right) \\
&= 1 - N(z(\alpha)). \\
\frac{\alpha}{2} &= P(N_{at}^* \leq z_l^a(\alpha)) \\
&= P(\log(N_{at}^*) \leq \log(z_l^a(\alpha))) \\
&= P\left(\frac{\log(N_{at}^*) - E[\log(N_{at}^*)]}{\text{std}(\log(N_{at}^*))} \leq -z(\alpha)\right) \\
&= P\left(\frac{W_t}{\sqrt{t}} \leq -z(\alpha)\right) \\
&= N(-z(\alpha))
\end{aligned}$$

giving $z(\alpha) = N^{-1}(1 - \frac{1}{2}\alpha)$ and

$$\begin{aligned}
z_u^a(\alpha) &= \exp(E[\log(N_{at}^*)] + z(\alpha)\text{std}(\log(N_{at}^*))) \\
&= \left(\frac{y^*}{a_t}\right)^{-1/R} G(T_r \wedge t, T_r) f(\eta) \\
&\quad \times \exp\left(g_1 t + N^{-1}\left(1 - \frac{1}{2}\alpha\right) g_2\sqrt{t}\right)
\end{aligned}$$

$$\begin{aligned}
z_l^a(\alpha) &= \exp(E[\log(N_{at}^*)] - z(\alpha)\text{std}(\log(N_{at}^*))) \\
&= \left(\frac{y^*}{a_t}\right)^{-1/R} G(T_r \wedge t, T_r) f(\eta) \\
&\quad \times \exp\left(g_1 t - N^{-1}\left(1 - \frac{1}{2}\alpha\right) g_2\sqrt{t}\right).
\end{aligned}$$

Likewise,

$$\begin{aligned}
z_u^r(\alpha) &= \exp(E[\log(N_{rt}^*)] + z(\alpha)\text{std}(\log(N_{rt}^*))) \\
&= \left(\frac{y^*}{a_t}\right)^{-1/R} \phi^{1/R} \\
&\quad \times \exp(-g(1, R)(T_r - T_r \wedge t)) \\
&\quad \times\ G_0(T_r \vee t, T) \exp(g_3 t) \\
&\quad \times \exp\left(N^{-1}\left(1 - \frac{1}{2}\alpha\right) \frac{1}{R}\theta\sqrt{t}\right) \\
z_l^r(\alpha) &= \exp(E[\log(N_{rt}^*)] - z(\alpha)\text{std}(\log(N_{rt}^*))) \\
&= \left(\frac{y^*}{a_t}\right)^{-1/R} \phi^{1/R} \\
&\quad \times \exp(-g(1, R)(T_r - T_r \wedge t)) \\
&\quad \times\ G_0(T_r \vee t, T) \exp(g_3 t) \\
&\quad \times \exp\left(-N^{-1}\left(1 - \frac{1}{2}\alpha\right) \frac{1}{R}\theta\sqrt{t}\right).
\end{aligned}$$

This completes the proof of the proposition. □

The next corollary describes the effects of risk aversion on the confidence bands.

Corollary 4. *The impact of relative risk aversion on the confidence bands* $[z_l^a(\alpha), z_u^a(\alpha)]$ *and* $[z_l^r(\alpha), z_u^r(\alpha)]$ *for the wealth components is* *given by*

$$\frac{\partial z_u^a(\alpha)/\partial R}{z_u^a(\alpha)} = \frac{\partial Z_t^+(\alpha;\eta)}{\partial R} + K^a, \quad \frac{\partial z_l^a(\alpha)/\partial R}{z_l^a(\alpha)} = \frac{\partial Z_t^-(\alpha;\eta)}{\partial R} + K^a$$

$$\frac{\partial z_u^r(\alpha)/\partial R}{z_u^r(\alpha)} = \frac{\partial Z_t^+(\alpha;1)}{\partial R} + K^r, \quad \frac{\partial z_l^r(\alpha)/\partial R}{z_l^r(\alpha)} = \frac{\partial Z_t^-(\alpha;1)}{\partial R} + K^r,$$

where

$$K^a = \frac{\partial \log G(T_r \wedge t, T_r)}{\partial R} + \frac{\partial f(\eta)}{\partial R} - \frac{1}{R}\frac{\partial \log y^*}{\partial R} + \frac{1}{R^2}\log\left(\frac{y^*}{a_t}\right)$$

$$K^r = \frac{\partial \log G_0(T_r \vee t, T)}{\partial R} + \frac{\partial f(\eta)}{\partial R} - \frac{1}{R}\frac{\partial \log y^*}{\partial R} + \frac{1}{R^2}\log\left(\frac{y^*}{a_t}\right) - \frac{\partial g(1,R)(T_r - T_r \wedge t)}{\partial R}$$

$$\frac{\partial Z_t^{\pm}(\alpha;\eta)}{\partial R} \equiv \frac{1}{R^2}\left[\left((1-\eta)\left(\mu^w - \frac{1}{2}(\sigma^w)^2\right) - \left(r + \frac{1}{2}\theta^2\right)\right) t \pm N^{-1}\left(1 - \frac{1}{2}\alpha\right)((1-\eta)\sigma^w - \theta)\sqrt{t}\right]$$

$$\frac{\partial Z_t^{\pm}(\alpha;1)}{\partial R} \equiv \frac{1}{R^2}\left[-\left(r + \frac{1}{2}\theta^2\right) t \pm N^{-1}\left(1 - \frac{1}{2}\alpha\right)(-\theta)\sqrt{t}\right]$$

$$\frac{\partial G(t,T_r)}{\partial R} = \left[-G(t,T_r)\frac{1}{g(\eta,R)} + \frac{\exp(-g(\eta,R)(T_r - t))}{g(\eta,R)}(T_r - t)\right]\frac{\partial g(\eta,R)}{\partial R}$$

$$\frac{\partial G_0(t,T)}{\partial R} = \left[-G_0(t,T_r)\frac{1}{g(1,R)} + \frac{\exp(-g(1,R)(T_r - t))}{g(1,R)}(T_r - t)\right]\frac{\partial g(1,R)}{\partial R}$$

$$\frac{\partial f(\eta)}{\partial R} = -\frac{1-\eta}{R^2} f(\eta)\log\left(\frac{1-\eta}{\eta}\right)$$

$$\frac{\partial g(\eta,R)}{\partial R} \equiv -\frac{1}{R^2}\left[\beta - \left(r + \frac{1}{2}\theta^2\right) + (1-\eta)\left(\mu^w - \frac{1}{2}(\sigma^w)^2\right) + \rho(\theta - (1-\eta)\sigma^w)^2\right]$$

$$\frac{\partial g(1,R)}{\partial R} \equiv -\frac{1}{R^2}\left[\beta - \left(r + \frac{1}{2}\theta^2\right) + \rho\theta^2\right]$$

and the budget multiplier derivative is

$$\frac{\partial (y^*)^{-1/R}}{\partial R} = -\frac{((y^*)^{-1/R})^2}{x + H_0}\left(\left(\frac{1-\eta}{R^2} + \frac{\partial G(0,T_r)/\partial R}{G(0,T_r)} + \frac{\partial f(\eta)/\partial R}{f(\eta)}\right) w_0^{(1-\eta)\rho} G(0,T_r) f(\eta)\right.$$
$$\left. + \left(-\frac{\log\phi}{R^2} - \frac{\partial g(1,R)}{\partial R}T_r + \frac{\partial G_0(T_r,T)/\partial R}{G_0(T_r,T)}\right) \times \phi^{1/R}\exp(-g(1,R)T_r)G_0(T_r,T)\right).$$

The difference between the relative bound changes is

$$\frac{\partial z_u^a(\alpha)/\partial R}{z_u^a(\alpha)} - \frac{\partial z_l^a(\alpha)/\partial R}{z_l^a(\alpha)} = \frac{2}{R^2} N^{-1}\left(1-\frac{1}{2}\alpha\right)((1-\eta)\sigma^w - \theta)\sqrt{t}$$

$$\frac{\partial z_u^r(\alpha)/\partial R}{z_u^r(\alpha)} - \frac{\partial z_l^r(\alpha)/\partial R}{z_l^r(\alpha)} = \frac{2}{R^2} N^{-1}\left(1-\frac{1}{2}\alpha\right)(-\theta)\sqrt{t}.$$

Assume the market price of equity risk is positive ($\theta > 0$). The difference is then negative during retirement. It is positive (resp. negative) during the active phase if $(1-\eta)\sigma^w > \theta$ (resp. $(1-\eta)\sigma^w < \theta$). As risk aversion goes to infinity the width of the confidence intervals converges to

$$\begin{aligned} z_u^a(\alpha) - z_l^a(\alpha) &\to Y(\infty)G(T_r \wedge t, T_r; \infty) \\ &\quad \times f(\eta;\infty)(\exp(Z_t^+(\alpha;\eta,\infty)) \\ &\quad - \exp(Z_t^-(\alpha;\eta,\infty))) \\ z_u^r(\alpha) - z_l^r(\alpha) &\to 0 \end{aligned}$$

where the limiting values $Y(\infty)$, $G(t, T_r;\infty)$, $f(\eta;\infty)$, $Z^+(\alpha;\eta,\infty)$, and $Z^-(\alpha;\eta,\infty)$ are defined in the proof below.

Proof of Corollary 4. The computation of the derivatives with respect to risk aversion is straightforward. To show the limiting behavior of the confidence bounds note that

$$(y^*)^{-1/R} = \frac{x + H_0}{w_0^{(1-\eta)\rho} G(0, T_r) f(\eta) + \phi^{1/R}\exp(-g(1,R)T_r)G_0(T_r, T)}$$

converges to

$$(y^*)^{-1/R} \to \frac{x + H_0}{w_0^{1-\eta} G(0, T_r;\infty) f(\eta;\infty) + \exp(-g(1,\infty)T_r)G_0(T_r, T;\infty)} \equiv Y(\infty),$$

where

$$\begin{aligned} g(\eta,\infty) &= r + \frac{1}{2}\theta^2 - (1-\eta) \\ &\quad \times \left(\mu^w - \frac{1}{2}(\sigma^w)^2\right) \\ &\quad - \frac{1}{2}(\theta - (1-\eta)\sigma^w)^2 \\ &= r - (1-\eta)\left(\mu^w - \frac{1}{2}\eta(\sigma^w)^2\right) \\ &\quad + \theta(1-\eta)\sigma^w \\ G(0, T_r) &\to \frac{1-\exp(-g(\eta,\infty)T_r)}{g(\eta,\infty)} \\ &\equiv G(0, T_r;\infty) \end{aligned}$$

$$\begin{aligned} G_0(T_r, T) &\to \frac{1-\exp(-g(1,\infty)(T-T_r))}{g(1,\infty)} \\ &\equiv G_0(T_r, T;\infty) \\ f(\eta) &\to \left(\frac{1-\eta}{\eta}\right)^{-(1-\eta)}\frac{1}{\eta} \\ &\equiv f(\eta;\infty). \end{aligned}$$

Using $g(1,\infty) = r$ and

$$\begin{aligned} Z_t^{\pm}(\alpha;\eta) &\to (1-\eta)\left(\mu^w - \frac{1}{2}(\sigma^w)^2\right)t \\ &\quad \pm N^{-1}\left(1-\frac{1}{2}\alpha\right)(1-\eta)\sigma^w\sqrt{t} \\ &\equiv Z^{\pm}(\alpha;\eta,\infty) \\ Z_t^{\pm}(\alpha;1) &\to 0 \end{aligned}$$

it follows that

$$z_u^a(\alpha) \to Y(\infty)G(T_r \wedge t, T_r; \infty) f(\eta; \infty) \exp(Z_t^+(\alpha; \eta, \infty))$$

$$z_l^a(\alpha) \to Y(\infty)G(T_r \wedge t, T_r; \infty) f(\eta; \infty) \exp(Z_t^-(\alpha; \eta, \infty))$$

$$z_u^r(\alpha) \to Y(\infty) \exp(-r(T_r - T_r \wedge t))G_0(T_r \vee t, T; \infty)$$

$$z_l^r(\alpha) \to Y(\infty) \exp(-r(T_r - T_r \wedge t))G_0(T_r \vee t, T; \infty)$$

and the width of the confidence bands becomes $z_u^a(\alpha) - z_l^a(\alpha) \to Y(\infty)G(t, T_r; \infty) f(\eta; \infty)$ $(\exp(Z_t^+(\alpha; \eta, \infty)) - \exp(Z_t^-(\alpha; \eta, \infty)))$ and $z_u^r(\alpha) - z_l^r(\alpha) \to 0$. This completes the proof. □

Proposition 4. *For $s \geq t$, let $P(t, s) \equiv \exp(-\int_t^s \lambda(v)dv)$ be the conditional survival probability at time t (i.e., the conditional probability at time t of surviving past time s, $s \geq t$). Conditional on survival, optimal consumption and leisure (c_t^*, l_t^*) are given by* (A.1) *and* (A.2) *where*

$$\Psi(w_t, t) = \frac{w_t^{(1-\eta)\rho} f(\eta)G(t, T_r)}{w_t^{(1-\eta)\rho} f(\eta)G(t, T_r) + \phi^{1/R} \exp(-g(1, R)(T_r - t))P(t, T_r)G_0(T_r, T)} \tag{A.29}$$

$$G(t, T_r) \equiv \int_t^{T_r} \exp(-g(\eta, R)(s - t))P(t, s)ds \tag{A.30}$$

$$G_0(t, T) \equiv \int_t^{T} \exp(-g(1, R)(s - t))P(t, s)ds \tag{A.31}$$

with $f(\eta)$ and $g(\eta, R)$ as in Proposition 1. Total wealth N_t^* and accumulation wealth N_{at}^* are given by (A.8) *and* (A.9) *with $G(t, T_r)$ as in* (A.30)*. Retirement wealth is*

$$N_{rt}^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1/R} \phi^{1/R} \exp\left(-\int_{T_r \wedge t}^{T_r} g(1, R, v)dv\right) P(T_r \wedge t, T_r)G_0(T_r \vee t, T) \tag{A.32}$$

with $G_0(T_r \vee t, T)$ as in (A.31)*. The endogenous budget multiplier is*

$$y^* = \left(\frac{x + w_0\overline{h}\left(\int_0^{T_r} \exp(-(r - \mu^w + \sigma^w\theta)(s - t))P(0, s)ds\right)}{w_0^{(1-\eta)\rho} G(0, T_r) f(\eta) + \phi^{1/R} \exp(-g(1, R)T_r)P(0, T_r)G_0(T_r, T)}\right)^{-R}. \tag{A.33}$$

The optimal portfolio and its components are given by the formulas in Proposition 1*. In the event of death, $\{N_t \geq 1\}$, consumption and leisure are null.*

Proof of Proposition 4. The household maximizes

$$E\left[\int_0^{T_r} a_s u^a(c_s, l_s)1_{\{N_s=0\}}ds + \phi \int_{T_r}^{T} a_s u^r(c_s, \overline{h})1_{\{N_s=0\}}ds\right]$$

subject to

$$x + H_0 = E\left[\int_0^{T_r} \xi_s (c_s + l_s w_s) 1_{\{N_s=0\}} ds\right] + E\left[\int_{T_r}^{T} \xi_s c_s 1_{\{N_s=0\}} ds\right]$$

$$H_0 = \overline{h} E\left[\int_0^{T_r} \xi_s w_s 1_{\{N_s=0\}} ds\right].$$

It can be shown that this optimization problem is equivalent to

$$E\left[\int_0^{T_r} a_s u^a(c_s, l_s) P(0, s) ds + \phi \int_{T_r}^{T} a_s u^r(c_s, \overline{h}) P(0, s) ds\right]$$

subject to

$$x + H_0 = E\left[\int_0^{T_r} \xi_s (c_s + l_s w_s) P(0, s) ds\right] + E\left[\int_{T_r}^{T} \xi_s c_s P(0, s) ds\right]$$

$$H_0 = \overline{h} E\left[\int_0^{T_r} \xi_s w_s P(0, s) ds\right],$$

where $P(0, s) = \exp(-\int_0^s \lambda(v) dv)$ is the probability of survival to time s. In turn, this is equivalent to a model with modified time-dependent subjective discounting rate and interest rate given by $\widehat{\beta}(t) \equiv \beta + \lambda(t)$ and $\widehat{r}(t) \equiv r + \lambda(t)$. The optimal solution is obtained by following the steps in Bodie *et al.* (2009), using this modified formulation. To retrieve the wealth process and its components, it is useful to note that the ratio ξ/a is not affected by the survival probability and the intensity of death arrival. The endogenous multiplier (A.33) is obtained by substituting optimal policies in the static budget constraint, using the formulas for the wealth components and

$$H_t = \overline{h} E_t\left[\int_t^{T_r} \xi_{t,s} w_s P(t, s) ds\right] = w_t \overline{h} \left(\int_t^{T_r} \exp(-(r - \mu^w + \sigma^w \theta)(s - t)) P(t, s) ds\right)$$

for human capital, all evaluated at $t = 0$ and solving for y^*. □

Notes

1 Total wealth is financial wealth plus human capital. It is possible for the fraction of total wealth to increase with age at the same time as the fraction of financial wealth decreases.

2 The notion of nonretirement wealth (NRW) is also sometimes employed. NRW is defined as the difference between financial wealth and retirement wealth. Accumulation wealth is the difference between total wealth and retirement wealth. It can also be expressed as the sum of human capital and NRW. For example, if human capital is $H_t = 2,000,000$, financial wealth is $X_t = 500,000$, and retirement wealth is $N_{rt}^* = 400,000$, then nonretirement wealth equals $NRW_t = 100,000$ and accumulation wealth amounts to $N_{at}^* = 2,100,000$. Accumulation wealth is $N_t^* - N_{rt}^* = 2,500,000 - 400,000$. It also equals $H_t + NRW_t = 2,000,000 + 100,000$.

3 Strategies involving leverage at a young age are advocated in Ayres and Nalebuff (2010). The analysis above shows that high leverage is optimal in the one-phase model studied here. As shown in Section 4.2, the same holds true in the presence of a retirement period.

4 Optimal pension plans can be manufactured by financial intermediaries seeking to fill the retirement needs of households. The analysis in this section identifies the optimal plans and discusses their properties.

5 Milevsky and Robinson (2005) stress the importance of taking mortality risk into account for sound asset allocation in a life cycle model.

6 An early study taking unspanned labor income into account is Viceira (2001).

References

Ayres, I. and Nalebuff, B. (2010). *Lifecycle Investing: A New, Safe, and Audacious Way to Improve the Performance of Your Retirement Portfolio*. Basic Books.

Barberis, N. (2000). "Investing for the Long Run When Returns are Predictable." *Journal of Finance* **55**, 225–264.

Benzoni, L., Collin-Dufresne, P., and Goldstein, R. S. (2007). "Portfolio Choice Over the Lifecycle When the Stock and Labor Markets Are Co-Integrated." *Journal of Finance* **62**, 2123–2167.

Betermier, S., Jansson, T., Parlour, C. A., and Walden, J. (2010). "Hedging Labor Income Risk." Working Paper, Berkeley.

Bick, B., Kraft, H., and Munk, C. (2009). "Investment, Income, and Incompleteness." Working Paper, Goethe University Frankfurt.

Bodie, Z., Detemple, J., and Rindisbacher, M. (2009). "Life Cycle Finance and the Design of Pension Plans." *Annual Review of Financial Economics* **1**, 249–286.

Bodie, Z., Detemple, J., Otruba, S., and Walter, S. (2004). "Optimal Consumption-Portfolio Choices and Retirement Planning." *Journal of Economic Dynamics and Control* **28**, 1115–1148.

Bodie, Z., Merton, R., and Samuelson, W. F. (1992). "Labor Supply Flexibility and Portfolio Choice in a Life-Cycle Model." *Journal of Economic Dynamics and Control* **16**, 427–449.

Bodie, Z., Ruffino, D., and Treussard, J. (2008). "Contingent Claims Analysis and Life-Cycle Finance." *American Economic Review* **98**, 291–296.

Bodie, Z. and Treussard, J. (2007). "Making Investment Choices as Simple as Possible but not Simpler." *Financial Analysts Journal* **63** (May-June), 42–47.

Bovenberg, L., Koijen, R., Nijman, T., and Teulings, C. (2007). "Saving and Investing Over the Life Cycle and the Role of Collective Pension Funds." *De Economist* **155**, 347–415.

Campbell, J. Y. (2006). "Houshold Finance." *Journal of Finance* **61**, 1553–1604.

Campbell, J. Y., Chacko, G., Rodriguez, J., and Viceira, L. M. (2004). "Strategic Asset Allocation in a Continuous-Time VAR Model." *Journal of Economic Dynamics and Control* **28**, 2195–2214.

Campbell, J. Y. and Viceira, L. M. (1999). "Consumption and Portfolio Decisions When Expected Returns are Time Varying." *The Quarterly Journal of Economics* **114**, 433–495.

Chai, J., Horneff, W., Maurer, R., and Mitchell, O. S. (2010). "Optimal Portfolio Choice Over the Life-Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts." Working Paper.

Cvitanic, J., Goukasian, L., and Zapatero, F. (2007). "Optimal Risk Taking with Flexible Income." *Management Science* **55**, 1594–1603.

Detemple, J. B., Garcia, R., and Rindisbacher, M. (2003). "A Monte-Carlo Method for Optimal Portfolios." *Journal of Finance* **58**, 401–446.

Detemple, J. and Serrat, A. (2003). "Dynamic Equilibrium with Liquidity Constraints." *Review of Financial Studies* **16**, 597–629.

Edwards, R. D. (2008). "Health Risk and Portfolio Choice." *Journal of Business and Economic Statistics* **26**, 472–485.

El Karoui, N. and Jeanblanc-Picque, M. (1998). "Optimization of Consumption with Labor Income." *Finance and Stochastics* **2**, 404–440.

Gomes, F. J., Kotlikoff, L. J., and Viceira, L. M. (2008). "Optimal Life-Cycle Investing with Flexible Labor Supply: A Welfare Analysis of Life-Cycle Funds." *American Economic Review* **98**, 297–303.

Heaton, J. and Lucas, D. (1997). "Market Frictions, Savings Behavior, and Portfolio Choice." *Macroeconomic Dynamics* **1**, 76–101.

Hugonnier, J. N., St-Amour, P., and Pelgrin, F. (2009). "Health and (Other) Asset Holdings." Swiss Finance Institute Research Paper No. 09-18. Available at SSRN: http://ssrn.com/abstract=1417364.

Jagannathan, R. and Kocherlakota, N. R. (1996). "Why Should Older People Invest Less in Stock than Younger People?" *Quarterly Review, Federal Reserve Bank of Minneapolis*, (Summer), 11–23.

Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case." *Review of Economics and Statistics* **51**, 247–257.

Merton, R. C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous Time Model." *Journal of Economic Theory* **3**, 373–413.

Milevsky, M. A. and Robinson, C. (2005). "A Sustainable Spending Rate without Simulation." *Financial Analysts Journal* **61**, 89–100.

Samuelson, P. A. (1969). "Lifetime Portfolio Selection by Dynamic Stochastic Programming." *Review of Economics and Statistics* **51**, 239–246.

Viceira, L. M. (2001). "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income." *Journal of Finance* **56**, 433–470.

Wachter, J. and Warusawitharana, M. (2009). "Predictable Returns and Asset Allocation: Should a Skeptical Investor Time the Market?" *Journal of Econometrics* **148**, 162–178.

Yogo, M. (2009). "Portfolio Choice in Retirement: Health Risk and the Demand for Annuities, Housing, and Risky Assets." NBER Working Paper 15, 307.

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