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A SIMPLE MODEL FOR TIME-VARYING EXPECTED RETURNS ON THE S&P 500 INDEX

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This paper presents a parsimonious, implementable model for the estimation of the short- and long-term expected rates of return on the S&P 500 stock market Index. Sufficient statistics for the expected return on the S&P 500 Index consist of the risk-free rate of interest, the option market's (priced) implied volatility on the S&P 500 Index, and a measure of the economy's wealth level.

The short- and long-term risk-free rates of interest reflect the impact of the level and slope of the risk-free term structure. The implied volatility captures a forward-looking measure of uncertainty. Utility-function assumed decreasing relative risk aversion gives rise to an increased willingness to invest in risky assets when current wealth level is "high." The model's empirical parameters are estimated using Livingston/Philadelphia Fed growth rates substituted into a dividend-discount model.

Specifically, conditioning on a one-year treasury bill and a 30-year treasury bond, $r_{S,t}$ *and* $r_{L,t}$, *the implied volatility* VIX_t *on the Index, and the* average *realized S&P 500 Index rate of return over the past five-six years* $S\&P\ 500_t / S\&P\ 500_{t-5,t-6}$*—the model generates prospective expected rates of return* μ_t *of the form:*

$$\mu_t = \begin{cases} r_{S,t} + \left(\lambda_0 + \lambda_1 \dfrac{S\&P\ 500_t}{S\&P\ 500_{t-5,t-6}}\right) VIX_t & \text{for a one-year horizon} \\ r_{S,t} + \widehat{\beta}(r_{L,t} - r_{S,t}) + a_0 + a_1 VIX_t + a_2 VIX_t \dfrac{S\&P\ 500_t}{S\&P\ 500_{t-5,t-6}} & \\ \quad + a_3 \dfrac{S\&P\ 500_t}{S\&P\ 500_{t-5,t-6}} & \text{for the long-term,} \end{cases}$$

where the current prevailing Sharpe Ratio is $\lambda_0 + \lambda_1\ (S\&P\ 500_t / S\&P\ 500_{t-5,t-6})$, *and* $\lambda_0, \lambda_1, \widehat{\beta}, a_0, a_1, a_2$, *and* a_3 *are coefficients estimated from the data.*

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In contrasting ex-ante risk-premium with their ex-post counterparts, our model reveals little information regarding the significantly more-noisy realized returns. Similar to other authors, we find that over the January 1986–December 2004 period we examine, average realized returns were substantially higher than their ex-ante counterparts.

1 Introduction

Ever since financial markets began trading equity securities, especially in the 20th century when equity markets reflected a broad swath of the economy, financial economists have struggled to understand the equity risk-premium, both domestically and across international markets. Summarizing as it does the expected rate of return on investable risky wealth in the economy, the equity risk-premium—the expected rate of return on equities in excess of the risk-free rate of interest—has constituted the compensation for equity investments; it has been studied as measures of the economy's well-being, and as a broad measure of the success of market-based economies.[1]

Estimating the contemporaneous (conditional) expected rate of return on equity markets, and properly capturing its intertemporal variation, has been the source of much modern analyses. We seek to add to the literature by addressing a market-efficient estimate of time-varying expected returns, grounded in theory but parsimonious in application. We estimate a parametric form for the Market Price of Risk, or the Sharpe Ratio, of the S&P 500 Index. In addition to short- and long-term risk-free rates of interest, which reflect the impact of slope of the risk-free term structure, our empirical implementation makes use of two forward-looking measures. The first is the option market's (priced) implied volatility on the S&P 500 Index. Second, our model accounts for realized wealth levels by modeling and estimating an assumed *decreasing relative risk aversion,* which gives rise to an increased willingness to invest in risky assets when the realized rate of return for the recent past is "high."[2]

Using a dividend-discount valuation approach applied to the Livingston/Philadelphia Fed series growth rates, we compute a time-series of expected returns. This time-series then allows us to *calibrate* our model, to obtain estimates of the time-varying, stochastic Market Price of Risk in US equity markets. This calibration entails using our parsimonious model whose sufficient statistics are the term structure of interest rates, the implied volatility on the market portfolio and a measure of recent (five-year) wealth appreciation. Thus calibrated, the model gives rise to a time-series of historical—and, by inference, *prospective*—expected rates of return.

Time-varying expected returns find at least two uses in practical applications. In the investments arena, various asset-allocation models require as input an expected rate of return on the "market portfolio." In capital-budgeting decisions, a key ingredient is the asset's expected rate of return, which uses as (one of the) inputs, the expected rate of return on the market.

The paper is structured as follows. Section 2 reviews the literature on market risk-premia and their intertemporal variation. Section 3 follows with a presentation of the theoretical models we wish to estimate. Section 4 contains the empirical tests, and their associated results. Section 5 includes robustness checks. Section 6 summarizes and concludes.

2 The US equity risk-premium

The current literature pertaining to the market risk-premium starts with the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965). In deriving a relationship between equity returns and a market-wide risk factor, these authors laid the foundation for the numerous theoretical and empirical asset pricing articles that have become the staple of the financial economics literature. One of the most heavily tested and examined aspects of the model is the notion of the equity risk-premium, or alternatively, the market price of risk or Sharpe ratio: Sharpe (1966) introduced this notion of reward to variability ratio in describing mutual funds, later denoted the Sharpe ratio or measure.[3] As described in Sharpe (1994), and similar to our work here, the *ex-ante* Sharpe Ratio measures expected returns—in contrast to the distinct, *ex-post* realized returns. This leads to several questions often debated within the literature: Estimation of the intertemporal equity price risk-premium, and the factors that determine the magnitude of the premium. In this paper, we attempt to address these issues.

While the theoretical implications of the CAPM suggest that there is positive relationship between the level of volatility and the size of the risk-premium, the empirical evidence is mixed. Campbell (1987) and Glosten *et al.* (1993) have documented a negative relationship between the conditional volatility and the risk-premium, contrary to economic theory, while Harvey (1989), and Turner, Startz and Nelson (1989) found a positive relationship. Scruggs (1998) decomposed the CAPM model into a partial relation in a two-stage estimation, and is able to explain away the negative relationship of Campbell (1987) and Glosten *et al.* (1993). Brandt and Kang (2004) attempt to resolve these differences in the literature regarding contemporaneous correlation by implementing a VAR technique. By incorporating time-varying volatility, their conclusions suggest that these differences can be explained by the conditional and unconditional correlations. Our findings suggest that it is necessary to separate, from both a modeling and empirical estimation perspectives, short- and long-term expected rates of return.

By separating returns into short- and long-term components, we are better able to estimate the expected market price of risk. Additionally, coming up with a simple measure of investor sentiment helps explain why expected returns are time varying, such as in the works of LeRoy and Porter (1981) and Fama and French (1988a, 1988b). Finding a time-varying risk-premium is consistent with the results of Ferson and Harvey (1991a) and Evans (1994). Our measure negatively relates the market risk-premium to perceived investor wealth levels: The higher the perceived wealth of the representative investor, the lower the market price of risk.

Campbell and Viceira (2005) take the time variation in expected returns a step further by suggesting that investors, particularly aggressive investors, may want to engage in market-timing (or tactical asset allocation) strategies aimed at maximizing short-term returns, based on the predictions of their return forecasting model. There remains considerable uncertainty about the degree of asset return predictability, as noted by Pástor and Stambaugh (2001), making it hard to identify the optimal market timing strategy. Attempting to capture the time variation of expected returns has been extensively examined. Using a multi-beta asset pricing model, Ferson and Harvey (1991b) incorporate risk exposure to the market as well as the interest rate and inflation to explain realized returns. Others, such as Lewellen (1999, 2004), have used explanatory variables such as the dividend yield, short rate, term-premium, Book-to-Market, and the default premium; however, in light of the statistical issues brought up in Boudoukh and Richardson (1993), Stambaugh (1999), and Ferson *et al.* (2003), the validity of the results are still in question. While we cannot claim to be able to forecast future market movements, our findings show a strong link to investor sentiment and the *expected* risk-premium. This link is especially useful for calculating expected returns and subsequent decision for asset allocation.

Finally, in their influential piece, Mehra and Prescott (1985) first documented the equity premium *puzzle*. They found that the annualized rate of return on stocks in excess of the risk-free rate is higher than can be explained by the classical theories in financial economics, by about 6.8%. Mehra (2003) further decomposed the fundamental pricing relationship and demonstrated that the growth rate of consumption does not vary enough to be consistent with the observed high equity premium. In calibrating the model, using upper-bound levels for risk-aversion generates a risk-free rate that is too high and a risk-premium that is too low. While this is troubling, Mehra (2003) points out that this is a quantitative puzzle, and that current theory is consistent. Many authors have attempted to resolve this puzzle, including Campbell and Cochrane (1999) and Constantinides (1990), by altering preferences, using incomplete markets, survivorship bias, and omitting rare events. As of yet, there is no solution to this problem. Our findings are similar to the Mehra and Prescott (1985) findings. The predicted value of the expected return is still approximately 4%–5% less than the average realized returns over the given evaluation period, consistent with the findings of Fama–French (2002). The magnitude of "noise" in financial markets is considerable: While we are able to explain as much as 50% of the variation in expected returns, we have little power to explain realized returns.

3 The models

Our model is a straightforward combination of the Gordon-Shaprio (1956)–Williams (1938) dividend-growth model and the Sharpe–Lintner security market line. The alternate models we postulate and test vary by their:

1. Interpretation of the Livingston/Phildelphia Fed short- and long-term growth rates.
2. Use of the VIX implied volatility value in terms of a biased-vs.-unbiased predictor of realized volatility.
3. Examination of the long- vs. one-year expected rates of return on the S&P 500.

3.1 A short-*term expected return model*

Consider first the interpretation of the one-year Livingston growth rate $g_{S,t}$ as both a short-term dividend growth rate as well as the capital-gains component of the S&P 500. In that case, we have

$$\mu_t = \frac{D_{0,t}(1+g_{S,t})}{P_t} + g_{S,t}, \qquad (1)$$

where

$\mu_t =$ Expected/required rate of return on the equity asset as of date t.
$P_t =$ Price of equity asset at date t.
$D_{0,t} =$ Dividends payable over the past 12 months, as of date t, and assumed to satisfy the relationship $D_{1,t} = D_{0,t}(1+g_{S,t})$.
$g_{S,t} =$ One-year dividend growth rate as of date t *and* capital-gains forecast over the next twelve months.

From the capital market line, we have, at date t,

$$\mu_t = r_{S,t} + \lambda_t\sigma_t, \qquad (2)$$

where

$\lambda_t =$ Market Price of Risk, or Sharpe Ratio, as of date t.
$r_{S,t} =$ One-year Treasury Bill rate of interest as of date t.
$\sigma_t =$ Equity asset's annualized volatility, as of date t.

Assuming that $\text{VIX}_t = \sigma_t$, we can equate Eq. (1) to Eq. (2) to yield

$$\frac{D_{0,t}}{P_t}(1 + g_{S,t}) + g_{S,t} = r_{S,t} + \lambda_t \text{VIX}_t,$$

which can be solved for λ_t:

$$\frac{(D_{0,t}/P_t)(1 + g_{S,t}) + g_{S,t} - r_{S,t}}{\text{VIX}_t} = \lambda_t. \quad (3)$$

Assuming the observability of $\{D_{0,t}, P_t, g_{S,t}, r_{S,t}, \text{VIX}_t\}$, the LHS of (3) is an observable datum.

Whereas the LHS of (3) is unquestionably *observable*, we would like to relate the market price of risk to utility theory, and in particular, to the commonly assumed and intuitively plausible attribute of decreasing relative risk aversion. To do this, we provide a functional, estimable and empirically testable form for λ_t by relying on financial theory for guidance: Assuming the representative investor exhibits the attribute of decreasing relative risk aversion, λ_t can be modeled as a function of investable (presumably, per capita) wealth. We proxy for that per capita wealth level by making λ_t a function of the past realized return on the S&P 500:

$$\lambda_t = \begin{cases} \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \qquad + \lambda_2 \left(\dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}\right)^2, \end{cases} \quad (4)$$

where $\lambda_0 > 0, \lambda_1 < 0$, and $\lambda_2 > 0$ are coefficients to be estimated, for some value of T. The postulated negative sign of λ_1 reflects our assumption that, as $\text{S\&P } 500_t/\text{S\&P } 500_{t-T}$ increases and investors feel "wealthier," their required compensation per unit standard deviation declines.[4] When modeled in its *quadratic* form, the postulated $\lambda_2 > 0$ reflects the standard assumption of the second derivative sign differing from the first—i.e., a *decreasing* marginal effect of wealth on the Sharpe Ratio.

Thus, the empirical relationships we will test combine the two Eqs. (3) and (4) to produce[5]:

$$\frac{(D_{0,t}/P_t)(1 + g_{S,t}) + g_{S,t} - r_{S,t}}{\text{VIX}_t} = \begin{cases} \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \qquad + \lambda_2 \left(\dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}\right)^2. \end{cases} \quad (5)$$

Interpreting this variant of the model as a test of a short-term, one-year Sharpe ratio—in contrast to the models in Sections 3.2 through 3.4 below—we will seek to explore how well the simple model of the form (5) explains the time-varying changes in our proxy for expected returns, whether the estimated signs and magnitudes of $\{\lambda_0, \lambda_1, \lambda_2\}$ conform to economic intuition. Given the results obtained in (5), the quadratic model's implied *ex-ante* expected rate of return on the S&P is given by

$$\mu_t = r_{S,t} + \left[\lambda_0 + \lambda_1 \frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} + \lambda_2 \left(\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}\right)^2\right] \text{VIX}_t, \quad (6)$$

with the linear model obtaining with its respective $\{\lambda_0, \lambda_1\}$ parameters.

For completeness, we will contrast those results with analogous results for the intertemporal variation in *realized* returns.[6]

3.2 Two-growth rate model

In taking explicit notice that post-June 1990 the Livingston data provide both a one-year growth GDP growth-rate forecast g_S *as well as* a ten-year forecast g_L, the next model takes explicit cognizance of these two growth rates. Specifically, we now interpret these two growth rates as a one-year and infinite-maturity dividend growth rates, giving rise to the valuation model

$$P_t = \frac{D_{1,t}}{\mu_t - g_{L,t}} = \frac{D_{0,t}(1+g_{S,t})}{\mu_t - g_{L,t}},$$

which can be sequentially inverted to solve for $\lambda_t \equiv (\mu_t - r_{S,t})/\mathrm{VIX}_t$:

$$\mu_t = \frac{D_{0,t}(1+g_{S,t})}{P_t} + g_{L,t} \tag{7}$$

$$\lambda_t = \frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t}}{\mathrm{VIX}_t}. \tag{8}$$

For the post-June 1990 for which the full set of data (i.e., $g_{L,t}$) are available, the linear and quadratic testable versions of (8) are, respectively, given by:

$$\frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t}}{\mathrm{VIX}_t} = \begin{cases} \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \lambda_0 + \lambda_1 \dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} \\ \qquad + \lambda_2 \left(\dfrac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}\right)^2. \end{cases} \tag{9}$$

The linear and quadratic *ex-ante* expected returns are once again given by expression (6), with the respective $\{\lambda_0, \lambda_1, \lambda_2\}$ parameters having been estimated using (9).

3.3 *Two-growth rate, "term structure-adjusted" model*

In beginning to make the transition to a long-term expected return model, consider the issue of the *slope* of the term structure of interest rates, of which investors are presumably aware and one which they may well take into account in establishing equity required expected rates of return.[7] Thus, consider a re-formulation of (9) which permits us to elicit from the data the combination of short- and long-term rates which investors contemplate.

Proceeding from (7), we have the risk-premium equation

$$\begin{aligned} \mu_t - r_{S,t} - \beta(r_{L,t} - r_{S,t}) &= (D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t} \\ &\quad - \beta(r_{L,t} - r_{S,t}), \end{aligned} \tag{10}$$

where the long maturity can be $L \in \{10, 30\}$. Thus, by estimating which β best explains the market price of risk, we can infer how investors "optimally" choose their target interest rate (and maturity).[8] We proceed to estimate β and infer its implication, in two steps:

1. If we divide the LHS of (10) by VIX_t, we obtain a term structure-adjusted measure of the market price of risk λ_t. Thus, we have

$$\lambda_t \equiv \frac{\mu_t - r_{S,t} - \beta(r_{L,t} - r_{S,t})}{\mathrm{VIX}_t} = \frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t} - \beta(r_{L,t} - r_{S,t})}{\mathrm{VIX}_t}. \tag{11}$$

Having in (4) modeled the market price of risk as linearly and quadratically dependent on the wealth accumulation factor $\text{S\&P } 500_t/\text{S\&P } 500_{t-T}$, we can apply that model to the LHS of (11):

$$\frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t} - \beta(r_{L,t} - r_{S,t})}{\mathrm{VIX}_t} = \lambda_0 + \lambda_1 \frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}. \tag{12}$$

2. To obtain an empirical estimate of β in the linear model, we first transpose it to the RHS of (12):

$$\frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t}}{\mathrm{VIX}_t} = \lambda_0 + \lambda_1 \frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} + \beta \frac{r_{L,t} - r_{S,t}}{\mathrm{VIX}_t}. \tag{13}$$

Substituting the estimated parameter $\widehat{\beta}$ from (13) back into (12) produces the estimated $\{\lambda_0, \lambda_1\}$ parameters.

The estimated coefficients obtained from constraining $\beta = \widehat{\beta}$ are of course identical to the ones obtained in the regression (13), but their *interpretation* is now different: We have elicited the optimal term structure adjustment to the expected rate of return μ_t. Specifically, with this information in hand, the quadratic model's prospective date t expected rate of return is given by

$$\mu_t = r_{S,t} + \beta(r_{L,t} - r_{S,t}) + \left[\lambda_0 + \lambda_1 \frac{\text{S\&P }500_t}{\text{S\&P }500_{t-T}} + \lambda_2 \left(\frac{\text{S\&P }500_t}{\text{S\&P }500_{t-T}}\right)^2\right] \text{VIX}_t.$$

3.4 *A long-term expected rate of return model: two-growth rate, term structure adjustment, stochastic volatility model*

In seeking a long-term expected return model, we recognize the short-term, one-month nature of VIX. In accordance with the previous work of Doran and Ronn (2006) and others, for longer-term-periods, especially those exceeding one year, investors are cognizant of the well-documented mean-reversion in VIX. This final model will take cognizance of this mean-reversion.

Incorporating stochastic volatility helps capture the long-run component of volatility, as well as the level to which volatility reverts. In doing so, the weight w on current versus long-run volatility can be determined endogenously. Accounting for mean-reversion requires the adjustment of the volatility variable to

$$w\text{VIX}_t + (1-w)\sqrt{\theta}, \tag{14}$$

instead of dividing through by VIX_t. As shown in Doran and Ronn (2006), the stochastic model for volatility changes is given by

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \xi\sigma_t dz, \tag{15}$$

where κ is the speed of mean-reversion, θ is the level to which volatility reverts, and ξ is the variation in volatility. The relationship between κ and w in (14) is given by the weighting $w = \exp\{-\kappa T\}$ for whatever T value investors have "in mind." Whereas this model has been empirically verified, for our purpose here we need not discretize (15), but rather use its analytical *implication* (14).

With the expression (14) replacing VIX_t in (11), the expression (11) becomes

$$\begin{aligned}\lambda_t &\equiv \frac{\mu_t - r_{S,t} - \beta(r_{L,t} - r_{S,t})}{w\text{VIX}_t + (1-w)\sqrt{\theta}}\\ &= \frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t} - \beta(r_{L,t} - r_{S,t})}{w\text{VIX}_t + (1-w)\sqrt{\theta}}\\ &= \lambda_0 + \lambda_1 \frac{\text{S\&P }500_t}{\text{S\&P }500_{t-T}} + \beta \frac{r_{L,t} - r_{S,t}}{w\text{VIX}_t + (1-w)\sqrt{\theta}}.\end{aligned} \tag{16}$$

Since the parameters $\{w, \theta\}$ are unknown, the estimation procedure for (16) is altered. Multiplying through by the "blended" volatility measure $w\text{VIX}_t + (1-w)\sqrt{\theta}$, we have

$$\begin{aligned}&\frac{D_{0,t}(1+g_{S,t})}{P_t} + g_{L,t} - r_{S,t}\\ &\quad = \left(\lambda_0 + \lambda_1 \frac{\text{S\&P }500_t}{\text{S\&P }500_{t-T}}\right) \times [w\text{VIX}_t + (1-w)\sqrt{\theta}] + \beta(r_{L,t} - r_{S,t})\\ &\quad = \lambda_0(1-w)\sqrt{\theta} + \lambda_0 w\text{VIX}_t + \lambda_1 w\text{VIX}_t \frac{\text{S\&P }500_t}{\text{S\&P }500_{t-T}}\end{aligned}$$

$$+ \lambda_1(1-w)\sqrt{\theta}\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}$$
$$+ \beta(r_{L,t} - r_{S,t})$$
$$\equiv a_0 + a_1\text{VIX}_t + a_2\text{VIX}_t\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}$$
$$+ a_3\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}} + \beta(r_{L,t} - r_{S,t}). \tag{17}$$

The regression formulation in (17) merits several comments:

1. The dependent variable is no longer λ_t, but rather the term-structured adjusted risk-premium.
2. The regression is not unconstrained, as there is a linkage amongst the regression parameters $\{a_1, a_2, a_3, a_4\} : a_1/a_2 = \lambda_0/\lambda_1 = a_0/a_3$.
3. The regression formulation does not permit the distinct, separate identification of all variables of interest $\{\lambda_0, w, \theta, \lambda_1, \beta\}$—the regression is in that sense underidentified—but it does permit their estimation in the form they are required in order to calculate the expected return $\mu_t : \{a_0, a_1, a_2, a_3, \beta\}$. To see this, note that the (linear model's) expected return μ_t is now given by

$$\mu_t = r_{S,t} + \widehat{\beta}(r_{L,t} - r_{S,t}) + \lambda_0(1-w)\sqrt{\theta}$$
$$+ \lambda_0 w\text{VIX}_t + \lambda_1 w\text{VIX}_t\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}$$
$$+ \lambda_1(1-w)\sqrt{\theta}\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}$$
$$\equiv r_{S,t} + \widehat{\beta}(r_{L,t} - r_{S,t}) + a_0 + a_1\text{VIX}_t$$
$$+ a_2\text{VIX}_t\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}$$
$$+ a_3\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}.$$

4. The R^2 of the regression formulation (17) is not meaningful, in that it includes the regressor $r_{L,t} - r_{S,t}$ on its *RHS.* Rather, using $\widehat{\beta}$ as determined from regression (17), what *is* meaningful is the R^2 of $(D_{0,t}/P_t)(1 + g_{S,t}) + g_{L,t} - r_{S,t} - \widehat{\beta}(r_{L,t} - r_{S,t})$ regressed on the remaining RHS variables $\{\text{VIX}_t, \text{VIX}_t \cdot \text{S\&P } 500_t/\text{S\&P } 500_{t-T}, \ \text{S\&P } 500_t/\text{S\&P}500_{t-T}\}$.

3.5 Realized returns

In order to properly compare and contrast the results obtained under (5), we will perform analogous results for realized returns, which will replace expected returns in the LHS of (5): For monthly annualized *realized* returns given by $R_t^m \equiv [(\text{S\&P}_{t+1/12} + D_{t+1/12})/\text{S\&P}_t]^{12} - 1$, we will perform tests of the type

$$\frac{R_t - r_t}{\text{VIX}_t} = \lambda_0 + \lambda_1\frac{\text{S\&P } 500_t}{\text{S\&P } 500_{t-T}}.$$

This process will then be repeated for the three models outlined in Eqs. (9), (13), and (17).

4 Empirical results

4.1 Data

To derive the expected Sharpe ratio, or λ, daily prices of the S&P 500 and the VIX/VXO index are collected from CRSP and the CBOE, respectively, from January 1986 through December 2004. VIX is a key measure of market expectations of near-term volatility conveyed by S&P 500 stock index option prices. Since its introduction, VIX has been considered the premier barometer of investor sentiment and prospective market volatility. The VXO index is substituted up until 1990 since there are no observations of VIX prior to that data. This is done in part to infer estimates of expected λ prior to the October 1987 crash.

The data for the dividend yield come from Standard and Poors dividend bulletin, which reports quarterly dividends. To construct the dividend yield, the sum of the prior four dividends is calculated prior to dividing by the current level of S&P 500.[9] For the short-term risk-free rate, daily one-year Treasury bill yields are collected from the Federal Reserve. For the long-term rate, the 10-year T-note yields are used. In addition, a mixture of the 30-year and 20-year T-bill rate are collected to provide an alternative measure.[10]

To proxy for the long-term and short-term dividend growth rates, the Livingston Survey is used, which provides a semi-annual GDP forecast from economists from industry, government, banking, and academia. The feasibility of using the Livingston data as an accurate forecasting tool has been examined by many authors, noted in the summary piece by Croushore (1997).[11] In deriving the implied short-term dividend/capital gains growth rate, the forecast for the current year and next year of nominal GDP are used to construct a one-year expected growth rate.[12] Since the data's frequency is semi-annual, this potentially gives rise to estimation problems if the growth forecast is not constant over the semi-annual period. This potential measurement error will be accounted for in robustness checks presented in Section 5 and presented in detail in the Appendix. For the long-term growth rate, the Livingston data provides a 10-year forecast. However, one particular drawback in using the 10-year forecast is that the data only spans June 1990 through December 2004, reducing the number of observations and eliminates the October 1987 crash period. The summary statistics for all the data are provided in Table 1.

4.2 *Estimation procedure for short-term expected returns*

To explain the time-variation in the expected Sharpe ratio, a measure of perceived wealth is constructed. Our proxy for investor sentiment is the ratio of the current level of the S&P 500 to some prior level of the index. Our hypothesis holds that there is a negative relationship between this ratio and the expected risk-premium. If the intercept term is positive, as should be expected, then the higher this ratio is, the lower the expected rate of return required to satisfy aggregate investor preferences.

Table 1 Summary statistics.

Variable	Obs.	Mean	St.dev.	Min	Max
Dividend yield	4795	2.4%	0.8%	1.1%	4.0%
VIX	4795	20.6%	7.6%	9.3%	150.2%
Treasury yield one-year	4795	5.1%	2.1%	0.9%	9.9%
10-Year	4795	6.5%	1.5%	3.1%	10.2%
30-Year	4795	6.8%	1.5%	3.1%	10.3%
Growth rate					
one-year	4795	5.6%	1.0%	2.7%	7.3%
10-year	3679	5.9%	1.0%	5.0%	6.7%
$\frac{\text{S\&P}_t}{\text{S\&P}_{t-5,\,t-6}}$	4795	1.79	0.55	0.76	3.21
$\frac{\text{S\&P}_{t,\,t-1}}{\text{S\&P}_{t-5,\,t-6}}$	4795	1.89	0.54	0.86	3.30

This table reports the summary statisitics of the variables from January 1986 through December 2004. The number of observations, 4795, assumes that the previously published growth forecast prevails until the publication of its successor in six-months' time.

Recalling the hypotheses stated in Eq. (5)

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{S,t}-r_{S,t}}{\text{VIX}_t} = \begin{cases} \lambda_0+\lambda_1 x_t+e_t \\ \lambda_0+\lambda_1 x_t+\lambda_2 x_t^2+e_t, \end{cases}$$

where

x_t is alternately defined to be $\text{S\&P}_t/\text{S\&P}_{t-5,t-6}$ or $\text{S\&P}_{t,t-1}/\text{S\&P}_{t-5,t-6}$,

$\text{S\&P}_{t-5,t-6}$ is the *average value* of the S&P index over the time period between five and six years ago,

$\text{S\&P}_{t,t-1}$ is the *average value* of the S&P index over the last year,

e_t is the regression's error term, driven by the RHS *approximation* of the wealth effect by the linear $\lambda_0 + \lambda_1 x_t$ or quadratic forms $\lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2$.

This formulation provides an intuitive representation of investors perceptions. It is possible that investors have shorter, or perhaps longer, time horizons, but going back five to six years captures a limited-memory aspect in that the market is aware of past market highs and lows, but its memory is finite.[13]

The mean value over the period for $\text{S\&P}_t/\text{S\&P}_{t-5,t-6}$ is 1.79, equivalent to an annual return of 12.4%. The maximum and minimum values are 3.21 (26.2%) and .76 (−5.31%), respectively. In testing the hypothesis (5), a quadratic formulation is included to capture decreasing marginal wealth effects.

In reporting the results of Table 2, we are cognizant of price-level variables on the LHS, in the form of $1/P_t$, and RHS, in the form of $\text{S\&P}_t \equiv P_t$, of the regression. We take two steps to address this issue:

1. We compute the correlation between $(1/P_t, P_t)$ and found it to equal −.1529. The implied R^2 of such a correlation, $(-.1529)^2 = 2.3\%$, is significantly lower than the 48.2% we report in the regression results. Thus, the results we report are not simply the results of spurious correlations between the LHS and RHS of the regression.
2. More importantly, we seek to reduce the impact of the spot equity price P_t by replacing the numerator of $\text{S\&P}_t/\text{S\&P}_{t-5,t-6}$ by

Table 2 Estimation of Model 3.1: One-growth rate, Short-term interest rate, VIX model.

	Model 1: $x_t = \frac{\text{S\&P}_t}{\text{S\&P}_{t-5,t-6}}$				Model 2: $x_t = \frac{\text{S\&P}_{t,t-1}}{\text{S\&P}_{t-5,t-6}}$			
	All obs.		Semi-annual obs.		All obs.		Semi-annual obs.	
λ_1	−0.162	−0.253	−0.174	−0.268	−0.179	−0.313	−0.18	−0.346
	(84.36)**	(22.28)**	(6.48)**	(1.71)	(67.17)**	(20.90)**	(6.17)**	(2.10)*
λ_2		0.023		0.024		0.036		0.045
		(8.44)**		(0.61)		(9.08)**		(1.02)
λ_0	0.469	0.552	0.494	0.576	0.488	0.604	0.487	0.627
	(104.36)**	(50.11)**	(9.34)**	(3.95)**	(97.74)**	(44.11)**	(8.96)**	(4.26)**
Obs.	4795	4795	32	32	4795	4795	32	32
R^2	47.1%	48.2%	58.3%	59.2%	48.5%	49.3%	57.0%	57.6%

Robust t-statistics in parentheses.
*significant at 5%; **significant at 1%.
This Table reports the parameter estimates of the OLS regression of Eq. (5) shown below. The market price of risk is inferred from the dividend-growth model specified in Eq. (1) and is regressed on two x_t proxies of "perceived wealth": (1) $\text{S\&P}_t/\text{S\&P}_{t-5,t-6}$ and (2) $\text{S\&P}_{t,t-1}/\text{S\&P}_{t-5,t-6}$. Each panel contains linear and quadratic regressions. We report regressions including *all* observations, as well as observations separated by the distinct growth rates reported every six-months. t-stats are listed in parentheses.

$$\text{Linear relationship:} \quad \frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{S,t} - r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + e_t$$

$$\text{Quadratic relationship:} \quad \frac{(D_{0,t}/P_t)(1+g_{S,t}) + g_{S,t} - r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2 + e_t.$$

its latest-year *average*: $S\&P_{t,t-1}/S\&P_{t-6}$. This altered the correlation minutely, to -0.17—the computed correlation of $1/P_t$ with the RHS'S $S\&P_{t,t-1}/S\&P_{t-5,t-6}$ is even lower at -0.0696—and as reported in Table 2, resulted in only minute changes in the estimated coefficients and explanatory power R^2.

The results reported in Table 2 confirm the positive intercept and negative slope coefficients consistent with our hypothesis. Both slope and intercept terms are statistically significant, with high R^2s. This holds regardless of sample size. Figure 1 demonstrates the relationship through time between the predicted values, $\tilde{\lambda}$, using the regression coefficient estimates, and the expected values, $E(\lambda_t)$, from Eq. (3). As can be seen the model has tremendous explanatory power in capturing the short-term variation in expected returns.

The mean expected λ is .179 for mean values of $S\&P_t/S\&P_{t-5,t-6}$ and .149 for mean values of

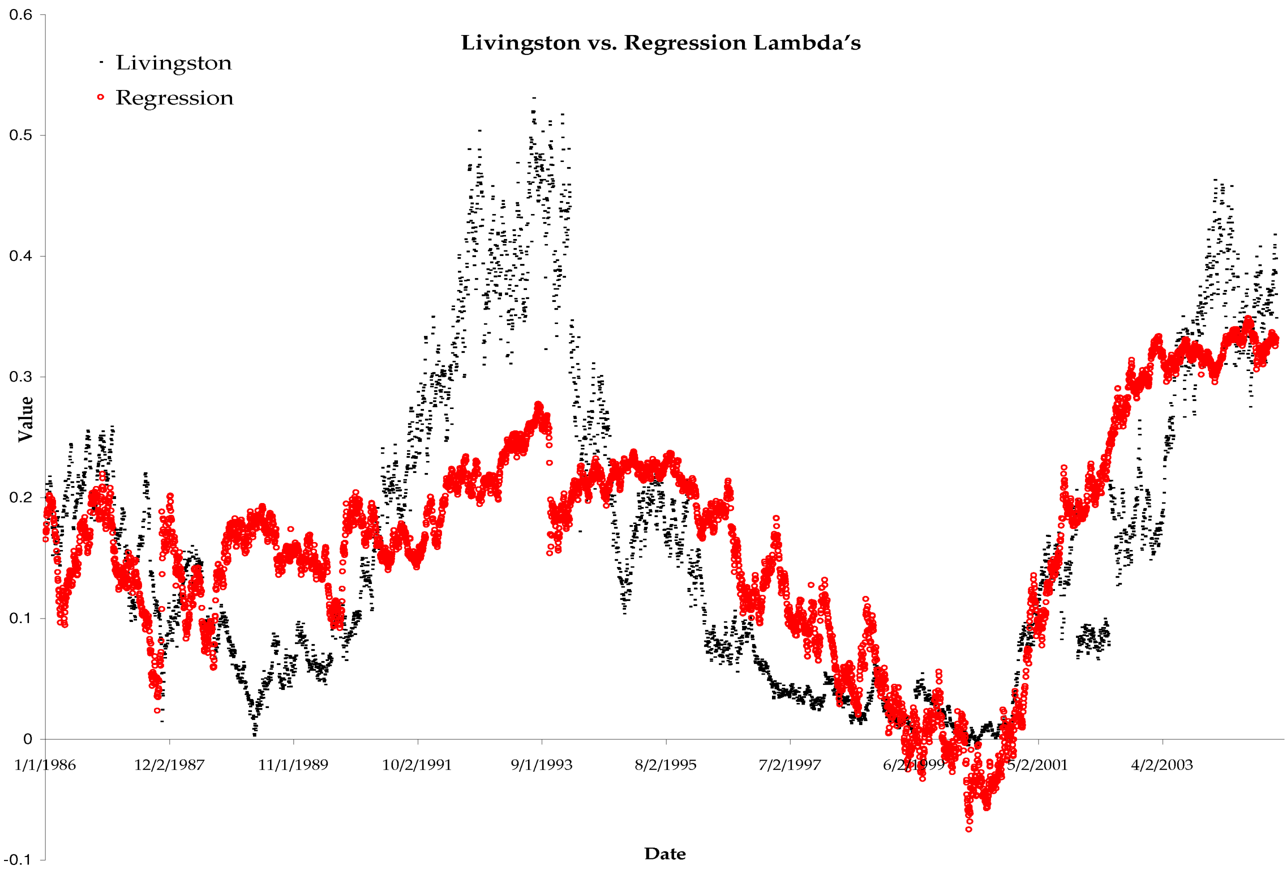

Figure 1 Sharpe ratios using the short-term model 3.1.

Figure 1 demonstrates the relationship over January 1986 to December 2004 between the Livingston/Philadelphia Fed Sharpe ratio, $[(D_{0,t}/P_t)(1+g_{S,t})+g_{S,t}-r_t]/\mathrm{VIX}_t$, and the regression-predicted Sharpe ratio, $\lambda_0+\lambda_1 x_t = 0.469 - .162x_t$, using coefficient estimates from regression Eq. (5). The control variable in this case is $x_t = \frac{S\&P_t}{S\&P_{t-5,t-6}}$.

$$\text{Livingston/Philadelphia Fed Sharpe Ratio} = \frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{S,t}-r_t}{\mathrm{VIX}_t}$$

$$\text{Regression-predicted Sharpe Ratio} = 0.469 - .162x_t.$$

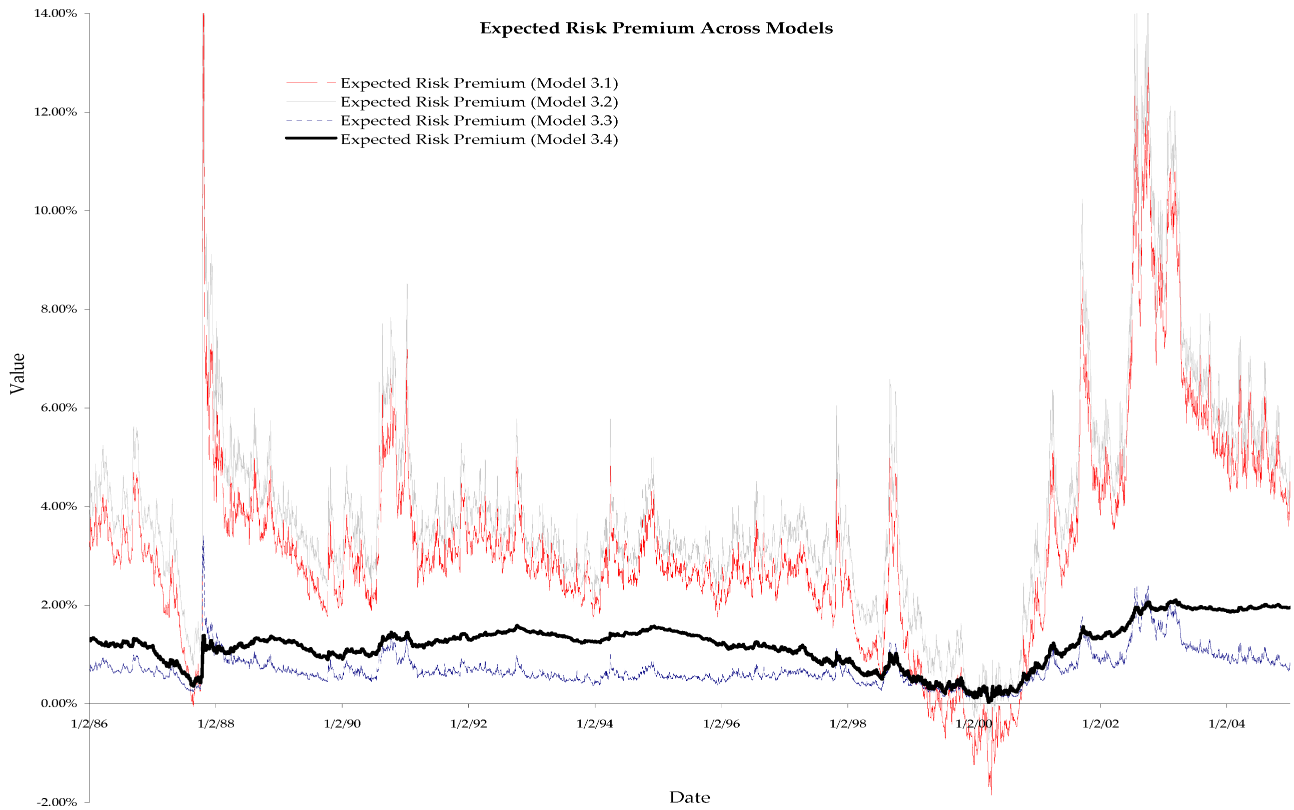

Figure 2 Short- and long-term expected risk premiums.

Figure 2 demonstrates the relationship over January 1986 to December 2004 over the predicted value for the expected risk-premium using the four models. (Models 3.1 and 3.2 differ only in the parameter values of the estimated λs, as a consequence of the two models' estimation uses in 3.2 or neglects in 3.1 the *two*-growth rate model.) For all models, $x_t \equiv \frac{\text{S\&P}_t}{\text{S\&P}_{t-5,t-6}}$.

$$\text{Models 3.1 and 3.2:} \quad \mu_t = r_{S,t} + (\lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2)\text{VIX}_t$$

$$\text{Model 3.3:} \quad \mu_t = r_{S,t} + \beta(r_{L,t} - r_{S,t}) + (\lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2)\text{VIX}_t$$

$$\text{Model 3.4:} \quad \mu_t = r_{S,t} + \widehat{\beta}(r_{L,t} - r_{S,t}) + a_0 + a_1 \text{VIX}_t + a_2\text{VIX}_t x_t + a_3 x_t.$$

$\text{S\&P}_{t,t-1}/\text{S\&P}_{t-5,t-6}$. The quadratic results for the two measures of investor sentiment produce mean λs of .173 and .133. Using these expected values, and multiplying by VIX, we derive estimates for the time-varying expected risk-premia. As shown in Figure 2, the expected risk-premia varied from a high of 14.3% on 10/21/87 (2 days after the crash) to a low of −1.8% on 4/11/00 (12 trading days after the S&P 500 high watermark of 1527.46). On average the estimated expected risk-premia over this period is 3.19%. Combined with the mean short-term risk-free rate corresponds to a expected rate of return of 8.34%, which is 4.3% less than the mean realized return on the S&P 500 calculated over the same period.[14]

Examining the maximum and minimum values for $\text{S\&P}_t/\text{S\&P}_{t-5,t-6}$, results in Sharpe ratios of −.043 and .313. The −.043 negative value is an indication that at high perceived wealth levels, investors are willing to accept negative equity risk-premium, an apparent manifestation of risk-seeking behavior. Such a phenomenon is not accommodated in our standard utility functions: Standard utility functions permit investors to devote an increasing proportion of their wealth to the risky asset as their wealth increases, but they do not give rise to risk-*seeking* behavior. Of course, it may well be that, in practice, a sufficiently long run of positive returns on the S&P does indeed give rise to seemingly risk-seeking behavior: Such a negative equity risk-premium may, in other words, constitute a sufficient condition for a "bubble."

4.3 *Estimation of two-growth rate model*

The previous estimation results are predicated on using only short-term interest and growth rates to arrive at an expected market price of risk. It more likely that multiple rates are used in deriving current prices, as is demonstrated in Eq. (9). As outlined in Section 3.2, incorporating both a long- and short-term growth rate will change the calculation of the expected market price of risk, but will not change the econometric specification. Thus, the regression outlined in Eq. (5) is used to test the multiple growth rate specification.

Since there appears to be little difference in the results in using multiple definitions of x_t, all remaining estimation will use $S\&P_t/S\&P_{t-5,t-6}$ as the proxy for per capita wealth level. The results of the regressions on the new dependent variable are shown in Table 3. Incorporating the long-term growth rate into the model provides about a 4% increase in performance, with an inferred mean value for the expected market price of risk equal to .199. This value is higher than the estimate from model 3.1, but not surprising given that the long-term growth rate is higher on average than the short-term rate. However, the small relative performance improvement suggests that including multiple growth rates has little added explanatory power.[15] This is demonstrated in Figure 2, where the expected risk-premium inferred from Model 3.1 and Model 3.2 exhibit almost identical time variation.

Table 3 Estimation of Model 3.2: Two-growth rate, short-term interest rate, VIX model.

	All obs.		Semi-annual obs.	
	Linear model	Quadratic model	Linear model	Quadratic model
λ_1	−0.162	−0.111	−0.176	−0.223
	(93.24)**	(9.08)**	(6.24)**	(1.29)
λ_2		−0.013		0.012
		(4.32)**		(0.28)
λ_0	0.506	0.46	0.501	0.542
	(117.81)**	(41.07)**	(9.16)**	(3.43)**
Obs.	3679	3679	27	27
R^2	51.27%	51.43%	61%	61%

Robust t-statistics in parentheses.
*significant at 5%; **significant at 1%.
This Table reports the parameter estimates of the OLS estimation of Model 3.2 given in Eq. (9). The dependent variable is the market price of risk computed using both one-year $g_{S,t}$ and ten-year $g_{L,t}$ growth rates. The estimation period is from June 1990 through December 2004. $x_t = \frac{S\&P_t}{S\&P_{t-5,t-6}}$. t-stats are listed in parentheses.

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{VIX_t} = \lambda_0 + \lambda_1 x_t + e_t$$

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{VIX_t} = \lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2 + e_t.$$

4.4 *Estimation of two-growth rate, "term structure-adjusted" model*

By allowing for multiple growth rates, but using only a short-term risk-free rate, we are able to isolate the impact of multiple growth rates on model performance. While the results are interesting, the model is misspecified since a long-term growth rate should be accompanied with a long-term risk-free rate. As shown in Section 3.3, the resulting model incorporates both the long- and short-term risk-free rate, but requires additional estimation.

The following regression will capture the weight placed on both the long ($r_{L,t}$)- and short-term ($r_{S,t}$)

risk-free rates

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{\mathrm{VIX}_t} = \begin{cases} \lambda_0+\lambda_1 x_t+\beta\dfrac{r_{L,t}-r_{S,t}}{\mathrm{VIX}_t}+e_t \\ \lambda_0+\lambda_1 x_t+\lambda_2 x_t^2+\beta\dfrac{r_{L,t}-r_{S,t}}{\mathrm{VIX}_t}+e_t. \end{cases}$$

Estimating β reveals the sensitivity to the term-premium investors incorporate within their expectations. To estimate β, the following estimations use two proxies for the long- and short-term rates. First, the 10-year rate and one-year Treasury rates are used as the initial long- and short-term rate, respectively. Second, the long-term rate will encompass the 30-year rate up through 2002, and then 20-year rate after that, to capture additional term-premium that is not contained in the 10-year note.

The estimation procedure requires two steps. The model's first stage, outlined in Eq. (13), is estimated to capture the coefficient estimates for β. After $\widehat{\beta}$ is estimated, the second stage is estimated, where the dependent variable is adjusted by $\widehat{\beta}(r_{L,t}-r_{S,t})/\mathrm{VIX}_t$ as shown in Eq. (12). This second stage is run to determine the regression's explanatory power independent of the information in the risk-free rate slope $(r_{L,t}-r_{S,t})$. The results of each estimation using the three proxies for the long- and short-term rates are shown in Table 4.

What is immediately obvious is the performance of the model when a term-structure adjustment is incorporated. It is interesting to note that regardless of the proxy chosen for the long-term rate, the coefficient on the term-structure spread is greater than one. This suggests investors are extremely sensitive to the spread between long and short rates, with higher spreads inducing greater required rates of return for equivalent levels of risk.

The second-stage results test the original model, but the dependent variable is adjusted to account for the term-premium. While the model performance is moderate, the coefficients on the wealth-premium have maintained some explanatory power, sign direction, and significance. The resulting expected market price of risk after controlling for the yield spread suggests a mean value of .059. However, as shown in Figure 2, the expected risk-premium demonstrates substantially less variation once the term-premium has been accounted for. This finding is entirely intuitive: The long-term risk-premium should indeed be relatively *in*sensitive to short-term fluctuations.

4.5 Incorporating stochastic volatility in the estimation of a long-term expected risk-premium

The final estimation accounts for the mean-reversion in volatilities. As such, all prior estimations have used the current level of VIX for volatility. Estimating the model in Section 3.4 can be done in two ways. It is possible to first estimate the mean-reverting parameters as given in Doran and Ronn (2006), and then use the resulting mean-reverting parameters to find the market price of risk. However, given our model, it is possible to estimate both parameters without a separate estimation. Given Eq. (17) we have the first-stage regression of the form:

$$\frac{D_{t,0}(1+g_{S,t})}{P_t}+g_{L,t}-r_{S,t} = a_0+a_1\mathrm{VIX}_t+a_2\mathrm{VIX}_t x_t+a_3 x_t + \widehat{\beta}(r_{L,t}-r_{S,t})+e_t,$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \lambda_0(1-w)\sqrt{\theta} \\ a_1 &= \lambda_0 w \\ a_2 &= \lambda_1 w \\ a_3 &= \lambda_1(1-w)\sqrt{\theta}. \end{aligned}$$

Similar to the previous section, after solving for the coefficient on the term-premium, the dependent

Table 4 Estimation of Model 3.3: Two-growth rate, blended interest rates, VIX model.

	r_L =10 year Bond and r_S = one year Bill				r_L =30 year Bond and r_S = one year Bill			
	1st stage	2nd stage	1st stage	2nd stage	1st stage	2nd stage	1st stage	2nd stage
β	1.654		1.659		1.449		1.448	
	(96.33)**		(96.58)**		(124.76)**		(126.46)**	
x_t	−0.032	−0.032	−0.059	−0.059	−0.024	−0.024	−0.010	−0.010
	(20.19)**	(36.24)**	(8.83)**	(9.01)**	(18.11)**	(31.51)**	(1.76)	(1.79)
x_t^2			0.007	0.007			−0.004	−0.004
			(4.37)**	(4.28)**			(2.62)**	(2.58)**
λ_0	0.121	0.119	0.143	0.143	0.081	0.081	0.069	0.069
	(27.68)**	(59.46)**	(19.39)**	(23.70)**	(22.16)**	(48.54)**	(10.76)**	(13.17)**
Obs.	3679	3679	3679	3679	3679	3679	3679	3679
R^2	89.7%	**16.1%**	89.7%	**16.5%**	93.0%	**13.9%**	93.0%	**14.1%**

Robust t-statistics in parentheses.
*significant at 5%; ** significant at 1%.
This table reports the parameter estimates of the two-stage estimation of Model 3.3 given in Eqs. (12) and (13). In the first-stage regression, the dependent variable is adjusted by $-\beta(r_L - r_S)/\text{VIX}_t$. The dependent variable is the two-growth rate market price of risk, $\left[\left(D_{0,t}/P_t\right)\left(1+g_{S,t}\right)+g_{L,t}-r_{S,t}\right]/\text{VIX}_t$. The estimation period is from June 1990 through December 2004. $x_t = \text{S\&P}_t/\text{S\&P}_{t-5,t-6}$. t-stats are listed in parentheses.
First stage:

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + \beta\frac{r_{L,t}-r_{S,t}}{\text{VIX}_t} + e_t$$

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2 + \beta\,\frac{r_{L,t}-r_{S,t}}{\text{VIX}_t} + e_t.$$

Second stage:

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{\text{VIX}_t} - \widehat{\beta}\,\frac{r_{L,t}-r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + e_t$$

$$\frac{(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}}{\text{VIX}_t} - \widehat{\beta}\,\frac{r_{L,t}-r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + \lambda_2 x_t^2 + e_t,$$

As explained in the text, the R^2 for the first-stage is not meaningful, but it is for the second-stage's bold-faced values.

variable is adjusted by $\widehat{\beta}(r_{L,t} - r_{S,t})$, and the regression is re-run. In addition, there is one nonlinear constraint,

$$\frac{a_1}{a_2} = \frac{a_0}{a_3},$$

that must be imposed on both regressions. The results for both regressions using the two proxies for the long- and short-term risk-free rates are shown in Table 5.

The parameter values are not directly comparable to the other models, but suggest a fixed component to the long-run risk-premium of around 2.7%. The mean value for the risk-premium is 1.1% given the mean value of x_t and VIX of 1.79% and 21%, respectively. This model suggests that even at the highest wealth levels, the risk-premium never fell below zero, reaching a minimum value on 4/12/2000. This is in sharp contrast to the short-term findings, which found negative short-term

Table 5 Estimation of Model 3.4: Two-growth rate, blended interest rates, blended volatility model.

	r_L = 10 year Bond and r_S = one year Bill		r_L = 30 year Bond and r_S = one year Bill	
	1st stage	2nd stage	1st stage	2nd stage
$\lambda_0(1-w)\theta$	0.02896 (35.66)**	0.02891 (49.26)**	0.02571 (35.75)**	0.02573 (48.49)**
$\lambda_0 w$	0.02899 (10.16)**	0.02899 (10.16)**	0.00939 (3.69)**	0.0094 (3.70)**
β	1.26908 (89.39)**		1.15754 (111.74)**	
$\lambda_1 w$	−0.00814 (9.98)**	−0.00814 (9.99)**	−0.00282 (3.67)**	−0.00282 (3.69)**
$\lambda_1(1-w)\theta$	−0.00813 (30.54)**	−0.00814 (42.01)**	−0.00771 (31.83)**	−0.00772 (41.49)**
Obs.	3679	3679	3679	3679
R^2	90.62%	**49.76%**	93.23%	**49.80%**

Robust t-statistics in parentheses.
*significant at 5%; ** significant at 1%.
This table reports the parameter estimates of the first- and second-stage linear estimation with nonlinear constraints of Model 3.4 given in Eq. (17). For both regressions the condition that $a_1/a_5 = a_2/a_4$ is imposed. Using the first-stage estimate $\widehat{\beta}$, in the second-stage regression the dependent variable is adjusted by $-\widehat{\beta}(r_L - r_S)$ and $a_3 = 0$. The dependent variable is the risk premium $(D_{0,t}/P_t)(1+g_{S,t}) + g_{L,t} - r_{S,t}$. The coefficients are equal to: $a_1 = \lambda_0(1-w)\sigma$, $a_2 = \lambda_0 w$, $a_3 = \beta$, $a_4 = \lambda_1 w$, $a_5 = \lambda_1(1-w)\sigma$. The estimation period is from June 1990 through December 2004. t-stats are listed in parentheses.
First stage:

$$\frac{D_{0,t}(1+g_{S,t})}{P_t} + g_{L,t} - r_{S,t} = a_0 + a_1\mathrm{VIX}_t + a_2\mathrm{VIX}_t x_t + a_3 x_t + \beta(r_L - r_S) + e_t.$$

Second stage:

$$\frac{D_{0,t}(1+g_{S,t})}{P_t} + g_{L,t} - r_{S,t} - \widehat{\beta}(r_L - r_S) = a_0 + a_1, \mathrm{VIX}_t + a_2\,\mathrm{VIX}_t x_t + a_3 x_t + e_t.$$

expected Sharpe ratios. These difference highlight why there are bubbles and troughs over short-term intervals, while over the long-term, the market produces positive expected risk-premiums.

In terms of performance, the findings for the first-stage regression are similar as those for Model 3.3. However, there is vast improvement in the second-stage regression, demonstrated in Figure 3, suggesting the importance of accounting for mean-reversion in implied volatility. While it is not possible to infer the true value of w, if we assume a mean-value for long-run volatility of 21% and r_L is a 30-year T-bond, suggesting a value of less than 10% on current volatility. This is quite surprising, as it suggests that investors focus more

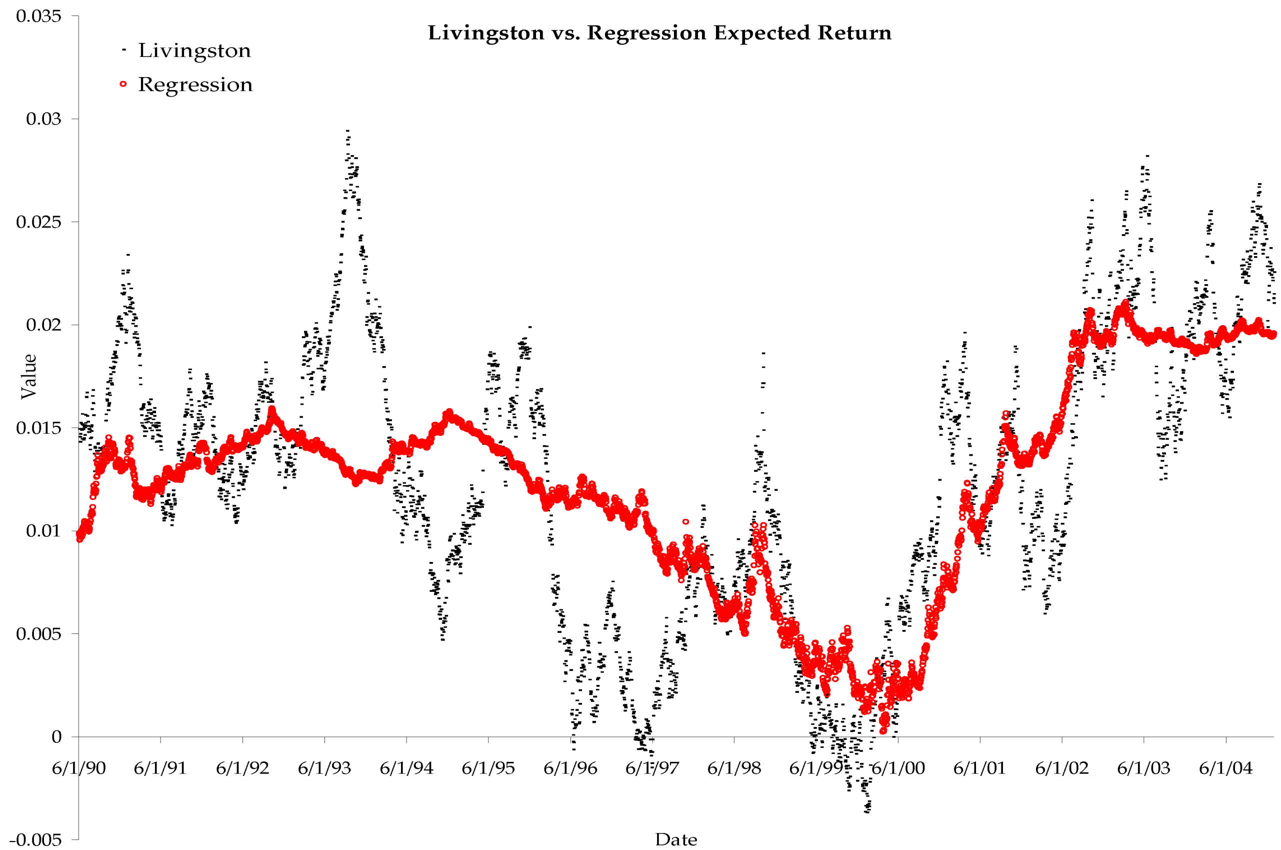

Figure 3 Expected risk-premia using model 3.4.

Figure 3 demonstrates the relationship over January 1986 to December 2004 between the Livingston/Philadelphia Fed expected risk-premia, $(D_{0,t}/P_t)(1+g_{S,t})+g_{L,t}-r_{S,t}-\widehat{\beta}(r_L-r_S)$, and the regression-predicted expected risk-premia, $0.0257+0.0094\,\mathrm{VIX}_t-0.0028\,\mathrm{VIX}_t\,x_t-0.0077x_t$, using coefficient estimates from regression equation (17). The control variable in this case is $x_t=\frac{\mathrm{S\&P}_t}{\mathrm{S\&P}_{t-5,t-6}}$.

$$\text{Livingston/Philadelphia Fed expected return} = \frac{D_{0,t}(1+g_{S,t})}{P_t}+g_{L,t}-r_{S,t}-\widehat{\beta}(r_L-r_S)$$

$$\text{Regression-predicted expected return} = 0.0257+0.0094\,\mathrm{VIX}_t-0.0028\mathrm{VIX}_t\,x_t-0.0077x_t.$$

on long-term volatility and are relatively insensitive short-term volatility swings. In this light, our ability to explain those relatively minor changes in the long-term risk-premium is of particular interest.

This is highlighted in Figure 2. By controlling for the mean-reverting nature of VIX, we have reduced the variation in the expected risk-premia beyond Model 3.3, demonstrating an almost permanent component, which only fluctuates moderately with perceived levels of wealth and contemporaneous values of VIX. Figure 4 decomposes the total rate of return implied by Model 3.4 into two components, the blended risk-free rate and the expected risk-premium. As can be seen, the expected risk-premium is a small component of the rate of return, and demonstrates relatively little time-variation. By comparison, the blended risk-free rate, which includes the short-term rate plus term-premium as estimated by the unconstrained regression, captures most of the time-variation. As several authors have noted, there is a general concern that the risk-premium has been declining over recent years. However, the evidence here seems to suggest the

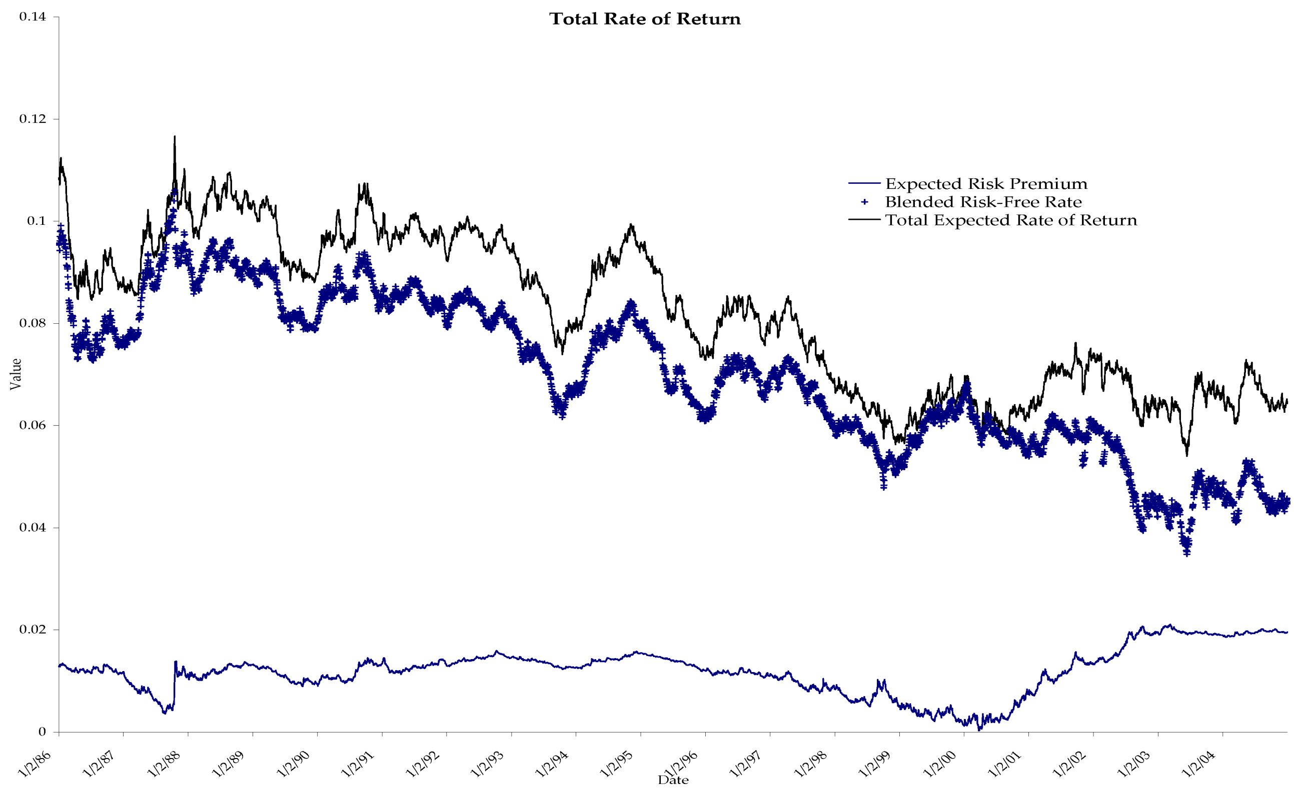

Figure 4 Components for the long-term expected rate of return.
Figure 4 demonstrates the relationship over January 1986 to December 2004 from the fitted value of the expected risk-premium from Model 3.4. The total expected rate of return is broken down into the expected risk-premium, and the blended risk-free component:

$$\text{Blended risk-free rate} = r_1 + 1.157(r_{30} - r_1)$$

$$\text{Expected risk-premium} = 0.0257 + 0.0094\,\text{VIX}_t - 0.0028\,\text{VIX}_t\,x_t - 0.0077x_t.$$

contrary. Since the tech-bubble burst in 2000, the expected risk-premium is on the rise; what has fallen is the spread between long- and short-term treasuries.

4.6 Realized returns

We now wished to test all four models on realized returns. Given the current state of the literature on the equity premium puzzle, our expectation is that each model should perform quite poorly, which would be consistent with other empirical findings and the notion of unpredictability of asset returns. In addition, we want to demonstrate that the realized Sharpe ratio is higher than the expected, confirming the findings of Mehra and Prescott (1985). The realized returns for day t are calculated in two ways:

$$R_t^m \equiv \left(\frac{\text{S\&P}_{t+1/12} + D_{t+1/12}}{\text{S\&P}_t}\right)^{12} - 1$$

$$R_t^a \equiv \left(\frac{\text{S\&P}_{t+1} + D_{t+1}}{\text{S\&P}_t}\right) - 1,$$

where $\text{S\&P}_{t+1/12}$ is the level of the S&P 500 one-month ahead and $D_{t+1/12}$ is the dividend payout. We have calculated one-month returns since VIX is a one-month estimate of implied volatility. One-year returns are also calculated since typical holding periods are longer than one-month, and one-month variation may incorporate short-term shocks outside of our current model. The annualized value of

R_t^m in the one-month case is then adjusted by the annualized risk-free rate and VIX to create a realized market price of risk:

$$\lambda_t = \frac{R_t - r_t}{\mathrm{VIX}_t}.$$

This value is regressed on the same dependent variables as in the prior section to make the interpretation comparable. The results reported in Tables 6 and 7 are analogous to those reported in Sections 4.2–4.5, but using realized returns.

Table 6 Estimation of realized market price of risk using a 30-day window.

	Models 3.1 & 3.2		Model 3.3				Model 3.4	
x_t	−0.627	0.66	−0.95	−0.95	−0.433	−0.432		
	(9.24)**	(2.01)*	(11.65)**	(15.19)**	(1.23)	(1.23)		
x_t^2		−0.326			−0.133	−0.133		
		(3.90)**			(1.49)	(1.51)		
λ_0	1.609	0.436	2.694	2.693	2.246	2.244		
	(12.63)**	(1.4)	(13.30)**	(22.50)**	(6.55)**	(6.81)**		
$\lambda_0(1-w)\sigma$							0.33	0.335
							(4.66)**	(5.67)**
$\lambda_0 w$							0.761	0.76
							(2.63)**	(2.64)**
$\lambda_1 w$							−0.256	−0.255
							(2.60)**	(2.61)**
$\lambda_1(1-w)\sigma$							−0.111	−0.112
							(4.51)**	(5.36)**
β			−6.171		−6.269		−5.449	
			(9.33)**		(9.34)**		(6.71)**	
Obs.	4775	4775	3657	3657	3657	3657	3657	3560
R^2	1.88%	2.14%	3.05%	5.47%	3.11%	5.60%	2.15%	3.83%

Robust t-statistics in parentheses.
*significant at 5%; **significant at 1%.
This table reports the parameter estimates using realized returns in all four models. The realized returns were calculated over a 30 day window, and the market price of risk was inferred using the 30-day implied volatility from VIX and the one-year Treasury Bill. The realized market price of risk was then regressed on $x_t \equiv \mathrm{S\&P}_t/\mathrm{S\&P}_{t-5,t-6}$. All independent variables are calculated in the same fashion as those in the expected market price of risk regression. t-stats are listed in parentheses.

$$\frac{[(\mathrm{S\&P}_{t+1/12} + D_{t+1/12})/\mathrm{S\&P}_t]^{12} - 1 - r_{S,t}}{\mathrm{VIX}_t} = \lambda_0 + \lambda_1 x_t + e_t$$

$$\frac{[(\mathrm{S\&P}_{t+1/12} + D_{t+1/12})/\mathrm{S\&P}_t]^{12} - 1 - r_{S,t}}{\mathrm{VIX}_t} = \lambda_0 + \lambda_1 x_t + \beta\frac{r_{L,t} - r_{S,t}}{\mathrm{VIX}_t} + e_t$$

$$\left(\frac{\mathrm{S\&P}_{t+1/12} + D_{t+1/12}}{\mathrm{S\&P}_t}\right)^{12} - 1 - r_{S,t} = a_0 + a_1\mathrm{VIX}_t + a_2\mathrm{VIX}_t x_t + a_3 x_t + \beta(r_{L,t} - r_{S,t}) + e_t.$$

Table 7 Estimation of realized market price of risk using one-year window.

	Models 3.1 & 3.2		Model 3.3				Model 3.4	
x_t	−0.61	0.96	−0.90	−0.90	1.05	1.05		
	(33.61)**	(11.48)**	(35.43)**	(46.47)**	(11.17)**	(11.04)**		
x_t^2		−0.39			−0.49	−0.49		
		(18.26)**			(19.62)**	(19.87)**		
λ_0	1.59	0.13	2.59	2.59	0.84	0.84		
	(43.29)**	(1.70)	(40.06)**	(68.64)**	(9.88)**	(10.43)**		
$\lambda_0(1-w)\sigma$							0.52	0.53
							(23.92)**	(29.62)**
$\lambda_0 w$							0.04	0.04
							(0.52)	(0.54)
$\lambda_1 w$							−0.02	−0.02
							(0.52)	(0.54)
$\lambda_1(1-w)\sigma$							−0.18	−0.18
							(22.39)**	(26.58)**
β			−5.01		−5.24		−5.61	
			(24.70)**		(26.62)**		(23.90)**	
Obs.	4678	4678	3560	3560	3560	3560	3560	3560
R^2	14.79%	17.79%	19.73%	31.29%	24.70%	36.39%	22.77%	37.35%

Robust t-statistics in parentheses.

*significant at 5%; **significant at 1%.

This table reports the parameter estimates using realized returns in all four models. The realized returns were calculated over a one-year window, and the market price of risk was inferred using the 30-day implied volatility from VIX and the one-year Treasury Bill. The realized market price of risk was then regressed on $x_t = \text{S\&P}_t/\text{S\&P}_{t-5,t-6}$. All independent variables are calculated in the same fashion as those in the expected market price of risk regression. t-stats are listed in parenthesis.

$$\frac{(\text{S\&P}_{t+1}+D)/\text{S\&P}_t - 1 - r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + e_t$$

$$\frac{(\text{S\&P}_{t+1}+D)/\text{S\&P}_t - 1 - r_{S,t}}{\text{VIX}_t} = \lambda_0 + \lambda_1 x_t + \beta\frac{r_{L,t} - r_{S,t}}{\text{VIX}_t} + e_t$$

$$\frac{\text{S\&P}_{t+1}+D}{\text{S\&P}_t} - 1 - r_{S,t} = a_0 + a_1\text{VIX}_t + a_2\text{VIX}_t x_t + a_3 x_t + \beta(r_L - r_S) + e_t.$$

Regardless of holding period, in each model, the R^2 is lower, except for the constrained regression of Model 3.3 using one-year holding period returns. This is interesting as it suggests that there is a limited explanatory power in the term-premium for forecasting realized results. While the variability in the expected Sharpe ratio is a function of the implied volatility, growth rate, risk-free rate, and dividend yield, the variability in realized Sharpe ratio appears unrelated to these factors.[16] More appropriately, the inability to forecast future returns seem to suggest that there are additional factors that have yet been identified or is a function of random error.

The coefficient estimates from the regression imply a realized Sharpe ratio of .41 for each of the three models and is consistent with the simple historical calculation. The finding for the realized Sharpe ratio is almost 2.5 times greater than the expected Sharpe ratio over the same period. This results in a difference in the risk-premia of roughly 4%. This is less that the Mehra and Prescott (1985) finding, where their reported difference between the realized and expected premium was roughly 6%.[17] The differences in expected, model predicted (using Model 3.1), and realized returns are shown in Table 8.

Table 8 reveals an interesting contrast between the coefficients obtained for the expected and realized regressions. For purpose of comparison, standardizing by the mean value of $\text{S\&P } 500_t/\text{S\&P}_{t-5,t-6} = 1.79$, the *expected* market price of risk is 0.179, whereas the *realized* analog is 0.441. This quantitative disparity highlights the difference between the expected and realized Sharpe ratios over this time period.

Table 8 Realized versus expected returns.

Variable	Obs.	Mean (%)	SD (%)	Min (%)	Max (%)
R_t	4795	11.91	53.08	−327.2	238.9
$\widetilde{R}_t$	4795	12.39	6.83	−8.3	40.6
$E(R_t)$	4795	7.97	1.68	4.0	11.4
$E(\widetilde{R}_t)$	4795	7.96	2.08	4.5	20.2

This table reports the realized and expected returns over the time period. The realized returns, R_t, are calculated as the annualized 30-day return, $R_t^m = [(\text{S\&P}_{t+1/12} + D_{t+1/12})/\text{S\&P}_t]^{12} - 1$. The −327.2% annualized loss was over the 9/23/87–10/26/87 period, where the index fell 28.5%, from 319.72 to 227.42. The expected returns, $E(R_t)$ are the model-dependent returns derived from the dividend-growth model $(D_{0,t}/P_t)(1 + g_{S,t}) + g_{S,t} - r_{S,t}$ as given in Section 3.1. The *estimated* realized returns, $\widetilde{R}_t$, and *estimated* expected returns, $E(\widetilde{R}_t)$, are the predicted values derived from coefficient estimates from the regression of Eq. (5).

Annualizing the monthly returns by a factor of 252/22, we observe an annualized difference between expected versus realized return of about 4%. The regression's predicted difference is 4.4%. What is most revealing is how the results here again highlight the difference in predicting expected versus realized returns. There is little to no difference in the mean, standard deviation, minimum, and maximum values in the predicted and actual expected returns, while there are drastic differences in the predicted and actual realized returns.[18] This suggests that the model effectively captures the *ex-ante* performance in the market, but reveals little information on ex-post returns. This result appears consistent with the notion of market efficiency, as time-varying expected returns are not inconsistent with a poor ability to forecast highly-noisy ex-post realized returns.

5 Modeling the measurement error in g_t and VIX

In using the data in this fashion, it is possible to introduce heteroscedasticity since there is the potential measurement error in the infrequently observed growth rate. To address this problem, we adjust for the measurement error as shown in the Appendix and re-run Eq. (5) on the prior dependent variables. The resulting specification eliminates the intercept term by dividing through by $\hat{\sigma}_{t,\epsilon}$, and results in a homoscedastic regression. The results of this regression for both expected and realized Sharpe ratios using all three measures of investor sentiment can be found in Table 9.

The findings are similar to the findings in the prior tables. The inferred Sharpe ratios for the expected and realized returns are around .14 and .41, respectively. In addition there is no change in the statistical significance of the estimates. We conclude that measurement error in the variables is of little concern.

Table 9 Estimation of realized and expected market price of risk with heteroscedasticity correction.

	Model 1: $x_t = \frac{S\&P_t}{S\&P_{t-5}}$		Model 2: $x_t = \frac{S\&P_t}{S\&P_{t-6}}$		Model 3: $x_t = \frac{S\&P_t}{S\&P_{t-5,t-6}}$	
	$Et(\widehat{\lambda}_t)$	λ_t	$E(\widehat{\lambda}_t)$	λ_t	$E(\widehat{\lambda}_t)$	λ_t
λ_1	−0.128	−0.373	−0.143	−0.723	−0.139	−0.634
	(64.23)**	(5.51)**	(77.58)**	(10.52)**	(73.83)**	(9.27)**
λ_0	0.373	1.082	0.432	1.871	0.406	1.617
	(98.54)**	(8.49)**	(110.84)**	(13.02)**	(108.25)**	(11.95)**
Obs.	4795	4795	4795	4795	4795	4795
R^2	81.2%	3.23%	83.7%	4.72%	82.9%	4.04%

Absolute t-statistics in parentheses.
*significant at 5%; **significant at 1%.
This table reports the parameter estimates of the heteroscedasticity corrected OLS estimation of Eq. (A.3) using realized and expected returns. The algorithm for correcting the measurement error in the growth rate, g_t, can be found in Appendix. The results for all three proxies of wealth are reported. There is no significant statistical difference between these estimates, and those found in Table 2. The results for quadratic regression are available upon request. t-stats are listed in parentheses.

$$\frac{E(\widehat{\lambda}_t)}{\widehat{\sigma}_{t,\epsilon}} \equiv \frac{[(D_{0,t}/P_t)(1+g_{S,t}) + g_{S,t} - r_{S,t}]/\text{VIX}_t}{\widehat{\sigma}_{t,\epsilon}} = \lambda_0 \frac{1}{\widehat{\sigma}_{t,\epsilon}} + \lambda_1 \frac{x_t}{\widehat{\sigma}_{t,\epsilon}} + e_t$$

$$\frac{\lambda_t}{\widehat{\sigma}_{t,\epsilon}} \equiv \frac{[\text{S\&P}_{t+1} + D)/\text{S\&P}_t - 1 - r_{S,t}]/\text{VIX}_t}{\widehat{\sigma}_{t,\epsilon}} = \lambda_0 \frac{1}{\widehat{\sigma}_{t,\epsilon}} + \lambda_1 \frac{x_t}{\widehat{\sigma}_{t,\epsilon}} + e_t.$$

6 Conclusion

This paper presented a parsimonious, easily implementable model for the estimation of the short- and long-term expected rates of return on the S&P 500 stock market Index. Using as our primary variable of interest the Market Price of Risk, or Sharpe Ratio, of the S&P 500 Index, we used as predictive variables the risk-free rate of interest, the economy's growth rate estimate, and the option market's implied volatility on the S&P 500 Index. The model explicitly accounted for an assumed increasing relative risk aversion by incorporating and estimating the impact of past S&P 500 returns.

Conditioning on four variables—the risk-free rate of interest r_t, the slope of the yield curve $r_{L,t} - r_{S,t}$, the implied volatility VIX_t on the Index, and the *realized* S&P 500 Index rate of return over the past five to six years S&P 500_t/S&P $500_{t-5,t-6}$—the model generated expected rates of return μ_t given by expressions of the form:

$$\mu_t = \begin{cases} r_{S,t} + \left(0.46 - 0.162\dfrac{\text{S\&P }500_t}{\text{S\&P }500_{t-5,t-6}}\right)\text{VIX}_t & \text{for a } \textit{one-year} \text{ horizon} \\ r_{S,t} + 1.158(r_{L,t} - r_{S,t}) + 0.0257 + 0.0094\text{VIX}_t & \\ \quad - 0.00282\text{VIX}_t\dfrac{\text{S\&P }500_t}{\text{S\&P }500_{t-5,t-6}} - .00772\dfrac{\text{S\&P }500_t}{\text{S\&P }500_{t-5,t-6}} & \text{for the long-term.} \end{cases} \tag{18}$$

In examining the implications of (18) for short- and long-term expected rates of return, we find that:

1. Short-term expected rates of return are quite volatile, due to changes in VIX, the term structure of interest rates $\{r_{S,t}, r_{L,t}\}$ and the accumulated wealth factor S&P 500_t/S&P $500_{t-5,t-6}$.
2. The *behavior* of the short-term μ_t presents an interesting "history" of the past 20 years, with the risk-premium peaking immediately subsequent to the 1987 stock market crash and reaching a low point—a *negative* risk-premium just as the stock market reached its recent March 2000 high-water, possibly "bubble," mark.
3. The long-term expected risk-premium is remarkably stable, as indeed befits a long-term predictor of the excess return on the US stock market: Any transitory effects would be *expected* to dissipate in the long-term.
4. Whereas the long-term risk-premium unsurprisingly reached a low point in the March–April 2000 time period, it remained slightly positive and never fell into negative territory.
5. While we believe our model effectively captures *ex-ante* market information, such is by no means inconsistent with market efficiency, since time-varying expected returns are entirely consistent with a poor ability to forecast highly noisy *ex-post* realized returns.

With respect to the relationship between expected and realized returns, we find the work Fama and French (2002), "The Equity Premium," particularly relevant to our results. Quoting from their Abstract:

"We estimate the equity premium using dividend and earnings growth rates to measure the expected rate of capital gain. Our estimates for 1951 to 2000, 2.55% and 4.32%, are much lower than the equity premium produced by the average stock return, 7.43%. Our evidence suggests that high average return for 1951 to 2000 is due to a decline in discount rates that produces a large unexpected capital gain. Our main conclusion is that the average stock return of the last half-century is a lot higher than expected."

Our work has used *prospective* data on growth rates and volatility for the time period available, January 1986 to December 2004, with a parsimonious expected-return model, and come to strikingly similar results.

In conclusion, this suggests to us two qualitative results:

1. A negative implied equity risk-premium, such as manifested themselves (in our data period) in October 1987 and the 2000 period are strongly suggestive of "irrational exuberance" giving rise to unsustainably high asset prices.
2. Overall, in this period January 1986 to December 2004 there were positive shocks to the system that resulted in realized returns exceeding their expected values. While it is tempting to suggest two of these shocks were the "peace dividend" following 1991 and the productivity shocks induced by improved computer technology in the 90s, such attribution must at this time remain speculative.

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Appendix Modeling the measurement error in g_t and VIX

1. Recall the basic equation we are examining is:

$$\frac{D_{0,t}(1+g_t)}{P_t} + g_t = r_t + \lambda_t \sigma_t. \tag{A.1}$$

x_t is our wealth-relative variable, e.g., $x_t = \text{S\&P}_t/\text{S\&P}_{t-6}$ or $x_t = \text{S\&P}_t/\text{S\&P}_{t-5}$. Now, combine (A.1) with

$$\lambda_t = \lambda_0 + \lambda_1 x_t.$$

If we rearrange (A.1), we obtain

$$\frac{1}{\sigma_t}\left(\frac{D_{0,t}}{P_t}+1\right)g_t + \frac{D_{0,t}}{\sigma_t P_t} - \frac{r_t}{\sigma_t} = \lambda_0 + \lambda_1 x_t. \tag{A.2}$$

2. If realized volatility at date t is σ_t, then for a negative market price of volatility risk, we have $\text{VIX}_t > \sigma_t$. Assume then that

$$\sigma_t = a\text{VIX}_t e_V$$

for some $a < 1$ and an error term e_V satisfying $E(\ln e_V) = 0$.

3. Because of the quarterly (hence less than monthly) observation frequency for the growth rate g_t, assume the growth rate is measured with error, resulting in the substitution in Eq. (A.2) of g_t with $g_t + e_g$.

4. Substituting these two expressions into (A.2) results in

$$\begin{aligned}\frac{1}{a\text{VIX}_t e_V}\left(\frac{D_{0,t}}{P_t}+1\right)(g_t+e_g) \\ + \frac{D_{0,t}}{a\text{VIX}_t e_V P_t} - \frac{r_t}{a\text{VIX}_t e_V} \\ = \lambda_0 + \lambda_1 x_t + e_t,\end{aligned}$$

where e_t is the regression's error term. Now, multiplying through by ae_V results in

$$\begin{aligned}\frac{1}{\text{VIX}_t}\left(\frac{D_{0,t}}{P_t}+1\right)(g_t+e_g) + \frac{D_{0,t}}{\text{VIX}_t P_t} - \frac{r_t}{\text{VIX}_t} \\ = \lambda_0 a e_V + \lambda_1 a x_t e_V + a e_V e_t.\end{aligned}$$

Transpose the e_g term to the RHS:

$$\begin{aligned}\frac{1}{\text{VIX}_t}\left(\frac{D_{0,t}}{P_t}+1\right)g_t + \frac{D_{0,t}}{\text{VIX}_t P_t} - \frac{r_t}{\text{VIX}_t} \\ = \lambda_0 a e_V + \lambda_1 a x_t e_V + a e_V e_t \\ - \frac{1}{\text{VIX}_t}\left(\frac{D_{0,t}}{P_t}+1\right)e_g.\end{aligned} \tag{A.3}$$

Clearly, regression (A.3) is subject to heteroscedasticity, since (with the relevant statistical assumptions)

$$\begin{aligned}\text{Var}\Big[\lambda_0 a e_V + \lambda_1 a x_t e_V + a e_V e_t \\ - \frac{1}{\text{VIX}_t}\left(\frac{D_{0,t}}{P_t}+1\right)e_g\Big] \\ = (\lambda_0^2 a^2 + \lambda_1^2 a^2 x_t^2)\sigma_V^2 + \Sigma_1^2 \\ + \frac{1}{\text{VIX}_t^2}\left(\frac{D_{0,t}}{P_t}+1\right)^2 \sigma_g^2 \\ \equiv \Sigma^2 + \lambda_1^2 a^2 x_t^2 \sigma_V^2 \\ + \frac{1}{\text{VIX}_t^2}\left(\frac{D_{0,t}}{P_t}+1\right)^2 \sigma_g^2,\end{aligned}$$

which shows the heteroscedasticity induced by x_t and $y_t \equiv \frac{1}{\text{VIX}_t}(\frac{D_{0,t}}{P_t}+1)$.

5. Adjusting for this heteroscedasticity is not simple, since it has a constant Σ as well as time-t dependent variables x_t and y_t:
The solution we implement, albeit one that is cumbersome, is the following:

   (a) Estimate σ_g from a time-series of g_t's: $\widehat{\sigma}_g =$ Std. Dev. (g_t).
   (b) Estimate σ_V from a regression of $\ln(\sigma_t/\text{VIX}_t)$ on a constant, which simultaneously produce an estimate of $\widehat{a}$ as well as $\widehat{\sigma}_V$.
   (c) Obtain an estimate of regression (A.3) slope coefficient $\widehat{\lambda_1 a}$ by running an OLS version of the regression (A.3).

(d) By (initially) setting $\Sigma \equiv 0$, obtain an estimate of the date t regression error (A.4) $\widehat{\sigma}_t$ by substituting $\widehat{\sigma}_g, \widehat{\lambda_1 a}$. and $\widehat{\sigma}_V$.

(e) Deflate both LHS and RHS of (A.3) by this estimate of $\widehat{\sigma}_t$, then run the regression (A.3).

(f) Calculate the std. dev. of the error term in that regression $\epsilon_t, \widehat{\sigma}_\epsilon$. On average, we would expect that

$$\widehat{\sigma}_\epsilon^2 = \Sigma^2 + \frac{1}{T}\left[\sum_t \lambda_1^2 a^2 x_t^2 \sigma_V^2 + \frac{1}{\mathrm{VIX}_t^2}\left(\frac{D_{0,t}}{P_t} + 1\right)^2 \sigma_g^2\right]. \quad \text{(A.4)}$$

(g) In (A.4), if $\Sigma \geq 0$, substitute Σ into (A.3), deflate by the new $\widehat{\sigma}_t$ and run regression (A.3) one more time.

Notes

1 For example, financial economists have sought to understand the US equity risk-premium in comparison to markets where a breakdown of trading occurred in the second and fourth decades of the 20th century.

2 We are of course not the first to posit nonconstant relative risk aversion; such analyses date back to the Arrow (1970)–Pratt (1964) definitions of risk aversion.

3 Merton (1980) uses a similar but not identical measure—the ratio of the risk-premium to variance, whereas we posit the ratio of risk-premium to the square root of variance, namely standard deviation.

4 In the final analysis, whether our market-price-of-risk assumption holds is a matter of empirical testing: In our tests, we seek to determine whether the sign of the coefficient on the wealth proxy is negative (consistent with our assumption), zero (consistent with a constant market price of risk), or positive.

5 In the discussion of the empirical proxies, note that we take care to devise the RHS variable S&P 500_t/S&P 500_{t-T} in a way that will not diminish our ability to estimate Eq. (5).

6 In addition, a robustness test will be offered to examine measurement errors in the growth-rate g and a relaxation of whether VIX is the proper proxy for volatility.

7 With respect to the effect of the term structure spread, a related effect was documented by Fama–French (1993), who consider a regression of the form:

$$e_p(t+1) = k_0 + k_1 \text{Div. Yield} + k_2 \text{Default Spread} + k_3 \text{Term Spread} + k_4 \text{Risk-free rate},$$

where $e_p(t+1)$ is the date-$t+1$ residual for "our 25 stock and seven bond portfolios from the five-factor regression of table 7." There are two important differences between their paper and ours: (1) Fama–French document a *cross-sectional* effect; (2) Whereas Fama–French document a cross-sectional difference, our analysis considers the impact on the *market portfolio.*

8 For example, if $L = 30$ and $\beta = 1$, then $r_{S,t} + \beta(r_{L,t} - r_{S,t}) = r_{L,t} = r_{30,t}$, then the "target maturity" is $L = 30$ years.

9 We had sought to use futures contracts on the S&P to obtain implied future dividend yields. Unfortunately, active futures contracts do not extend the full one-year maturity required to calculate such an implied dividend yield. Using long-dated LEAPS to infer the value of long-dated S&P futures contracts might provide an implied *ex-ante* dividend yield, albeit in this case one driven by the risk-neutral rather than statistical expectations.

10 The last observed date for the 30-year T-bill is on 2/15/2002. All subsequent dates use the 20-year T-bill yields.

11 Most of the noted flaws in using the Livingston data have focused on the CPI forecasts, and not GDP. In particular, Dokko and Edelstein (1989) find that the Livingston stock market surveys are unbiased estimators of realized stock returns.

12 Base values are not used as there are discrepancies in the data as noted by the Philadelphia Fed.

13 Such limited-memory allows us to indirectly model the *per capita* wealth which we seek to proxy, and which a long-term upward-drift in the S&P would fail to capture. We also examined alternate proxies, $\text{S\&P}_t/\text{S\&P}_{t-1}$, $\text{S\&P}_t/\text{S\&P}_{t-5}$, $\text{S\&P}_t/\text{S\&P}_{t-6}$, and $\text{S\&P}_t/\text{S\&P}_{t-10}$ and have those results available upon request.

14 The realized returns are calculated using daily return and dividends.

15 Moreover, the negative sign of $\widehat{\lambda}_2$ is not confirmed in the truncated, nonoverlapping observations sample.

16 Moreover, the coefficients on x_t and x_t^2 often have the wrong sign—i.e., different from the model's predictions.

17 This may be a direct result of the 2000–2004 period, where the average difference is actually negative and is not included in Mehra (2003).

18 The rolling one-year realized returns report a similar mean, but an annualized standard deviation of 16.2%, a minimum value of −32.6%, and a maximum of 52.6%.

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